भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 9

Swaprava Nath

Slide preparation acknowledgments: Rounak Dalmia

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

Task Allocation Domain

- Unit amount of task to be shared among n agents
- Agent i gets a share $s_i \in [0, 1]$ of the job, $\sum_{i \in N} s_i = 1$
- Agent **payoff**: every agent has a most preferred share of work.
- **Example:**
  - The task has rewards, e.g., wages per unit time $= w$
  - if agent i works for t_i time then gets $w \cdot t_i$
  - The task also has costs, e.g., physical tiredness/less free time, etc. Let the cost be quadratic = $c_i t_i^2$
  - Net payoff = $wt_i - c_i t_i^2 \implies$ **maximized** at $t_i = w/2c_i$, and **monotone** decreasing on both sides

Task Allocation Domain

- Net payoff = wt_i - $c_i t_i^2 \implies$ **maximized** at $t_i = w/2c_i$
- **Important:** This is single peaked over the **share of the task** and not over the alternatives
- Suppose, two alternatives are $(0.2, 0.4, 0.4)$ and $(0.2, 0.6, 0.2)$: player 1 likes both of them equally
- For 3 players, the set of alternatives is a simplex
- There cannot be a single common order over the alternatives s.t. the preferences are single-peaked for all agents

Task Allocation Domain and Pareto Efficiency

- Denote this **domain of task allocation** with T
- An allocation of the task is $a = (a_i \in [0,1], i \in N)$, set of all task allocations is A
- **SCF:** $f : T^n \to A$
- Let $P \in T^n$
  - $f(P) = (f_1(P), f_2(P), \dots, f_n(P))$
  - $f_i(P) \in [0,1], \ \forall i \in N$
  - $\sum_{i \in N} f_i(P) = 1$
- Player i has a peak p_i over the shares of the task

Definition (Pareto Efficiency)

An SCF f is *Pareto efficient* (PE) if there does not exist any profile P where there exists a task allocation $a \in A$ such that it is weakly preferred over $f(P)$ by all agents and strictly preferred by at least one. Mathematically,

$$\nexists a \in A \text{ s.t. } \begin{array}{ll} a \, R_i \, f(P) & \forall i \in N, \\ a \, P_j \, f(P) & \exists j \in N \end{array}$$

Implications of Pareto Efficiency

1. If $\sum_{i \in N} p_i = 1$, allocate tasks according to the peaks of the agents This is the unique PE allocation
2. If $\sum_{i \in N} p_i > 1$, there must exist $k \in N$, s.t. $f_k(P) < p_k$

Question

Can there be an agent j s.t. $f_j(P) > p_j$ if f is PE?

Answer

No. If such a j exists, increasing k's share of task and reducing j's makes both players strictly better off
Therefore, $\forall j \in N, f_j(P) \leqslant p_j$

3. If $\sum_{i \in N} p_i < 1$, by a similar argument, we conclude that $\forall j \in N, f_j(P) \geqslant p_j$

Task Allocation Domain and Anonymity

Definition (Anonymity)

An SCF f is *anonymous* (ANON) if for every agent permutation $\sum_{i \in N} : N \to N$, the task shares get permuted accordingly, i.e.,

$$\forall \sigma,\ f_{\sigma(j)}(P^{\sigma}) = f_j(P)$$

Example:

- $N = \{1, 2, 3\},\ \sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$
- $P = (0.7, 0.4, 0.3) \implies P^{\sigma} = (0.3, 0.7, 0.4)$
- $f_1(0.7, 0.4, 0.3) = f_2(0.3, 0.7, 0.4)$

Task Allocation Domain: Some Candidate SCFs

Definition (Serial Dictatorship)

A predetermined sequence of the agents is fixed. Each agent is given either her peak share or the leftover share of the task. If $\sum_{i \in N} p_i < 1$, then the last agent is given the leftover share.

Question

PE, SP, ANON?

Answer

Not ANON. Also quite unfair to the last agent.

Task Allocation Domain: Some Candidate SCFs

Definition (Proportional)

Every player is assigned a share that is c times their peaks, s.t. $c \sum_{i \in N} p_i = 1$

Question

PE, ANON, SP?

