



भारतीय प्रौद्योगिकी संस्थान मुंबई  
Indian Institute of Technology Bombay

# CS 6001: Game Theory and Algorithmic Mechanism Design

Week 11

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal



- ▶ Affine Maximizers
- ▶ Single Object Allocation
- ▶ Myerson's Lemma
- ▶ Illustration of Myerson's Lemma
- ▶ Optimal Mechanism Design

# Generalization of VCG mechanism



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Can we incorporate a larger class of DSIC mechanisms in the quasi-linear domain?

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## Special cases

- $\kappa \equiv 0$  and  $w_i = 1 \ \forall i \in N$  – **efficient**
- $\kappa \equiv 0$  and  $w_d = 1, w_i = 0 \ \forall i \neq d$  – **dictatorial**
- $w_i$ 's are different  $\implies$  not ANON
- $\kappa$  is a non-constant function  $\implies$  different importance is given to different allocations

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## Definition

An AM rule  $f^{AM}$  with weights  $w_i \forall i \in N$  and the function  $\kappa$  satisfies independence of non-influential agents (INA) if for all  $i \in N$  with  $w_i = 0$  we have

$$f^{AM}(\theta_i, \theta_{-i}) = f^{AM}(\theta'_i, \theta_{-i}), \quad \forall \theta_i, \theta'_i, \theta_{-i}$$

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- **Remark:** This is a tie-breaking requirement – the zero weight agent does not influence the allocation decision, hence it should not break any tie either

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## Example

If INA was not satisfied, then AM can be manipulated, e.g., suppose there is a tie when  $w_i = 0$  for some valuation profile, but the allocation is the less preferred one for agent  $i$

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Consider

$$p_i^{AM}(\theta_i, \theta_{-i}) = \begin{cases} \frac{1}{w_i} [h_i(\theta_{-i}) - \{\sum_{j \neq i} w_j \theta_j (f^{AM}(\theta)) + \kappa(f^{AM}(\theta))\}] & \forall i : w_i > 0, \\ 0, & \forall i : w_i = 0. \end{cases}$$

# (Almost) All Affine Maximizers are DSIC (contd.)



Proof.

Payoff of  $i$  if  $w_i > 0$

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$$\begin{aligned} &= \theta_i(f^{AM}(\theta_i, \theta_{-i})) - p_i^{AM}(\theta_i, \theta_{-i}) \\ &= \frac{1}{w_i} [\{ \sum_{j \in N} w_j \theta_j (f^{AM}(\theta_i, \theta_{-i})) + \kappa(f^{AM}(\theta_i, \theta_{-i})) \} - h_i(\theta_{-i})] \\ &\geq \frac{1}{w_i} [\{ \sum_{j \in N} w_j \theta_j (f^{AM}(\theta'_i, \theta_{-i})) + \kappa(f^{AM}(\theta'_i, \theta_{-i})) \} - h_i(\theta_{-i})] \\ &= \theta_i(f^{AM}(\theta'_i, \theta_{-i})) - p_i^{AM}(\theta'_i, \theta_{-i}) \end{aligned}$$

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Hence, payoff are identical for all types.



# Roberts' Theorem



- Similar to GS Theorem, we ask what if the valuations are **unrestricted**, i.e.,  $\Theta_i$  contains all possible valuation functions  $\theta_i : A \rightarrow \mathbb{R}$ , no restriction on the functions is imposed

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## Theorem (Roberts 1979)

*Let  $A$  be finite with  $|A| \geq 3$ . If the type space is unrestricted, then every ONTO and dominant strategy implementable allocation rule must be an affine maximizer*

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- **Proof reference:** Ron Lavi, Ahuva Mu'alem, and Noam Nisan. "Two simplified proofs for Roberts' theorem". In: Social Choice and Welfare 32 (2009), pp. 407–423.
- **Similarity with GS Theorem:** GS Theorem is restricting the class to dictatorships, but here it is restricting to affine maximizers



