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CS 6001: Game Theory and Algorithmic Mechanism Design

Week 11

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Knowledge is the supreme goal

Contents

Generalization of VCG mechanism

Question

Can we incorporate a larger class of DSIC mechanisms in the quasi-linear domain?

Definition (Affine Maximizer (AM) Allocation Rule)

$$f^{AM}(\theta) \in \arg\max_{a \in A} \left(\sum_{i \in N} w_i \theta_i(a) + \kappa(a) \right)$$

where, $w_i \geqslant 0 \ \ \forall i \in N$, (not all zero) – **different weight for players**
$\kappa : A \to \mathbb{R}$ is any arbitrary function – **translation**

Special cases

- $\kappa \equiv 0$ and $w_i = 1 \ \ \forall i \in N$ – **efficient**
- $\kappa \equiv 0$ and $w_d = 1, w_i = 0 \ \ \forall i \neq d$ – **dictatorial**
- w_i's are different $\implies$ not ANON
- κ is a non-constant function $\implies$ different importance is given to different allocations

Observations on Affine Maximizers

- Affine Maximizer is a super class of VCG (efficient) allocations, hence, it can satisfy more properties
- We can ask a characterization question (like GS Theorem) in the quasi-linear setting with public goods.
- **Independence of Non-influential Agents**

Definition

An AM rule f^{AM} with weights $w_i \ \forall i \in N$ and the function κ satisfies independence of non-influential agents (INA) if for all $i \in N$ with $w_i = 0$ we have

$$f^{AM}(\theta_i, \theta_{-i}) = f^{AM}(\theta_i', \theta_{-i}), \ \ \forall \ \theta_i, \theta_i', \theta_{-i}$$

- **Remark:** This is a tie-breaking requirement – the zero weight agent does not influence the allocation decision, hence it should not break any tie either

(Almost) All Affine Maximizers are DSIC

Example

If INA was not satisfied, then AM can be manipulated, e.g., suppose there is a tie when $w_i = 0$ for some valuation profile, but the allocation is the less preferred one for agent i

Theorem

An AM rule satisfying INA is implementable in dominant strategies.

Proof.

We need to construct a payment function p^{AM} to make (f^{AM}, p^{AM}) DSIC. Consider

$$p_i^{AM}(\theta_i, \theta_{-i}) = \begin{cases} \frac{1}{w_i}[h_i(\theta_{-i}) - \{\sum_{j \neq i} w_j \theta_j(f^{AM}(\theta)) + \kappa(f^{AM}(\theta))\}] & \forall i : w_i > 0, \\ 0, & \forall i : w_i = 0. \end{cases}$$

□

(Almost) All Affine Maximizers are DSIC (contd.)

Proof.

Payoff of i if $w_i > 0$

$$
\begin{aligned}
&= \theta_i(f^{AM}(\theta_i, \theta_{-i})) - p_i^{AM}(\theta_i, \theta_{-i}) \\
&= \frac{1}{w_i}[\{\sum_{j \in N} w_j \theta_j(f^{AM}(\theta_i, \theta_{-i})) + \kappa(f^{AM}(\theta_i, \theta_{-i}))\} - h_i(\theta_{-i})] \\
&\geqslant \frac{1}{w_i}[\{\sum_{j \in N} w_j \theta_j(f^{AM}(\theta_i', \theta_{-i})) + \kappa(f^{AM}(\theta_i', \theta_{-i}))\} - h_i(\theta_{-i})] \\
&= \theta_i(f^{AM}(\theta_i', \theta_{-i})) - p_i^{AM}(\theta_i', \theta_{-i})
\end{aligned}
$$

For i where $w_i = 0$, payments are zero and

$$
f^{AM}(\theta_i, \theta_{-i}) = f^{AM}(\theta_i', \theta_{-i}), \ \forall \theta_i, \theta_i', \theta_{-i}
$$