Answer

Not SP.
Suppose peaks are $0.2, 0.3, 0.1$ for 3 players, $c = 1/0.6$
Player 1 gets $1/3$ (more than its peak 0.2)
if the report is $0.1, 0.3, 0.1$, $c = 1/0.5$, player 1 gets 0.2

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

PE, ANON, and SP?

Question

How to ensure PE, ANON, and SP in task allocation domain?

Uniform Rule (Sprumont 1991)

- Suppose, $\sum_{i \in N} p_i < 1$
- Begin with everyone's allocation being 1 (infeasible)
- Keep reducing until $\sum_{i \in N} f_i = 1$
- On this path, if some agent's peak is reached, set the allocation for that agent to be its peak
- Symmetric for $\sum_{i \in N} p_i > 1$

Definition

1. **Case** $\sum_{i \in N} p_i = 1$: $f_i^u(P) = p_i$
2. **Case** $\sum_{i \in N} p_i < 1$: $f_i^u(P) = \max\{p_i, \mu(P)\}$, where $\mu(P)$ solves $\sum_{i \in N} \max\{p_i, \mu\} = 1$
3. **Case** $\sum_{i \in N} p_i > 1$: $f_i^u(P) = \min\{p_i, \lambda(P)\}$, where $\lambda(P)$ solves $\sum_{i \in N} \min\{p_i, \lambda\} = 1$

The Uniform Rule

Theorem (Sprumont 1991)

*The **uniform rule** SCF is ANON, PE, and SP*

- **ANON** is obvious: only the peaks matter and not their owners
- **PE**: the allocation is s.t.
  - $f_i^u(P) = p_i, \forall i \in N$, if $\sum_{i \in N} p_i = 1$
  - $f_i^u(P) \geqslant p_i, \forall i \in N$, if $\sum_{i \in N} p_i < 1$
  - $f_i^u(P) \leqslant p_i, \forall i \in N$, if $\sum_{i \in N} p_i > 1$
- This is PE from our previous observation on PE: *allocations should stay on the same side of the peaks for every agent*

The Uniform Rule: Strategyproofness

- **Case** $\sum_{i \in N} p_i = 1$: each agent gets her peak, no reason to deviate
- **Case** $\sum_{i \in N} p_i < 1$: then $f_i^u(P) \geqslant p_i, \forall i \in N$
- Manipulation, only for $i \in N$ s.t. $f_i^u(P) > p_i \implies \mu(P) > p_i$
- The only way i can change the allocation is by reporting $p_i' > \mu(P) > p_i$
- Leads to an worse outcome for i than $\mu(P)$
- A similar argument for case $\sum_{i \in N} p_i > 1$

The converse is also true, i.e.,

Theorem

An SCF in the task allocation domain is SP, PE, and ANON iff it is the uniform rule.

- See **Sprumont (1991) : Division problem with single-peaked preferences**
- **Envy-free (EF):** Agents do not envy each other's shares – also holds for uniform rule
- **SP, PE, ANON, EF, polynomial-time computable**

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

Mechanism Design with Transfers

- Social Choice Function $F : \Theta \to X$
- X: space of all **outcomes**
- In this domain, an outcome $x \in X$ has two components:
  - **allocation** a
  - **payment** $\pi = (\pi_1, \cdots, \pi_n)$, $\pi_i \in \mathbb{R}$
- Examples of allocations:
  1. A public decision to build a bridge, park, or museum. $a = \{\text{park}, \text{bridge}, \cdots\}$
  2. Allocation of a divisible good, e.g., a shared spectrum, $a = (a_1, a_2, \cdots, a_n)$, $a_i \in [0, 1]$, $\sum_{i \in N} a_i = 1$, here a_i : fraction of the resource i gets
  3. Single indivisible object allocation, e.g., a painting to be auctioned, $a = (a_1, a_2, \cdots, a_n)$, $a_i \in \{0, 1\}$, $\sum_{i \in N} a_i \leqslant 1$
  4. Partitioning indivisible objects, S = set of objects, $A = \{(A_1, \cdots, A_n) : A_i \subseteq S,\ \forall i \in N, A_i \cap A_j = \emptyset,\ \forall i \neq j\}$

Mechanism Design with Transfers

- Type of an agent i is $\theta_i \in \Theta_i$ this is a private information of i
- Agent's *benefit* from an allocation is defined via the **valuation function**
- Valuation depends on the **allocation** and the **type** of the player

$$v_i : A \times \Theta_i \to \mathbb{R} \qquad \textbf{(independent private values)}$$