- ▶ Affine Maximizers
- ▶ Single Object Allocation
- ▶ Myerson's Lemma
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# Setup for selling single indivisible object



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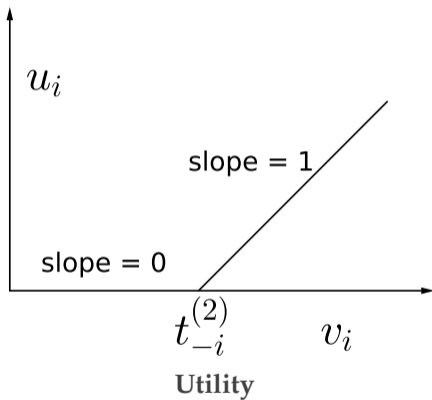
# Vickrey (Second Price) Auction



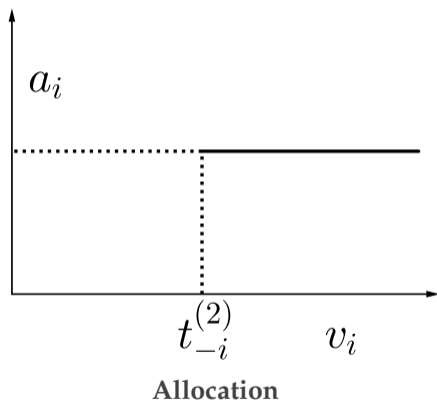
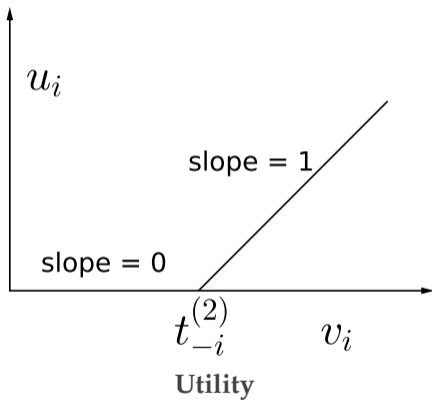
- 1 Define  $t_{-i}^{(2)} = \max_{j \neq i} \{v_j\}$
- 2 Agent  $i$  wins if  $v_i > t_{-i}^{(2)}$ , loses if  $v_i < t_{-i}^{(2)}$  and a tie breaking rule decides if there is an equality
- 3 Since payment is  $t_{-i}^{(2)}$  if  $i$  is the winner, the utility is zero in case of a tie

$$u_i(v_i, v_{-i}) = \begin{cases} 0 & \text{if } v_i \leq t_{-i}^{(2)} \\ v_i - t_{-i}^{(2)} & \text{if } v_i > t_{-i}^{(2)} \end{cases}$$

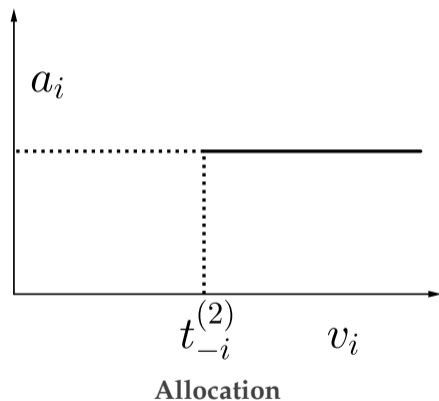
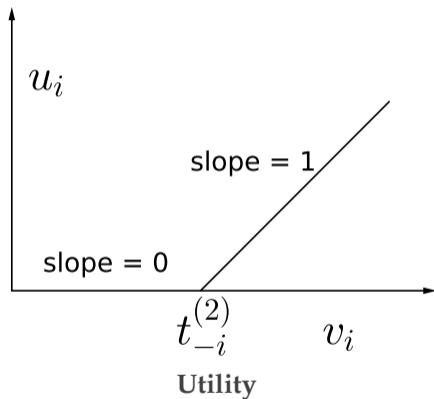
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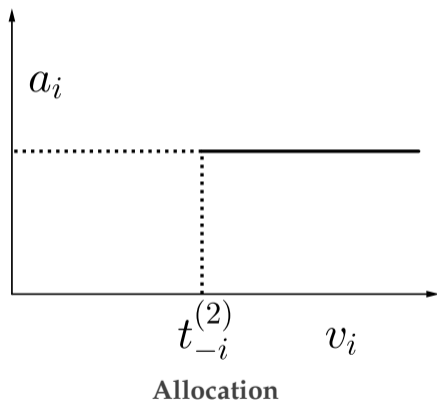
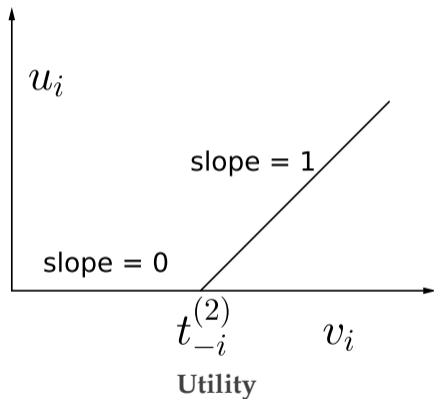


# Observations



- Utility is **convex**, derivative is zero if  $v_i < t_{-i}^{(2)}$  and 1 if  $v_i > t_{-i}^{(2)}$  (not differentiable at  $v_i = t_{-i}^{(2)}$ )

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- Whenever differentiable, it coincides with the allocation probability

# Brief review of convex functions



**Recall:** A function  $g : I \rightarrow \mathbb{R}$  (where  $I$  is an interval) is convex if for every  $x, y \in I$  and  $\lambda \in [0, 1]$

$$\lambda g(x) + (1 - \lambda)g(y) \geq g(\lambda x + (1 - \lambda)y)$$

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*Convex functions are continuous in the interior of its domain*

i.e., jumps can only occur at the boundaries



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Fact

*Convex functions are differentiable **almost everywhere***

i.e., the points where the function is not differentiable form a countable set (see the example before) - has measure zero

# Convex functions



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## Definition (Subgradient)

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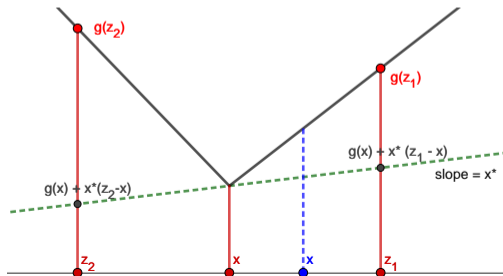
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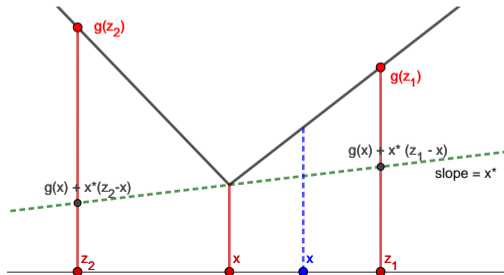
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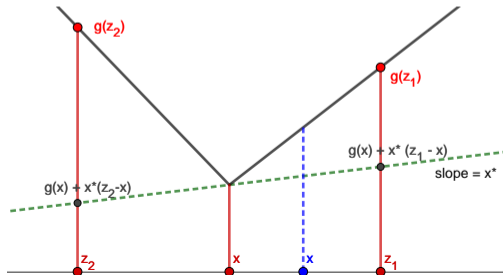
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## Question

- Always exists?
- Is it unique?



Proofs for the following lemmas can be found in any standard convex analysis text

## Lemma

*Let  $g : I \rightarrow \mathbb{R}$  be a convex function. Suppose  $x$  is in the interior of  $I$  and  $g$  is differentiable at  $x$ . The  $g'(x)$  is the unique subgradient of  $g$ .*



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## Lemma

*Let  $g : I \rightarrow \mathbb{R}$  be a convex function. Then for every  $x \in I$  a subgradient of  $g$  at  $x$  exists.*

## Standard results (contd.)



### Fact

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### Fact

The subgradients at  $x \in I \setminus I'$  is  $[g'_-(x), g'_+(x)]$

# Summary of the Lemmas



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- This result says that subgradient functions are monotone

# Summary of the Lemmas (contd.)



## Lemma

Let  $g : I \rightarrow \mathbb{R}$  be a convex function. Then for any  $x, y \in I$

$$g(x) = g(y) + \int_y^x \phi(z) dz$$

where  $\phi : I \rightarrow \mathbb{R}$  is such that  $\phi(z) \in \partial g(z) \forall z \in I$



- ▶ Affine Maximizers
- ▶ Single Object Allocation
- ▶ Myerson's Lemma
- ▶ Illustration of Myerson's Lemma
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# Monotonicity and Myerson's Lemma



## Definition

An allocation rule is non-decreasing if for every agent  $i \in N$  and  $t_{-i} \in T_{-i}$  we have  $f_i(t_i, t_{-i}) \geq f_i(s_i, t_{-i})$ ,  $\forall s_i, t_i \in T_i$ ,  $t_i > s_i$ .

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*Suppose  $T_i = [0, b_i]$ ,  $\forall i \in N$ , and the valuations are in the product form. An allocation rule  $f : T \rightarrow \Delta A$  and a payment rule  $(p_1, p_2, \dots, p_n)$  are DSIC iff*



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- $f$  is non-decreasing, and
- payments are given by

$$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i}) dx, \quad \forall t_i \in T_i, \forall t_{-i} \in T_{-i}, \forall i \in N$$

# Proof of Myerson's Lemma



## Remark: Difference with Roberts' theorem

Roberts' result gives a functional form, while Myerson's result is a more implicit property. Sometimes functional forms help answering questions in a more direct manner.

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- Given  $(f, p)$  is DSIC
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# Proof of Myerson's Lemma (contd.)



## Proof: Forward direction

- We have,  $u_i(t_i, t_{-i}) \geq u_i(s_i, t_{-i}) + (t_i - s_i)f_i(s_i, t_{-i})$

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- Hence, the above inequality can be written as

$$g(t_i) \geq g(s_i) + \phi(s_i)(t_i - s_i)$$



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- Thus,  $g$  is convex

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- Lemma 4  $\implies$

$$g(t_i) = g(0) + \int_0^{t_i} \phi(x)dx \implies u_i(t_i, t_{-i}) = u_i(0, t_{-i}) + \int_0^{t_i} f_i(x, t_{-i})dx$$

$$\implies t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) = -p_i(0, t_{-i}) + \int_0^{t_i} f_i(x, t_{-i})dx$$

$$\implies p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i})dx \implies \text{part 2 proved}$$

# Proof of Myerson's Lemma (contd.)



## Proof: Reverse direction

- Given  $f$  is non-decreasing and the payment formula

# Proof of Myerson's Lemma (contd.)



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# Proof of Myerson's Lemma (contd.)



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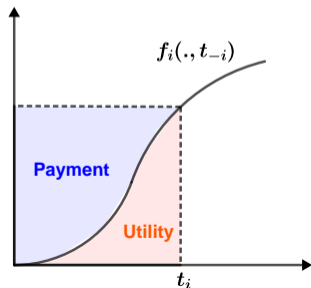
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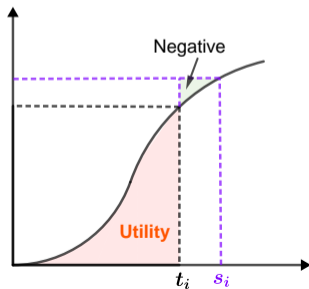
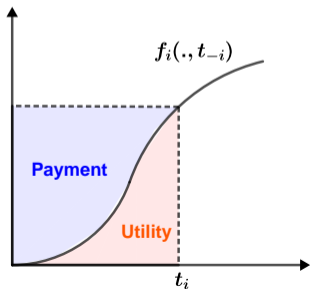




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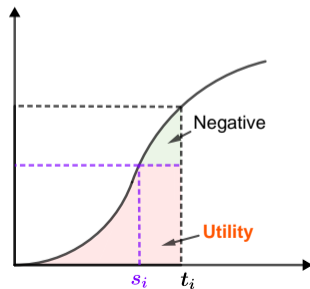
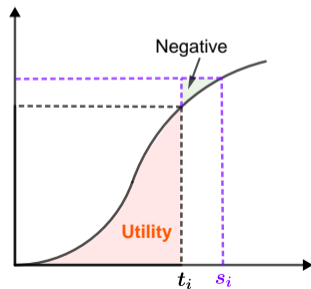
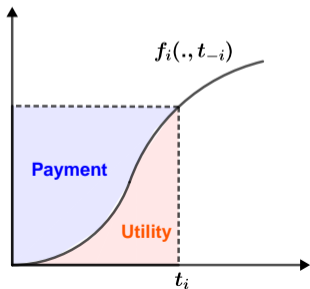




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## Corollary

*An allocation rule in a single object allocation setting is implementable in dominant strategies iff it is non-decreasing.*



- ▶ Affine Maximizers
- ▶ Single Object Allocation
- ▶ Myerson's Lemma
- ▶ Illustration of Myerson's Lemma
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# Examples of single object allocation



1. Constant allocation rule - non-decreasing, payment = constant (e.g. 0)

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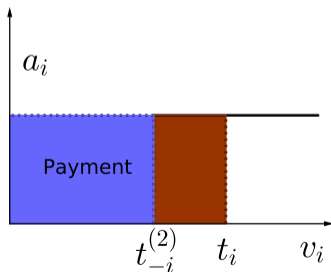
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# Examples of single object allocation

1. Constant allocation rule - non-decreasing, payment = constant (e.g. 0)
2. Dictatorial - give the object only to the dictator - non decreasing = constant / zero
3. Second price auction

$$p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_i) dx$$



Allocation for second price auction

# Examples of single object allocation



4. Efficient allocation with a reserve price is also non decreasing. If the highest value is below a reserve price  $r$ , nobody gets the object. Otherwise, the item goes to the highest bidder.

Allocated to  $i$  if  $v_i > \max\{t_{-i}^{(2)}, r\}$ . Payment =  $\{t_{-i}^{(2)}, r\}$

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5. Not so common allocation rule:  $N = \{1, 2\}$ ,  $A = \{a_0, a_1, a_2\}$  Given a type profile  $t = (t_1, t_2)$ , the seller computes  $u(t) = \max\{2, t_1^2, t_2^3\}$  - select  $a_0, a_1, a_2$  depending on which of the three expressions is the maxima - break ties in favour of  $0 > 1 > 2$

Player 1 gets the object if  $t_1 > \sqrt{\max\{2, t_2^3\}}$

Player 2 gets the object if  $t_2 > \sqrt[3]{\max\{2, t_1^2\}}$



## Definition

A mechanism  $(f, p)$  is **ex-post individually rational** if

$$t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) \geq 0, \quad \forall t_i \in T_i, t_{-i} \in T_{-i}, \forall i \in N$$



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**Ex-post:** Even after all agents have revealed their types, participating is weakly preferred.



## Lemma

*In the single object allocation setting, consider a DSIC mechanism  $(f, p)$*

- 1 It is IR iff  $\forall i \in N$  and  $\forall t_{-i} \in T_{-i}$ ,  $p_i(0, t_{-i}) \leq 0$
- 2 It is IR and satisfies no subsidy, i.e.,  $p_i(t_i, t_{-i}) \geq 0$ ,  $\forall t_i \in T_i, t_{-i} \in T_{-i}$ ,  $\forall i \in N$  iff  $\forall i \in N, t_{-i} \in T_{-i}, p_i(0, t_{-i}) = 0$



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- 1 Suppose  $(f, p)$  is IR, then  $0 - p_i(0, t_{-i}) \geq 0$ , hence  $p_i(0, t_{-i}) \leq 0$

# Implications of Individual Rationality



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Clearly if  $p_i(0, t_{-i}) = 0 \implies (f, p)$  is IR and no-subsidy.



# Non-Vickrey Auctions: Example 1

The object goes to the highest bidder, but the payment is such that everyone is compensated some amount. Assume, WLOG,  $t_1 > t_2 > \dots > t_n$

- 1 Highest and second highest bidders are compensated  $\frac{1}{n}$  of the third highest bid.

$$p_1(0, t_{-i}) = p_2(0, t_{-2}) = -\frac{1}{n}t_3$$

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**Deterministic mechanism that redistributes the money**

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**Randomized mechanism that redistributes the money**



- ▶ Affine Maximizers
- ▶ Single Object Allocation
- ▶ Myerson's Lemma
- ▶ Illustration of Myerson's Lemma
- ▶ **Optimal Mechanism Design**



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Accordingly, the notions of incentive compatibility and individual rationality need to change



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- Expected utility of agent  $i$ :  $u_i = t_i \alpha_i(s_i; t_i) - \pi_i(s_i; t_i)$



## Definition (Bayesian Incentive Compatibility (BIC))

A mechanism  $(f, p)$  is Bayesian incentive compatible (BIC) if  $\forall i \in N, \forall s_i, t_i \in T_i$

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Similarly,  $f$  is Bayesian implementable if  $\exists p$  s.t.  $(f, p)$  is BIC.

# Characterization of BIC mechanisms



Assume that **priors are independent**, i.e., agent  $i$ 's value is drawn from a distribution  $G_i$  (density  $g_i$ ) independently from other agents.

$$G(s_1, s_2, \dots, s_n) = \prod_{i \in N} G_i(s_i), \quad G(s_{-i} | t_i) = \prod_{j \neq i} G_j(s_j)$$

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## Definition

An allocation rule is **Non-decreasing in expectation (NDE)** if  $\forall i \in N, \forall s_i, t_i \in T_i$  with  $s_i < t_i$  we have  $\alpha_i(s_i) \leq \alpha_i(t_i)$

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Note: The rules that are non-decreasing (defined before) are always NDE, but there can be more rules that are NDE

# NDE but not ND



				1
$t_2$			1	
			1	1
		1		1
		$t_1$		

**Figure:** An allocation rule may be NDE but not non-decreasing

All five types are equally likely,  $\alpha_1(t_1)$  and  $\alpha_2(t_2)$  are monotone, but  $f(t_1, t_2)$  is not.

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## Theorem (Myerson 1981)

A mechanism  $(f, p)$  in the independent prior setting is BIC iff

- $f$  is NDE, and
- $p_i$  satisfies  $\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x) dx, \quad \forall t_i \in T_i, \forall i \in N$

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This is Bayesian version of the earlier Myerson theorem, proof proceeds in similar lines as before [exercise] □

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A mechanism  $(f, p)$  is **interim individually rational (IIR)** if for every bidder  $i \in N$ , we have  $t_i \alpha_i(t_i) - \pi_i(t_i) \geq 0, \forall t_i \in T_i$

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## Lemma

*A mechanism  $(f, p)$  is BIC and IIR iff*

- $f$  is NDE
- $p_i$  satisfies  $\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x) dx, \forall t_i \in T_i, \forall i \in N$
- $\forall i \in N, \pi_i(0) \leq 0$



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- So, the proof requires to show that IIR along with first two conditions is equivalent to third condition
- **Forward direction:** apply IIR at  $t_i = 0$  on second condition and get  $\pi_i(0) \leq 0$
- **Reverse direction:**  $t_i \alpha_i(t_i) - \pi_i(t_i) = -\pi_i(0) + \int_0^{t_i} \alpha_i(s_i) ds_i \geq 0$  if  $\pi_i(0) \leq 0$



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