Hence, payoff are identical for all types. □

Roberts' Theorem

- Similar to GS Theorem, we ask what if the valuations are **unrestricted**, i.e., Θ_i contains all possible valuation functions $\theta_i : A \to \mathbb{R}$, no restriction on the functions is imposed
- With this unrestricted space of valuations, we can characterize the class DSIC mechanisms in the quasi-linear domain
- Roberts' theorem

Theorem (Roberts 1979)

Let A be finite with $|A| \geqslant 3$. If the type space is unrestricted, then every ONTO and dominant strategy implementable allocation rule must be an affine maximizer

- **Proof reference:** Ron Lavi, Ahuva Mu'alem, and Noam Nisan. "Two simplified proofs for Roberts' theorem". In: Social Choice and Welfare 32 (2009), pp. 407–423.
- **Similarity with GS Theorem:** GS Theorem is restricting the class to dictatorships, but here it is restricting to affine maximizers

Contents

Setup for selling single indivisible object

- Type set of agent i : $T_i \subseteq \mathbb{R}$
- $t_i \in T_i$: value of agent i if she wins the object
- An allocation a is a vector of length n that represents the probability of winning the object by the respective agent (a_0 is probability of not selling the object)

$$\textbf{Set of allocations: } \Delta A = \{a \in [0,1]^n : \sum_{i=0}^{n} a_i = 1\}$$

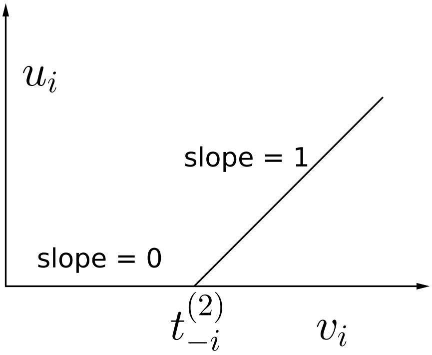
- **Allocation rule:** $f : T_1 \times T_2 \times \ldots \times T_n \to \Delta A$
- **Valuation:** $v_i(a, t_i) = a_i \cdot t_i$ (**product form**, expected valuation)
- Hence, $f_i(t_i, t_{-i})$ is agent i's probability of winning the object when the type profile is (t_i, t_{-i})
  — $f_0(t)$ is the probability of not selling the object

Vickrey (Second Price) Auction

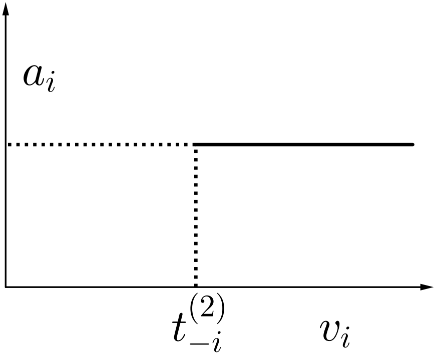
1. Define $t_{-i}^{(2)} = \max_{j \neq i}\{v_j\}$
2. Agent i wins if $v_i > t_{-i}^{(2)}$, loses if $v_i < t_{-i}^{(2)}$ and a tie breaking rule decides if there is an equality
3. Since payment is $t_{-i}^{(2)}$ if i is the winner, the utility is zero in case of a tie

$$u_i(v_i, v_{-i}) = \begin{cases} 0 & \text{if } v_i \leqslant t_{-i}^{(2)} \\ v_i - t_{-i}^{(2)} & \text{if } v_i > t_{-i}^{(2)} \end{cases}$$

Observations



Utility



Allocation

- Utility is **convex**, derivative is zero if $v_i < t_{-i}^{(2)}$ and 1 if $v_i > t_{-i}^{(2)}$ (not differentiable at $v_i = t_{-i}^{(2)}$)
- Whenever differentiable, it coincides with the allocation probability

Brief review of convex functions

Recall: A function $g : I \to \mathbb{R}$ (where I is an interval) is convex if for every $x, y \in I$ and $\lambda \in [0,1]$

$$\lambda g(x) + (1-\lambda)g(y) \geqslant g(\lambda x + (1-\lambda)y)$$

Some known facts from convex analysis (see e.g. Rockafeller (1980))