- Examples:
  - if i has a type 'environmentalist' θ_i^{env}, and $a \in \{\text{Bridge}, \text{Park}\}$, then $v_i(B, \theta_i^{\text{env}}) < v_i(P, \theta_i^{\text{env}})$
  - if type changes to 'business' θ_i^{bus}, $v_i(B, \theta_i^{\text{bus}}) > v_i(P, \theta_i^{\text{bus}})$

Payments = Monetary Transfers

- Unlike other domains, here we have an 'instrument' called **money** (also called **payment** or **transfers**)
- **Payments** $\pi_i \in \mathbb{R}, \forall i \in N$
- Payment vector $\pi = (\pi_1, \pi_2, \dots, \pi_n)$
- Utility of player i, when its type is θ_i, and the outcome is $x = (a, \pi)$ is given by

$$u_i((a, \pi), \theta_i) = v_i(a, \theta_i) - \pi_i \qquad \textbf{(quasi-linear payoff)}$$

Quasi Linear Domain

- Types θ_i that depend on the outcome $x = (a, \pi)$ this way belongs to the **quasi-linear domain**

$$u_i((a, \pi), \theta_i) = v_i(a, \theta_i) - \pi_i \qquad \textbf{(quasi-linear payoff)}$$

Question

Why is this a domain restriction?

Answer

- Consider two alternatives (a, π) and (a, π'), allocation is the same but payments are different
- Suppose $\pi'_i < \pi_i$ for some $i \in N$
- There **cannot** be any preference profile in the quasi-linear domain where (a, π) is more preferred than (a, π') for agent i
- Because $v_i(a, \theta_i)$ - $\pi'_i > v_i(a, \theta_i) - \pi_i, \forall \theta_i \in \Theta_i$
- In the complete domain, both preference orders would have been feasible
- This restriction opens up possibilities of several non-dictatorial mechanisms

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

Quasi Linear preferences

- The SCF $F \equiv (f, (p_1, p_2, \ldots, p_n)) \equiv (f, p)$ is decomposed into two components
- **Allocation rule component**

$$f : \Theta_1 \times \Theta_2 \times \cdots \Theta_n \to A$$

  When the types are $\theta_i, i \in N, f(\theta_1, \cdots, \theta_n) = a \in A$
- **Payment function**

$$p_i : \Theta_1 \times \Theta_2 \times \cdots \Theta_n \to \mathbb{R}, \forall i \in N$$

  When the types are $\theta_i, i \in N, p_i(\theta_1, \cdots, \theta_n) = \pi_i \in \mathbb{R}$

Example Allocation Rules

1. **Constant rule**, $f^c(\theta) = a, \ \forall \theta \in \Theta$
2. **Dictatorial rule**, $f^D(\theta) \in \arg\max_{a \in A} v_d(a, \theta_d), \forall \theta \in \Theta$, for some $d \in N$
3. **Allocatively efficient rule / utilitarian rule**

$$f^{AE}(\theta) \in \arg\max_{a \in A} \sum_{i \in N} v_i(a, \theta_i)$$

   Note: This is different from Pareto efficiency (PE is a property defined for the outcome which also considers the payment)

4. **Affine maximizer rule**:

$$f^{AM}(\theta) \in \arg\max_{a \in A} \Big(\sum_{i \in N} \lambda_i v_i(a, \theta_i) + \kappa(a) \Big), \text{ where } \lambda_i \geqslant 0, \text{ not all zero}$$

   — $\lambda_i = 1, \forall i \in N, \kappa \equiv 0$: allocatively efficient; $\lambda_d = 1, \lambda_j = 0, \forall j \in N \setminus \{d\}, \kappa \equiv 0$: dictatorial

5. **Max-min/egalitarian**

$$f^{MM}(\theta) \in \arg\max_{a \in A} \min_{i \in N} v_i(a, \theta_i)$$

Example Payment Rules

1. **No deficit:** $\sum_{i \in N} p_i(\theta) \geqslant 0, \ \forall \theta \in \Theta$
2. **No subsidy:** $p_i(\theta) \geqslant 0, \ \forall \theta \in \Theta, \forall i \in N$
3. **Budget balanced:** $\sum_{i \in N} p_i(\theta) = 0, \ \forall \theta \in \Theta$

