भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 11

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Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

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Knowledge is the supreme goal

Contents

Generalization of VCG mechanism

Question

Can we incorporate a larger class of DSIC mechanisms in the quasi-linear domain?

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Special cases

- $\kappa \equiv 0$ and $w_i = 1 \ \ \forall i \in N$ – **efficient**
- $\kappa \equiv 0$ and $w_d = 1, w_i = 0 \ \ \forall i \neq d$ – **dictatorial**
- w_i's are different $\implies$ not ANON
- κ is a non-constant function $\implies$ different importance is given to different allocations

Observations on Affine Maximizers

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- **Independence of Non-influential Agents**

Definition

An AM rule f^{AM} with weights w_i $\forall i \in N$ and the function κ satisfies independence of non-influential agents (INA) if for all $i \in N$ with $w_i = 0$ we have

$$f^{AM}(\theta_i, \theta_{-i}) = f^{AM}(\theta_i', \theta_{-i}), \;\; \forall\, \theta_i, \theta_i', \theta_{-i}$$

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- **Remark:** This is a tie-breaking requirement – the zero weight agent does not influence the allocation decision, hence it should not break any tie either

(Almost) All Affine Maximizers are DSIC

Example

If INA was not satisfied, then AM can be manipulated, e.g., suppose there is a tie when $w_i = 0$ for some valuation profile, but the allocation is the less preferred one for agent i

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Consider

$$p_i^{AM}(\theta_i, \theta_{-i}) = \begin{cases} \frac{1}{w_i}[h_i(\theta_{-i}) - \{\sum_{j \neq i} w_j \theta_j(f^{AM}(\theta)) + \kappa(f^{AM}(\theta))\}] & \forall i : w_i > 0, \\ 0, & \forall i : w_i = 0. \end{cases}$$

□

(Almost) All Affine Maximizers are DSIC (contd.)

Proof.

Payoff of i if $w_i > 0$

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Payoff of i if $w_i > 0$

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&= \theta_i(f^{AM}(\theta_i, \theta_{-i})) - p_i^{AM}(\theta_i, \theta_{-i}) \\
&= \frac{1}{w_i}[\{\sum_{j \in N} w_j \theta_j (f^{AM}(\theta_i, \theta_{-i})) + \kappa(f^{AM}(\theta_i, \theta_{-i}))\} - h_i(\theta_{-i})] \\
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$$f^{AM}(\theta_i, \theta_{-i}) = f^{AM}(\theta_i', \theta_{-i}), \ \forall \theta_i, \theta_i', \theta_{-i}$$

Hence, payoff are identical for all types. □

Roberts' Theorem

- Similar to GS Theorem, we ask what if the valuations are **unrestricted**, i.e., Θ_i contains all possible valuation functions $\theta_i : A \to \mathbb{R}$, no restriction on the functions is imposed

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Theorem (Roberts 1979)

Let A be finite with $|A| \geqslant 3$. If the type space is unrestricted, then every ONTO and dominant strategy implementable allocation rule must be an affine maximizer

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- **Similarity with GS Theorem:** GS Theorem is restricting the class to dictatorships, but here it is restricting to affine maximizers

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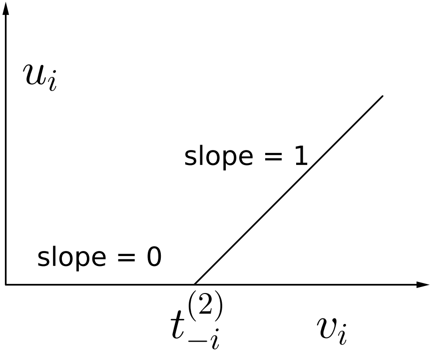
  - $f_0(t)$ is the probability of not selling the object

Vickrey (Second Price) Auction

1. Define $t_{-i}^{(2)} = \max_{j \neq i}\{v_j\}$
2. Agent i wins if $v_i > t_{-i}^{(2)}$, loses if $v_i < t_{-i}^{(2)}$ and a tie breaking rule decides if there is an equality
3. Since payment is $t_{-i}^{(2)}$ if i is the winner, the utility is zero in case of a tie

$$u_i(v_i, v_{-i}) = \begin{cases} 0 & \text{if } v_i \leqslant t_{-i}^{(2)} \\ v_i - t_{-i}^{(2)} & \text{if } v_i > t_{-i}^{(2)} \end{cases}$$

Observations



Utility

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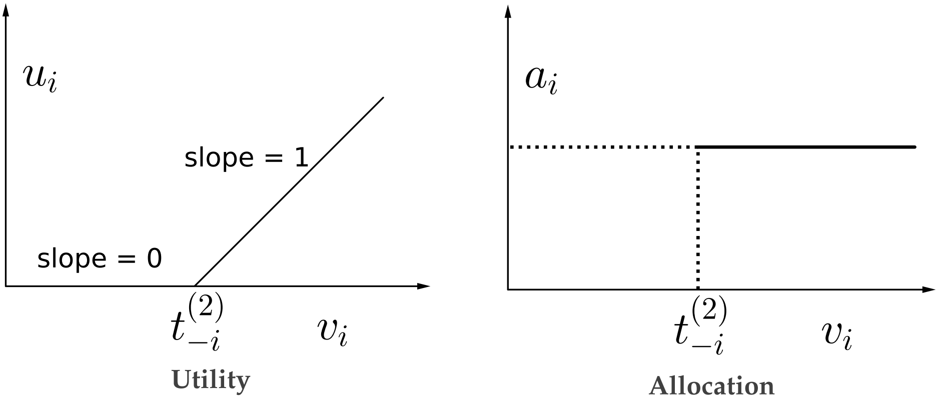

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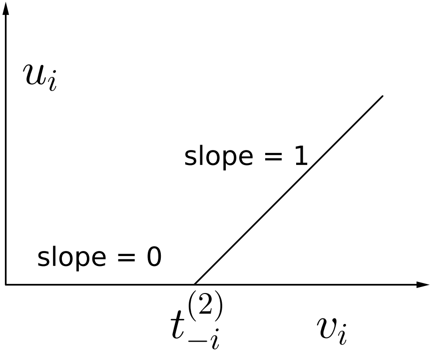

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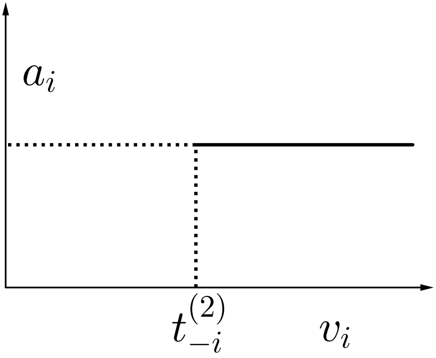

Allocation

- Utility is **convex**, derivative is zero if $v_i < t_{-i}^{(2)}$ and 1 if $v_i > t_{-i}^{(2)}$ (not differentiable at $v_i = t_{-i}^{(2)}$)

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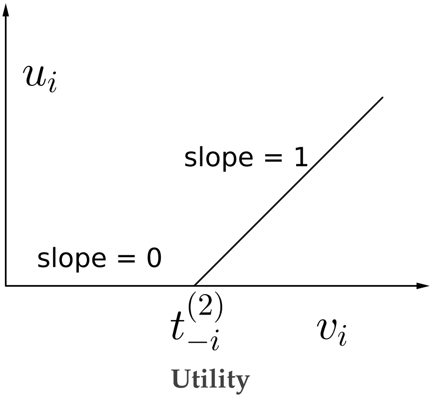

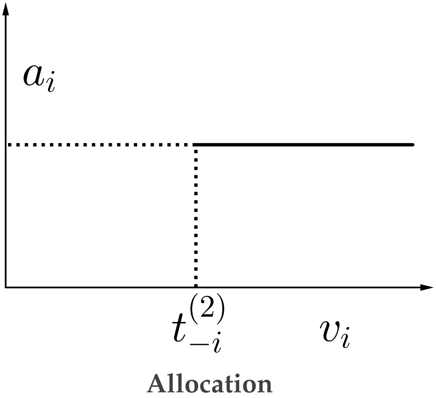

- Utility is **convex**, derivative is zero if $v_i < t_{-i}^{(2)}$ and 1 if $v_i > t_{-i}^{(2)}$ (not differentiable at $v_i = t_{-i}^{(2)}$)
- Whenever differentiable, it coincides with the allocation probability

Brief review of convex functions

Recall: A function $g : I \to \mathbb{R}$ (where I is an interval) is convex if for every $x, y \in I$ and $\lambda \in [0, 1]$

$$\lambda g(x) + (1 - \lambda) g(y) \geqslant g(\lambda x + (1 - \lambda) y)$$

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Fact

Convex functions are differentiable ***almost everywhere***

i.e., the points where the function is not differentiable form a countable set (see the example before) - has measure zero

Convex functions

If g is differentiable at $x \in I$, we denote the derivative by $g'(x)$

The following definition extends the idea of gradient

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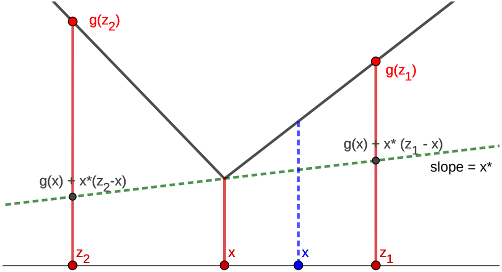

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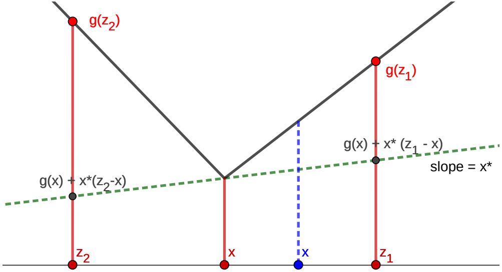

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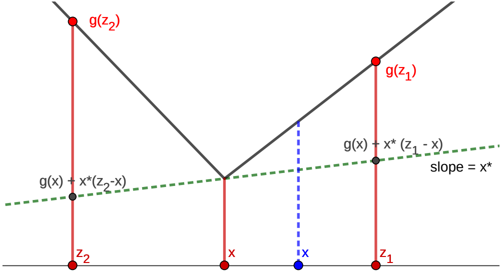

Question

- Always exists?
- Is it unique?

Standard results

Proofs for the following lemmas can be found in any standard convex analysis text

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Let $g : I \to \mathbb{R}$ be a convex function. Suppose x is in the interior of I and g is differentiable at x. The $g'(x)$ is the unique subgradients of g.

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Lemma

Let $g : I \to \mathbb{R}$ be a convex function. Then for every $x \in I$ a subgradient of g at x exists.

Standard results (contd.)

Fact

Let $I' \subseteq I$ be the set of points where g is differentiable. The set $I \setminus I'$ is of measure zero. The set of subgradients at a point forms a convex set.

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Define $g'_+(x)$ and $g'_-(x)$ as

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Fact

The subgradients at $x \in I \setminus I'$ is $[g'_-(x), g'_+(x)]$

Summary of the Lemmas

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Lemma

Let $g : I \to \mathbb{R}$ be a convex function. Let $\phi(z) \in \partial g(z), \forall z \in I$. Then for all $x, y \in I$ such that $x > y$, we have $\phi(x) \geqslant \phi(y)$.

Summary of the Lemmas

We will denote the set of subgradients of g at $x \in I$ as $\partial g(x)$

1. First lemma says $\partial g(x) = \{g'(x)\}, \forall x \in I'$
2. Second lemma says that $\partial g(x) \neq \emptyset, \forall x \in I$

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- $\phi(z)$ picks one value at every z (even if subgradients can be many)
- This result says that subgradient functions are monotone

Summary of the Lemmas (contd.)

Lemma

Let $g : I \to \mathbb{R}$ *be a convex function. Then for any* $x, y \in I$

$$g(x) = g(y) + \int_y^x \phi(z)dz$$

where $\phi : I \to \mathbb{R}$ *is such that* $\phi(z) \in \partial g(z)\ \forall z \in I$

Contents

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$$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i})\, dx, \quad \forall t_i \in T_i, \forall t_{-i} \in T_{-i}, \forall i \in N$$

Proof of Myerson's Lemma

Remark: Difference with Roberts' theorem

Roberts' result gives a functional form, while Myerson's result is a more implicit property. Sometimes functional forms help answering questions in a more direct manner.

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- Given (f, p) is DSIC
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- Since (f, p) is DSIC, we have

$$\begin{aligned} u_i(t_i, t_{-i}) = t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) &\geqslant t_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i}) \\ &= s_i f_i(s_i, t_{-i}) + (t_i - s_i) f_i(s_i, t_{-i}) - p_i(s_i, t_{-i}) \\ &= u_i(s_i, t_{-i}) + (t_i - s_i) f_i(s_i, t_{-i}) \end{aligned}$$

Proof of Myerson's Lemma (contd.)

Proof: Forward direction

- We have, $u_i(t_i, t_{-i}) \geqslant u_i(s_i, t_{-i}) + (t_i - s_i)f_i(s_i, t_{-i})$

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- DSIC implies

$$g(x_i) \geqslant g(y_i) + \phi(y_i)(x_i - y_i) \text{ and } g(z_i) \geqslant g(y_i) + \phi(y_i)(z_i - y_i)$$

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- Apply lemmas 3 and 4 from our review of convex analysis
- Lemma 3 $\implies$ $\phi = f_i(., t_{-i})$ is non-decreasing $\implies$ **part 1 proved**
- Lemma 4 $\implies$

$$g(t_i) = g(0) + \int_0^{t_i} \phi(x)dx \implies u_i(t_i, t_{-i}) = u_i(0, t_{-i}) + \int_0^{t_i} f_i(x, t_{-i})dx$$
$$\implies t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) = -p_i(0, t_{-i}) + \int_0^{t_i} f_i(x, t_{-i})dx$$
$$\implies p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i})dx \implies \textbf{part 2 proved}$$

Proof of Myerson's Lemma (contd.)

Proof: Reverse direction

- Given f is non-decreasing and the payment formula

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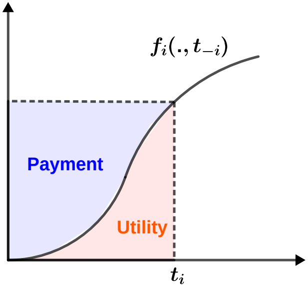

Figure: Proof by picture 1

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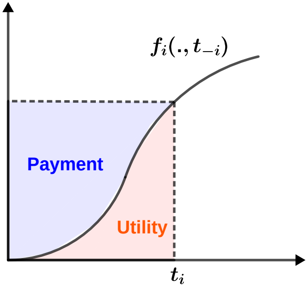

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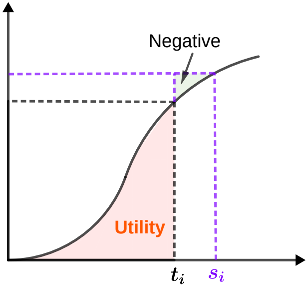

Figure: Proof by picture 2

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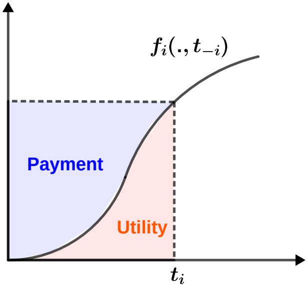

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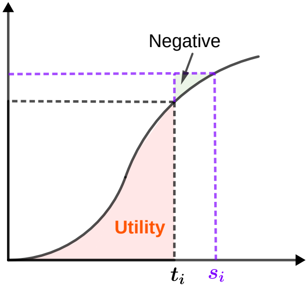

Figure: Proof by picture 2

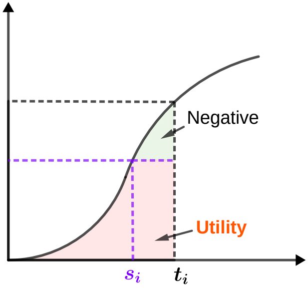

Figure: Proof by picture 3

Proof of Myerson's Lemma (contd.)

Proof: Reverse direction

- Given f is non-decreasing and the payment formula

Proof of Myerson's Lemma (contd.)

Proof: Reverse direction

- Given f is non-decreasing and the payment formula
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Corollary

An allocation rule in a single object allocation setting is implementable in dominant strategies iff it is non-decreasing.

Contents

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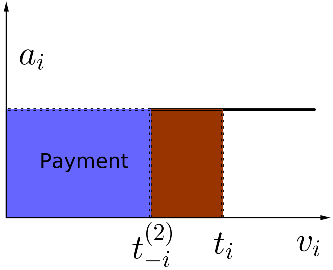

Allocation for second price auction

Examples of single object allocation

4. Efficient allocation with a reserve price is also non decreasing. If the highest value is below a reserve price r, nobody gets the object. Otherwise, the item goese to the highest bidder. Allocated to i if $v_i > \max\{t_{-i}^{(2)}, r\}$. Payment = $\{t_{-i}^{(2)}, r\}$

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5. Not so common allocation rule: $N = \{1,2\}$, $A = \{a_0, a_1, a_2\}$ Given a type profile $t = (t_1, t_2)$, the seller computes $u(t) = \max\{2, t_1^2, t_2^3\}$ - select a_0, a_1, a_2 depending on which of the three expressions is the maxima - break ties in favour of $0 > 1 > 2$

   Player 1 gets the object if $t_1 > \sqrt{\max\{2, t_2^3\}}$

   Player 2 gets the object if $t_2 > \sqrt[3]{\max\{2, t_1^2\}}$

Individual Rationality

Definition

A mechanism (f, p) is **ex-post individual rational** if

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Ex-post: Even after all agents have revealed their types, participating is weakly preferred.

Implications of Individual Rationality

Lemma

In the single object allocation setting, consider a DSIC mechanism (f, p)

1. *It is IR iff* $\forall i \in N$ *and* $\forall t_{-i} \in T_{-i}$, $p_i(0, t_{-i}) \leqslant 0$
2. *It is IR and satisfies no subsidy, i.e.,* $p_i(t_i, t_{-i}) \geqslant 0$, $\forall t_i \in T_i, t_{-i} \in T_{-i}$, $\forall i \in N$ *iff* $\forall i \in N, t_{-i} \in T_{-i}, p_i(0, t_{-i}) = 0$

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$$t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) = t_i f_i(t_i, t_{-i}) - p_i(0, t_{-i}) - t_i f_i(t_i, t_{-i}) + \int_0^{t_i} f_i(x, t_{-i}) dx \geqslant 0$$

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2. IR $\implies p_i(0, t_{-i}) \leqslant 0$, if $p_i(t_i, t_{-i}) \geqslant 0 \ \forall t_i \implies p_i(0, t_{-i}) = 0$

Implications of Individual Rationality

Lemma

In the single object allocation setting, consider a DSIC mechanism (f, p)

1. *It is IR iff $\forall i \in N$ and $\forall t_{-i} \in T_{-i}$, $p_i(0, t_{-i}) \leqslant 0$*
2. *It is IR and satisfies no subsidy, i.e., $p_i(t_i, t_{-i}) \geqslant 0$, $\forall t_i \in T_i, t_{-i} \in T_{-i}$, $\forall i \in N$ iff $\forall i \in N, t_{-i} \in T_{-i}, p_i(0, t_{-i}) = 0$*

Proof

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2. IR $\implies$ $p_i(0, t_{-i}) \leqslant 0$, if $p_i(t_i, t_{-i}) \geqslant 0 \; \forall t_i \implies p_i(0, t_{-i}) = 0$
   Clearly if $p_i(0, t_{-i}) = 0 \implies (f, p)$ is IR and no-subsidy.

Non-Vickrey Auctions: Example 1

The object goes to the highest bidder, but the payment is such that everyone is compensated some amount.

1. Highest and second highest bidders are compensated $\frac{1}{n}$ of the third highest bid.
   $p_1(0, t_{-i}) = p_2(0, t_{-2}) = -\frac{1}{n} t_3$
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WLOG $t_1 > t_2 > \ldots > t_n$

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Deterministic mechanism that redistributes the money

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Randomized mechanism that redistributes the money

Contents

Revenue Maximization

Question

How to maximize revenue earned by the auctioneer?

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Maximize w.r.t. what knowledge of the auctioneer?

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Answer

The common prior distribution over types

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Answer

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Accordingly, the notions of incentive compatibility and individual rationality have to change

Bayesian setup

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- Expected utility of agent i: $u_i = t_i \alpha_i(s_i|t_i) - \pi_i(s_i|t_i)$

Bayesian Incentive Compatibility

Definition (Bayesian Incentive Compatibility (BIC))

A mechanism (f, p) is Bayesian incentive compatible (BIC) if $\forall i \in N, \forall s_i, t_i \in T_i$

$$t_i \alpha_i(t_i|t_i) - \pi_i(t_i|t_i) \geqslant t_i \alpha_i(s_i|t_i) - \pi_i(s_i|t_i)$$

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Similarly, f is Bayesian implementable if $\exists p$ s.t. (f, p) is BIC.

Characterization of BIC mechanisms

Assume that **priors are independent**, i.e., agent i's value is drawn from a distribution G_i (density g_i) independently from other agents.

$$G(s_1, s_2, \ldots, s_n) = \prod_{i \in N} G_i(s_i), \qquad G(s_{-i}|t_i) = \prod_{j \neq i} G_j(s_j)$$

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An allocation rule is **Non-decreasing in expectation (NDE)** if $\forall i \in N, \forall s_i, t_i \in T_i$ with $s_i < t_i$ we have $\alpha_i(s_i) \leqslant \alpha_i(t_i)$

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Note: The rules that are non-decreasing (defined before) are always NDE, but there can be more rules that are NDE

NDE but not ND

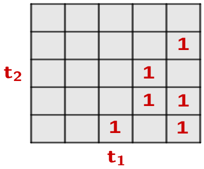

Figure: An allocation rule may be NDE but not non-decreasing

All five types are equally likely, $\alpha_1(t_1)$ and $\alpha_2(t_2)$ are monotone, but $f(t_1, t_2)$ is not.

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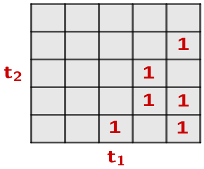

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Theorem (Myerson 1981)

A mechanism (f, p) in the independent prior setting is BIC iff

- *f is NDE, and*
- *p_i satisfies $\pi_i(t_i) = \pi_i(0) + t_i\alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx, \;\; \forall t_i \in T_i, \forall i \in N$*

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Proof.

This is Bayesian version of the earlier Myerson theorem, proof proceeds in similar lines as before [exercise] □

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Lemma

A mechanism (f, p) is BIC and IIR iff

- *f is NDE*
- *p_i satisfies $\pi_i(t_i) = \pi_i(0) + t_i\alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx, \ \ \forall t_i \in T_i, \forall i \in N$*
- *$\forall i \in N, \ \pi_i(0) \leqslant 0$*

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- **Forward direction:** apply IIR at $t_i = 0$ on second condition and get $\pi_i(0) \leqslant 0$
- **Reverse direction:** $t_i \alpha_i(t_i) - \pi_i(t_i) = -\pi_i(0) + \int_0^{t_i} \alpha_i(s_i)\, ds_i \geqslant 0$ if $\pi_i(0) \leqslant 0$

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