Fact

Convex functions are continuous in the interior of its domain

i.e., jumps can only occur at the boundaries

Fact

*Convex functions are differentiable **almost everywhere***

i.e., the points where the function is not differentiable form a countable set (see the example before) - has measure zero

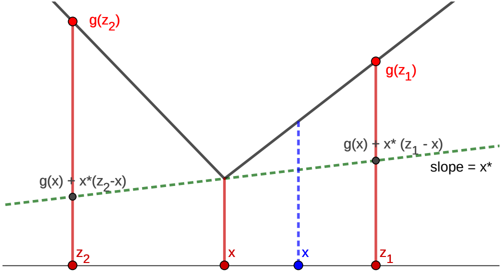
Convex functions

If g is differentiable at $x \in I$, we denote the derivative by $g'(x)$

The following definition extends the idea of gradient

Definition (Subgradient)

For any $x \in I$, x^* is a subgradient of g at x if $g(z) \geqslant g(x) + x^*(z - x), \forall z \in I$



Question

- Always exists?
- Is it unique?

Standard results

Proofs for the following lemmas can be found in any standard convex analysis text

Lemma

Let $g : I \to \mathbb{R}$ be a convex function. Suppose x is in the interior of I and g is differentiable at x. The $g'(x)$ is the unique subgradients of g.

Lemma

Let $g : I \to \mathbb{R}$ be a convex function. Then for every $x \in I$ a subgradient of g at x exists.

Standard results (contd.)

Fact

Let $I' \subseteq I$ be the set of points where g is differentiable. The set $I \setminus I'$ is of measure zero. The set of subgradients at a point forms a convex set.

Define $g'_+(x)$ and $g'_-(x)$ as

$$g'_+(x) = \lim_{z \to x,\ z > x} g'(z)$$
$$g'_-(x) = \lim_{z \to x,\ z < x} g'(z)$$

Fact

The subgradients at $x \in I \setminus I'$ is $[g'_-(x), g'_+(x)]$

Summary of the Lemmas

We will denote the set of subgradients of g at $x \in I$ as $\partial g(x)$

1. First lemma says $\partial g(x) = \{g'(x)\}, \forall x \in I'$
2. Second lemma says that $\partial g(x) \neq \emptyset, \forall x \in I$

Lemma

Let $g : I \to \mathbb{R}$ be a convex function. Let $\phi(z) \in \partial g(z), \forall z \in I$. Then for all $x, y \in I$ such that $x > y$, we have $\phi(x) \geqslant \phi(y)$.

- $\phi(z)$ picks one value at every z (even if subgradients can be many)
- This result says that subgradient functions are monotone

Summary of the Lemmas (contd.)

Lemma

Let $g : I \to \mathbb{R}$ be a convex function. Then for any $x, y \in I$

$$g(x) = g(y) + \int_y^x \phi(z)dz$$

where $\phi : I \to \mathbb{R}$ is such that $\phi(z) \in \partial g(z)\ \forall z \in I$

Contents

Monotonicity and Myerson's Lemma

Definition

An allocation rule is non-decreasing if for every agent $i \in N$ and $t_{-i} \in T_{-i}$ we have $f_i(t_i, t_{-i}) \geqslant f_i(s_i, t_{-i}), \ \forall s_i, t_i \in T_i, \ t_i > s_i$.

i.e., holding other agents' types fixed, the probability of allocation never decreases with valuation

Theorem (Myerson 1981)

Suppose $T_i = [0, b_i]$, $\forall i \in N$, and the valuations are in the product form. An allocation rule $f : T \to \Delta A$ and a payment rule $(p_1, p_2, \ldots, p_n)$ are DSIC iff

- *f is non-decreasing, and*
- *payments are given by*

$$p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i})\, dx, \quad \forall t_i \in T_i, \ \forall t_{-i} \in T_{-i}, \ \forall i \in N$$

Proof of Myerson's Lemma

Remark: Difference with Roberts' theorem

Roberts' result gives a functional form, while Myerson's result is a more implicit property. Sometimes functional forms help answering questions in a more direct manner.

Proof: Forward direction

- Given (f, p) is DSIC
- Utility of agent i at types t_i and s_i respectively:

$$u_i(t_i, t_{-i}) = t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}), \text{ and } u_i(s_i, t_{-i}) = s_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i})$$

- Since (f, p) is DSIC, we have

$$\begin{aligned} u_i(t_i, t_{-i}) = t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) &\geqslant t_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i}) \\ &= s_i f_i(s_i, t_{-i}) + (t_i - s_i) f_i(s_i, t_{-i}) - p_i(s_i, t_{-i}) \\ &= u_i(s_i, t_{-i}) + (t_i - s_i) f_i(s_i, t_{-i}) \end{aligned}$$

Proof of Myerson's Lemma (contd.)

Proof: Forward direction

- We have, $u_i(t_i, t_{-i}) \geqslant u_i(s_i, t_{-i}) + (t_i - s_i) f_i(s_i, t_{-i})$
- Fixing t_{-i}, define $g(t_i) = u_i(t_i, t_{-i})$, $\phi(t_i) = f_i(t_i, t_{-i})$
- Hence, the above inequality can be written as

$$g(t_i) \geqslant g(s_i) + \phi(s_i)(t_i - s_i)$$

- The above implies $\phi(s_i)$ is a sub-gradient of g at s_i, if g is convex
- Need to show that g is convex
- Pick $x_i, z_i \in T_i$ and define $y_i = \lambda x_i + (1-\lambda) z_i$, where $\lambda \in [0,1]$
- DSIC implies

$$g(x_i) \geqslant g(y_i) + \phi(y_i)(x_i - y_i) \text{ and } g(z_i) \geqslant g(y_i) + \phi(y_i)(z_i - y_i)$$

Proof of Myerson's Lemma (contd.)

Proof: Forward direction

$$g(x_i) \geqslant g(y_i) + \phi(y_i)(x_i - y_i) \text{ and } g(z_i) \geqslant g(y_i) + \phi(y_i)(z_i - y_i)$$
$$\implies \lambda g(x_i) + (1-\lambda)g(z_i) \geqslant g(y_i) + \phi(y_i)(\lambda x_i + (1-\lambda)z_i - y_i)$$
$$\implies \lambda g(x_i) + (1-\lambda)g(z_i) \geqslant g(\lambda x_i + (1-\lambda)z_i)$$

- Thus, g is convex
- Apply lemmas 3 and 4 from our review of convex analysis
- Lemma 3 $\implies$ $\phi = f_i(., t_{-i})$ is non-decreasing $\implies$ **part 1 proved**
- Lemma 4 $\implies$

$$g(t_i) = g(0) + \int_0^{t_i} \phi(x)dx \implies u_i(t_i, t_{-i}) = u_i(0, t_{-i}) + \int_0^{t_i} f_i(x, t_{-i})dx$$

$$\implies t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) = -p_i(0, t_{-i}) + \int_0^{t_i} f_i(x, t_{-i})dx$$

$$\implies p_i(t_i, t_{-i}) = p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_{-i})dx \implies \textbf{part 2 proved}$$

Proof of Myerson's Lemma (contd.)

Proof: Reverse direction

- Given f is non-decreasing and the payment formula
- Proof by pictures: assume $p_i(0, t_{-i}) = 0$

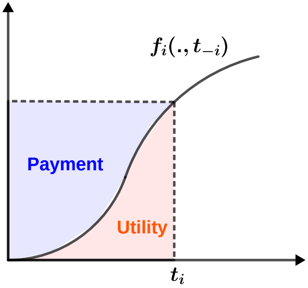


Figure: Proof by picture 1

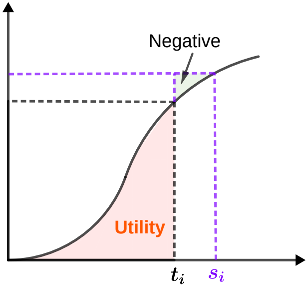


Figure: Proof by picture 2

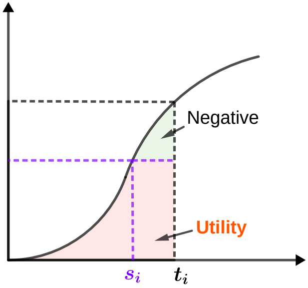


Figure: Proof by picture 3

Proof of Myerson's Lemma (contd.)

Proof: Reverse direction

- Given f is non-decreasing and the payment formula
- Proof by pictures: assume $p_i(0, t_{-i}) = 0$

$$[t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i})] - [t_i f_i(s_i, t_{-i}) - p_i(s_i, t_{-i})] = (s_i - t_i) f_i(s_i, t_{-i}) + \int_{s_i}^{t_i} f_i(x, t_{-i})\, dx \geqslant 0$$

Corollary

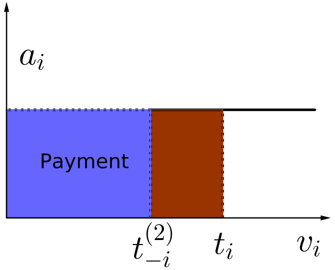
An allocation rule in a single object allocation setting is implementable in dominant strategies iff it is non-decreasing.

Contents

Examples of single object allocation

1. Constant allocation rule - non-decreasing, payment = constant (e.g. 0)
2. Dictatorial - give the object only to the dicatator - non decreasing = constant / zero
3. Second price auction

$$p_i(0, t_{-i}) + t_i f_i(t_i, t_{-i}) - \int_0^{t_i} f_i(x, t_i)dx$$



Allocation for second price auction

Examples of single object allocation

4. Efficient allocation with a reserve price is also non decreasing. If the highest value is below a reserve price r, nobody gets the object. Otherwise, the item goese to the highest bidder.
   Allocated to i if $v_i > \max\{t_{-i}^{(2)}, r\}$. Payment = $\{t_{-i}^{(2)}, r\}$
5. Not so common allocation rule: $N = \{1, 2\}$, $A = \{a_0, a_1, a_2\}$ Given a type profile $t = (t_1, t_2)$, the seller computes $u(t) = \max\{2, t_1^2, t_2^3\}$ - select a_0, a_1, a_2 depending on which of the three expressions is the maxima - break ties in favour of $0 > 1 > 2$
   Player 1 gets the object if $t_1 > \sqrt{\max\{2, t_2^3\}}$
   Player 2 gets the object if $t_2 > \sqrt[3]{\max\{2, t_1^2\}}$

Individual Rationality

Definition

A mechanism (f, p) is **ex-post individual rational** if

$$t_i f_i(t_i, t_{-i}) - p_i(t_i, t_{-i}) \geqslant 0, \ \forall t_i \in T_i, t_{-i} \in T_{-i}, \forall i \in N$$

Ex-post: Even after all agents have revealed their types, participating is weakly preferred.

Implications of Individual Rationality

Lemma

In the single object allocation setting, consider a DSIC mechanism (f,p)

1. *It is IR iff* $\forall i \in N$ *and* $\forall t_{-i} \in T_{-i}$, $p_i(0,t_{-i}) \leqslant 0$
2. *It is IR and satisfies no subsidy, i.e.,* $p_i(t_i,t_{-i}) \geqslant 0,\ \forall t_i \in T_i, t_{-i} \in T_{-i},\ \forall i \in N$ *iff* $\forall i \in N, t_{-i} \in T_{-i}, p_i(0,t_{-i}) = 0$

Proof

1. Suppose (f,p) is IR, then $0 - p_i(0,t_{-i}) \geqslant 0$, hence $p_i(0,t_{-i}) \leqslant 0$
   Conversely, if $p_i(0,t_{-i}) \leqslant 0$, then the payoff of i is

$$t_i f_i(t_i,t_{-i}) - p_i(t_i,t_{-i}) = t_i f_i(t_i,t_{-i}) - p_i(0,t_{-i}) - t_i f_i(t_i,t_{-i}) + \int_0^{t_i} f_i(x,t_{-i})dx \geqslant 0$$

2. IR $\implies p_i(0,t_{-i}) \leqslant 0$, if $p_i(t_i,t_{-i}) \geqslant 0\ \forall t_i \implies p_i(0,t_{-i}) = 0$
   Clearly if $p_i(0,t_{-i}) = 0 \implies (f,p)$ is IR and no-subsidy.

Non-Vickrey Auctions: Example 1

The object goes to the highest bidder, but the payment is such that everyone is compensated some amount.

1. Highest and second highest bidders are compensated $\frac{1}{n}$ of the third highest bid.
   $p_1(0, t_{-i}) = p_2(0, t_{-2}) = -\frac{1}{n} t_3$
2. Everyone else receives $\frac{1}{n}$ of the second highest bid
   $p_1(0, t_{-i}) = -\frac{1}{n}$ second highest in $\{t_j, j \neq i\}$

WLOG $t_1 > t_2 > \ldots > t_n$

- 1 pays $= -\frac{1}{n} t_3 + t_1 - \int_0^{t_1} f_1(x, t_{-1}) dx = -\frac{1}{n} t_3 + t_1 - (t_1 - t_2) = -\frac{1}{n} t_3 + t_2$
- 2 pays $= -\frac{1}{n} t_3$, all others $= -\frac{1}{n} t_2$
- Total payment $= -\frac{1}{n} t_3 + t_2 - \frac{1}{n} t_3 - \frac{n-2}{n} t_2 = \frac{2}{n}(t_2 - t_3)$, which tends to 0 for large n.

Deterministic mechanism that redistributes the money

Non-Vickrey Auctions: Example 2

- Allocate the object w.p. $(1 - \frac{1}{n})$ to the highest bidder and w.p. $\frac{1}{n}$ to the second highest bidder.
- $p_i(0, t_{-i}) = -\frac{1}{n}$ second highest bid in $\{t_j, j \neq i\}$
- WLOG $t_1 > t_2 > \ldots > t_n$
- 1 pays $= -\frac{1}{n}t_3 + (1 - \frac{1}{n})t_1 - \frac{1}{n}(t_2 - t_3) - (1 - \frac{1}{n})(t_1 - t_2) = (1 - \frac{2}{n})t_2$
- 2 pays $= -\frac{1}{n}t_3 + \frac{1}{n}t_2 - \frac{1}{n}(t_2 - t_3) = 0$
- All others $= -\frac{1}{n}t_2$.
- Together = 0

Randomized mechanism that redistributes the money

Contents

Revenue Maximization

Question

How to maximize revenue earned by the auctioneer?

Question

Maximize w.r.t. what knowledge of the auctioneer?

Answer

The common prior distribution over types

Accordingly, the notions of incentive compatibility and individual rationality have to change

Bayesian setup

Preliminaries

- $T_i = [0, b_i]$, Common prior G over $T = \times_{i=1}^{n} T_i$, g denotes the density
- $G_{-i}(s_{-i}|s_i)$ is the conditional distribution over s_{-i}, given i's type is s_i
- Similarly, $g_{-i}(s_{-i}|s_i)$ is derived via Bayes rule from g
- Every mechanism $(f, p_1, p_2, \ldots, p_n)$ induces an expected allocation and payment rule (α, π)

$$\alpha_i(\underbrace{s_i}_{\text{reported}} \mid \underbrace{t_i}_{\text{true}}) = \int_{s_{-i} \in T_{-i}} \underbrace{f_i(s_i, s_{-i})}_{\text{probabilistic allocation}} \underbrace{g_{-i}(s_{-i}|t_i)}_{\text{prior on types of others}} ds_{-i} \quad \textbf{expected allocation}$$

$$\pi_i(s_i|t_i) = \int_{s_{-i} \in T_{-i}} p_i(s_i, s_{-i}) g_{-i}(s_{-i}|t_i) ds_{-i} \quad \textbf{expected payment}$$

where s_i is the reported type and t_i is the true type.

- Expected utility of agent i: $u_i = t_i \alpha_i(s_i|t_i) - \pi_i(s_i|t_i)$

Bayesian Incentive Compatibility

Definition (Bayesian Incentive Compatibility (BIC))

A mechanism (f, p) is Bayesian incentive compatible (BIC) if $\forall i \in N, \forall s_i, t_i \in T_i$

$$t_i \alpha_i(t_i|t_i) - \pi_i(t_i|t_i) \geqslant t_i \alpha_i(s_i|t_i) - \pi_i(s_i|t_i)$$

Similarly, f is Bayesian implementable if $\exists p$ s.t. (f, p) is BIC.

Characterization of BIC mechanisms

Assume that **priors are independent**, i.e., agent i's value is drawn from a distribution G_i (density g_i) independently from other agents.

$$G(s_1, s_2, \ldots, s_n) = \prod_{i \in N} G_i(s_i), \qquad G(s_{-i}|t_i) = \prod_{j \neq i} G_j(s_j)$$

Because of independence, the conditional term can be dropped from the notation, i.e., $\alpha(s_i) = \alpha(s_i|t_i)$ – we will assume independence in all discussions from now on

Definition

An allocation rule is **Non-decreasing in expectation (NDE)** if $\forall i \in N, \forall s_i, t_i \in T_i$ with $s_i < t_i$ we have $\alpha_i(s_i) \leqslant \alpha_i(t_i)$

Note: The rules that are non-decreasing (defined before) are always NDE, but there can be more rules that are NDE

NDE but not ND

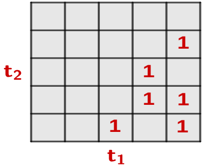


Figure: An allocation rule may be NDE but not non-decreasing

All five types are equally likely, $\alpha_1(t_1)$ and $\alpha_2(t_2)$ are monotone, but $f(t_1, t_2)$ is not.

Theorem (Myerson 1981)

A mechanism (f, p) in the independent prior setting is BIC iff

- *f is NDE, and*
- *p_i satisfies $\pi_i(t_i) = \pi_i(0) + t_i\alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx, \;\; \forall t_i \in T_i, \forall i \in N$*

Characterization of BIC rules

Proof.

This is Bayesian version of the earlier Myerson theorem, proof proceeds in similar lines as before [exercise] □

As we are in the Bayesian setting now, we can define an analog of individual rationality.

Definition

A mechanism (f, p) is **interim individually rational (IIR)** if for every bidder $i \in N$, we have $t_i \alpha_i(t_i) - \pi_i(t_i) \geqslant 0, \; \forall t_i \in T_i$

Lemma

A mechanism (f, p) is BIC and IIR iff

- f *is NDE*
- p_i *satisfies* $\pi_i(t_i) = \pi_i(0) + t_i \alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx, \;\; \forall t_i \in T_i, \forall i \in N$
- $\forall i \in N, \; \pi_i(0) \leqslant 0$

Characterization of BIC rules

Proof-sketch:

- The first two conditions uniquely identify a BIC mechanism
- So, the proof requires to show that IIR along with first two conditions is equivalent to third condition
- **Forward direction:** apply IIR at $t_i = 0$ on second condition and get $\pi_i(0) \leqslant 0$
- **Reverse direction:** $t_i \alpha_i(t_i) - \pi_i(t_i) = -\pi_i(0) + \int_0^{t_i} \alpha_i(s_i)\, ds_i \geqslant 0$ if $\pi_i(0) \leqslant 0$

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