Definition (DSIC)

A mechanism (f, p) is **dominant strategy incentive compatible (DSIC)** if

$$v_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i}) \geqslant v_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i', \tilde{\theta}_{-i}), \forall \tilde{\theta}_{-i} \in \Theta_{-i}, \theta_i', \theta_i \in \Theta_i, \forall i \in N$$

DSIC

- DSIC means truthtelling is a weakly DSE
- We say that the payment rule p implements an allocation rule f in dominant strategies (OR) f is implementable in dominant strategies (by a payment rule)
- In QL domain, we are often more interested in the allocation rule than the whole SCF (which also includes payment)

Question

What needs to be satisfied for a DSIC mechanism (f, p)?

Implications of DSIC

Question

What needs to be satisfied for a DSIC mechanism (f, p)?

Example

$N = \{1, 2\}, \Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}, f : \Theta_1 \times \Theta_2 \to A$. The following conditions must hold

Player 1:

$$v_1(f(\theta^H, \theta_2), \theta^H) - p_1(\theta^H, \theta_2) \geqslant v_1(f(\theta^L, \theta_2), \theta^H) - p_1(\theta^L, \theta_2), \forall \theta_2 \in \Theta_2 \quad (1)$$

$$v_1(f(\theta^L, \theta_2), \theta^L) - p_1(\theta^L, \theta_2) \geqslant v_1(f(\theta^H, \theta_2), \theta^L) - p_1(\theta^H, \theta_2), \forall \theta_2 \in \Theta_2 \quad (2)$$

Player 2:

$$v_2(f(\theta^H, \theta_1), \theta^H) - p_2(\theta^H, \theta_1) \geqslant v_2(f(\theta^L, \theta_1), \theta^H) - p_2(\theta^L, \theta_1), \forall \theta_1 \in \Theta_1 \quad (3)$$

$$v_2(f(\theta^L, \theta_1), \theta^L) - p_2(\theta^L, \theta_1) \geqslant v_2(f(\theta^H, \theta_1), \theta^L) - p_2(\theta^H, \theta_1), \forall \theta_1 \in \Theta_1 \quad (4)$$

Properties of the Payment

- Say (f, p) is incentive compatible, i.e., p implements f
- Consider another payment

$$q_i(\theta_i, \theta_{-i}) = p_i(\theta_i, \theta_{-i}) + h_i(\theta_{-i}), \forall \theta, \forall i \in N$$

- Question: Is (f, q) DSIC?

$$v_i(f(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i}) - h_i(\tilde{\theta}_{-i}) \geqslant v_i(f(\theta_i', \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i', \tilde{\theta}_{-i}) - h_i(\tilde{\theta}_{-i}),\ \forall \theta_i, \theta_i', \tilde{\theta}_{-i}, \forall i \in N$$

- If we can find a payment that implements an allocation rule, there exists uncountably many payments that can implement it
- **The converse question:** when do the payments that implement f differ only by a factor $h_i(\theta_{-i})$?

Properties of the Payment

- Suppose the allocation is same in two type profiles θ and $\tilde{\theta} = (\tilde{\theta}_i, \theta_{-i})$
- i.e., $f(\theta) = f(\tilde{\theta}) = a$, then
- if p implements f, then $p_i(\theta) = p_i(\tilde{\theta})$ **[exercise]**

Contents

- Task Allocation Domain
- The Uniform Rule
- Mechanism Design with Transfers
- Quasi Linear Preferences
- Pareto Optimality and Groves Payments

Pareto Optimality in Quasi-linear domain

Definition (Pareto Optimal)

A mechanism $(f, (p_1, \ldots, p_n))$ is **Pareto optimal** if at any type profile $\theta \in \Theta$, there does not exist an allocation $b \neq f(\theta)$ and payments $(\pi_1, \ldots, \pi_n)$ with $\sum_{i \in N} \pi_i \geqslant \sum_{i \in N} p_i(\theta)$ s.t.,

$$v_i(b, \theta_i) - \pi_i \geqslant v_i(f(\theta), \theta_i) - p_i(\theta), \forall i \in N,$$

with the inequality being strict for some $i \in N$

- Pareto optimality is meaningless if there is no restriction on the payment
- One can always put excessive subsidy to every agent to make everyone better off
- So, the condition requires to spend at least the same budget

Pareto Optimality in Quasi-linear Domain

Theorem

*A mechanism $(f, (p_1, \cdots, p_n))$ is **Pareto optimal** iff it is allocatively efficient*

- $(\Longleftarrow)$ we prove $\neg$PO $\implies$ $\neg$AE
- $\neg$PO, $\exists b, \pi, \theta$ s.t. $\sum_{i \in N} \pi_i \geqslant \sum_{i \in N} p_i(\theta)$
- $v_i(b, \theta_i) - \pi_i \geqslant v_i(f(\theta), \theta_i) - p_i(\theta), \forall i \in N$, strict for some $j \in N$
- summing over the all these inequalities

$$\sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} \pi_i > \sum_{i \in N} v_i(f(\theta), \theta_i) - \sum_{i \in N} p_i(\theta)$$
$$\sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i) > \sum_{i \in N} \pi_i - \sum_{i \in N} p_i(\theta) \geqslant 0$$

- f is $\neg$AE

Proof (contd.)

- $(\implies)$ $\neg$AE $\implies$ $\neg$PO
- $\neg$AE $\implies$ $\exists \theta, b \neq f(\theta)$ s.t. $\sum_{i \in N} v_i(b, \theta_i) > \sum_{i \in N} v_i(f(\theta), \theta_i)$
- Let $\delta = \sum_{i \in N} v_i(b, \theta_i) - \sum_{i \in N} v_i(f(\theta), \theta_i) > 0$
- Consider payment $\pi_i = v_i(b, \theta_i) - v_i(f(\theta), \theta_i) + p_i(\theta) - \delta/n, \forall i \in N$
- Hence, $(v_i(b, \theta_i) - \pi_i) - (v_i(f(\theta), \theta_i) - p_i(\theta)) = \delta/n > 0, \forall i \in N$
- also $\sum_{i \in N} \pi_i = \sum_{i \in N} p_i(\theta)$
- Hence f is not PO

Allocatively Efficient Rule is Implementable

- Consider the following payment: $p_i^G(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j)$, where $h_i : \Theta_{-i} \to \mathbb{R}$ is an arbitrary function: **Groves payment**

Example

- Single indivisible item allocation $N = \{1, 2, 3, 4\}$
- $\theta_1 = 10$, $\theta_2 = 8$, $\theta_3 = 6$, $\theta_4 = 4$, when they get the object, zero otherwise
- Let $h_i(\theta_{-i}) = \min \theta_{-i}$
- If everyone reports their true type, the values of h_i are $h_1 = 4$, $h_2 = 4$, $h_3 = 4$, $h_4 = 6$
- The efficient allocation gives the item to agent 1
- $p_1 = 4 - 0 = 4, p_2 = 4 - 10 = -6, p_3 = 4 - 10 = -6, p_4 = 6 - 10 = -4$, i.e., only player 1 pays, other get paid
- **Surprisingly, this is a truthful mechanism**

Groves mechanisms are Truthful

Theorem

Groves mechanisms are DSIC

- Consider player i
- $f^{AE}(\theta_i, \tilde{\theta}_{-i}) = a$, and $f^{AE}(\theta'_i, \tilde{\theta}_{-i}) = b$
- By definition, $v_i(a, \theta_i) + \sum_{j \neq i} v_j(a, \tilde{\theta}_j) \geqslant v_i(b, \theta_i) + \sum_{j \neq i} v_j(b, \tilde{\theta}_j)$
- utility of player i when he reports θ_i is

$$\begin{aligned}
& v_i(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta_i, \tilde{\theta}_{-i}) \\
& = v_i(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \\
& \geqslant v_i(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \theta_i) - h_i(\tilde{\theta}_{-i}) + \sum_{j \neq i} v_j(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \tilde{\theta}_j) \\
& = v_i(f^{AE}(\theta'_i, \tilde{\theta}_{-i}), \theta_i) - p_i(\theta'_i, \tilde{\theta}_{-i})
\end{aligned}$$

- Since player i was arbitrary, this holds for all $i \in N$. Hence the claim.

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay