



भारतीय प्रौद्योगिकी संस्थान मुंबई  
Indian Institute of Technology Bombay

# CS 6001: Game Theory and Algorithmic Mechanism Design

Week 12

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal



- ▶ Single Agent Optimal Mechanism Design
- ▶ Optimal Mechanism Design with Multiple Agents
- ▶ Examples of Optimal Mechanism Design
- ▶ Endnotes and Summary

# Mechanism Design for Single Agent



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- The expected revenue earned by a mechanism  $M$  is given by

$$\Pi^M := \int_0^\beta p(t)g(t)dt$$

# Optimal Mechanism for Single Agent



## Definition (Optimal Mechanism)

An optimal mechanism  $M^*$  for a single agent is a mechanism in the class of all IC and IR mechanisms, such that  $\Pi^{M^*} \geq \Pi^M, \forall M$

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- Since we want to maximize the revenue, hence  $p(0) = 0$

# Optimal Mechanism for Single Agent



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- **Expected revenue:**

$$\begin{aligned}\Pi^f &= \int_0^\beta p(t)g(t)dt \\ &= \int_0^\beta \left( tf(t) - \int_0^t f(x)dx \right) g(t)dt\end{aligned}$$



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- Need to maximize this w.r.t.  $f$

# The Optimization Problem



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# The Optimization Problem



## Lemma

*For any implementable allocation rule  $f$ , we have*

$$\Pi^f = \int_0^\beta \left( t - \frac{1 - G(t)}{g(t)} \right) g(t) f(t) dt$$

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- The following term is also called the **virtual valuation** of the agent

$$w(t) = \left( t - \frac{1 - G(t)}{g(t)} \right)$$

# The Modified Optimization Problem



- Hence the optimal mechanism finding mechanism reduces to

$$\text{OPT1: } \max_{f: f \text{ is non-decreasing}} \int_0^\beta \left( t - \frac{1 - G(t)}{g(t)} \right) g(t) f(t) dt$$

- Assumption:**  $G$  satisfies the monotone hazard rate condition (MHR), i.e.,  $\frac{g(x)}{1-G(x)}$  is non-decreasing in  $x$
- Standard distributions like **uniform** and **exponential** satisfy MHR condition

# Observation



Fact

*If  $G$  satisfies MHR condition, there is a solution to  $x = \frac{1-G(x)}{g(x)}$*

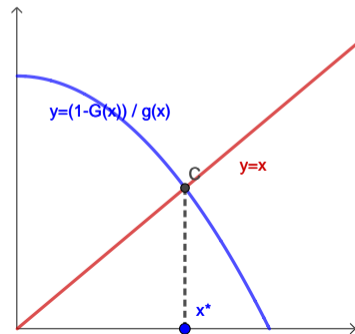


# Observation

## Fact

If  $G$  satisfies MHR condition, there is a solution to  $x = \frac{1-G(x)}{g(x)}$

- Let  $x^*$  be a solution of this equation
- Hence,  $w(x) = x - \frac{1-G(x)}{g(x)}$  is zero at  $x^*$
- $\implies w(x) \geq 0, \forall x > x^*$  and  $\leq 0, \forall x < x^*$



# Solution to the optimization problem



- The unrestricted solution to OPT1 is therefore

$$f(t) = \begin{cases} 0 & \text{if } t < x^* \\ 1 & \text{if } t > x^* \\ \alpha & \text{if } t = x^*, \alpha \in [0, 1] \end{cases} \quad (1)$$

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- $f$  is given by Equation (1) where  $x^*$  is a solution of  $x = \frac{1-G(x)}{g(x)}$ , and
- For all  $t \in T$ ,  $p(t) = \begin{cases} x^* & \text{if } t \geq x^* \\ 0 & \text{otherwise} \end{cases}$



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- This can be simplified to the following in a way similar to the earlier exercise

$$\int_0^{b_i} w_i(t_i) g_i(t_i) \alpha_i(t_i) dt_i$$

where,  $w_i(t_i) = t_i - \frac{1 - G_i(t_i)}{g_i(t_i)}$  (virtual valuation of player  $i$ ) and,

$$\alpha_i(t_i) = \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i}) dt_{-i}$$



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$$\sum_{i \in N} \int_T w_i(t_i) f_i(t) g(t) dt = \int_T \sum_{i \in N} (w_i(t_i) f_i(t)) g(t) dt$$

where  $\sum_{i \in N} (w_i(t_i) f_i(t))$  is the expected total virtual valuation

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- Hence, the optimal mechanism problem reduces to

$$\max \int_T \sum_{i \in N} (w_i(t_i) f_i(t)) g(t) dt, \text{ s.t. } f \text{ is NDE}$$



- As before, we try to solve the **unconstrained** optimization problem.

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geq w_j(t_j), \forall j, \text{ break ties arbitrarily} \\ 0, & \text{otherwise} \end{cases} \quad \text{(Sold)} \quad (2)$$

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A virtual valuation  $w_i$  is regular if  $\forall s_i, t_i \in T_i$  with  $s_i < t_i$ , it holds that  $w_i(s_i) \leq w_i(t_i)$ .

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- This condition is weaker than MHR condition as MHR implies regularity

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- i.e.,  $f_i$  is non-decreasing (hence NDE)



- Optimal Mechanism Design Problem

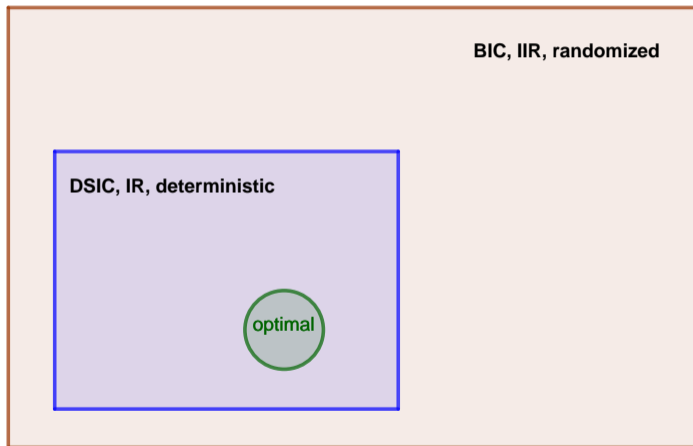
$$\max \int_T \left( \sum_{i \in N} w_i(t_i) f_i(t) \right) g(t) dt, \quad \text{such that } f \text{ is NDE}$$

Solution for **regular**  $w_i$ 's

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geq w_j(t_j), \forall j, \text{ break ties arbitrarily} \\ 0, & \text{otherwise} \end{cases} \quad \text{(Sold)} \quad (3)$$

$$f_i(t) = 0, \forall i \in N, \text{ if } w_i(t_i) < 0, \forall i \in N \quad \text{(Unsold)}$$

- We wanted to find an allocation that is NDE, but found an  $f$  that is non-decreasing
- It is also deterministic



Space of mechanisms with regular virtual valuations



## Theorem

*Suppose every agent's valuation is regular.*



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*Suppose every agent's valuation is regular. Then, for every type profile  $t$ , if  $w_i(t_i) < 0, \forall i \in N$ ,  $f_i(t) = 0, \forall i \in N$ .*



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Payments are given by  $p_i(t) = \begin{cases} 0 & \text{if } f_i(t) = 0 \\ \max\{w_i^{-1}(0), K_i^*(t_{-i})\} & \text{if } f_i(t) = 1, \end{cases}$

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where  $w_i^{-1}(0)$ : the value of  $t_i$  where  $w_i(t_i) = 0$ , and  $K_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i}) = 1\}$ ,

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where  $w_i^{-1}(0)$ : the value of  $t_i$  where  $w_i(t_i) = 0$ , and  $K_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i}) = 1\}$ , then  $(f, p)$  is an optimal mechanism.

# Optimal Mechanism: Allocation and Payment



## Theorem

Suppose every agent's valuation is regular. Then, for every type profile  $t$ , if  $w_i(t_i) < 0, \forall i \in N$ ,  $f_i(t) = 0, \forall i \in N$ .

Otherwise,  $f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geq w_j(t_j) \forall j \in N \\ 0 & \text{otherwise,} \end{cases}$

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**Note:**  $K_i^*(t_{-i})$  is the minimum of value of  $t_i$  where  $i$  begins to be the winner



- ▶ Single Agent Optimal Mechanism Design
- ▶ Optimal Mechanism Design with Multiple Agents
- ▶ **Examples of Optimal Mechanism Design**
- ▶ Endnotes and Summary



# Example 1

- 1 Two buyers :  $T_1 = [0, 12], T_2 = [0, 18]$
- 2 Uniform independent prior
- 3  $w_1(t_1) = t_1 - \frac{1-G(t)}{g(t)} = t_1 - \frac{1-\frac{t_1}{12}}{\frac{1}{12}} = 2t_1 - 12$
- 4  $w_2(t_2) = 2t_2 - 18$

$t_1$	$t_2$	Action	$p_1$	$p_2$
4	8	unsold	0	0
2	12	sold to 2	0	9
6	6	sold to 1	6	0
9	9	sold to 1	6	0
8	15	sold to 2	0	11

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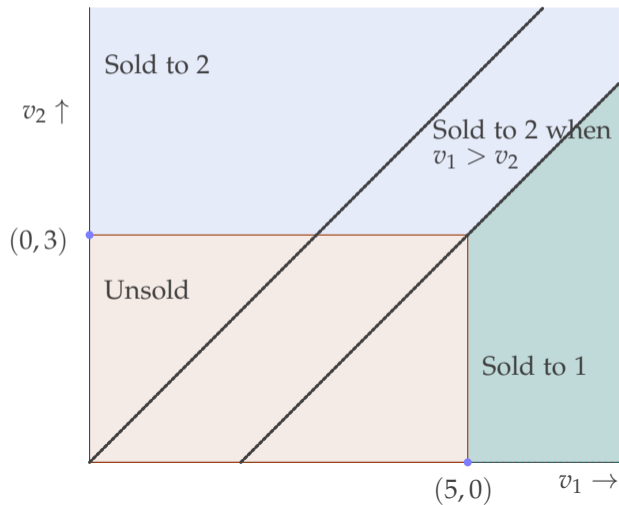
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- Second price auction with a reserve price, and is efficient when the object is sold.



## Example 3 : Efficiency and Optimality

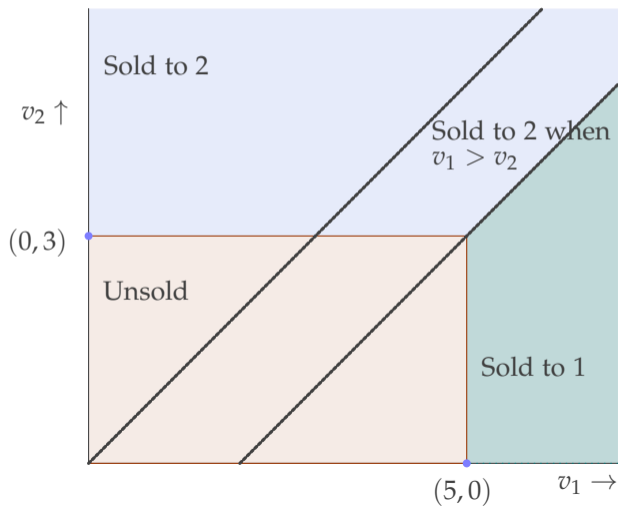
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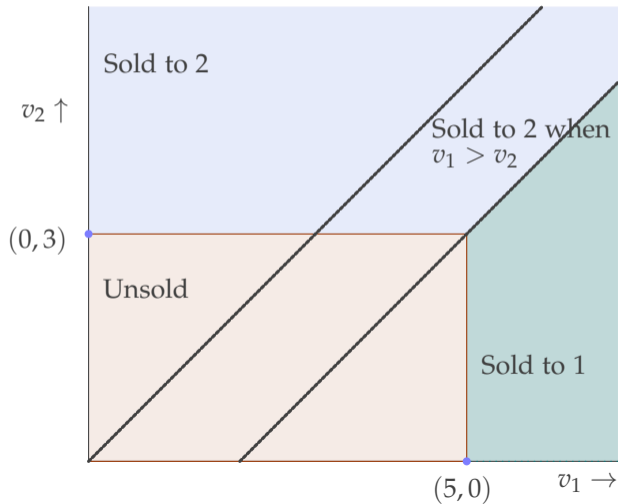
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- $w_1(t_1) = 2t_1 - 10$ ,  
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- Unsold is inefficient, also in the region of the plane where 1 has higher valuation but item is sold to 2





- ▶ Single Agent Optimal Mechanism Design
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  - Fix the valuations of other agents to  $t_{-i}$
  - Fix value of  $i$  at alternative  $b$  as  $t_i(b)$
- $\exists$  some threshold  $t_i^*(a)$  s.t.

$\forall t_i(a) \geq t_i^*(a)$ ,  $a$  is the outcome, and  $\forall t_i(a) < t_i^*(a)$ ,  $b$  is the outcome

## Proof sketch (contd.)



- Using DSIC for  $t_i^*(a) + \epsilon = t_i(a), \epsilon > 0$  we have,

$$t_i^*(a) + \epsilon - p_{ia} \geq t_i(b) - p_{ib} \quad (\text{Note: payment for a player has to be the same for an allocation})$$



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- Similarly,  $t'_i(a) = t_i^*(a) - \delta$ ,  $\delta > 0$  and

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- Since,  $\epsilon, \delta$  are arbitrary, then

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- But  $t_i^*(a)$  is the threshold of the efficient outcome, thus,

$$t_i^*(a) + \sum_{j \neq i} t_j(a) = t_i(b) + \sum_{j \neq i} t_j(b) \quad (5)$$



- From Equations (4) and (5)

$$p_{ia} - p_{ib} = \sum_{j \neq i} t_j(b) - \sum_{j \neq i} t_j(a)$$



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- Hence, the payment has to be of the form  $p_{ix} = h_i(t_{-i}) - \sum_{j \neq i} t_j(x)$

# Efficiency and Budget Balance



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Corollary

*If the valuation space is sufficiently rich, no efficient mechanism can be both DSIC and BB.*

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- Eliminating  $h_1(w_2)$ , we get  $w_2 = h_2(w_1^+) - h_2(w_1^-) - w_1^+$
- The RHS depends only on  $w_1$ , hence it is possible to alter  $w_2$  slightly to retain the inequalities, but then the above equality cannot hold.

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- To show budget balance, consider

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## Theorem

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## Theorem (Myerson, Satterthwaite (1983))

*In a bilateral trade (that involves two types of agents: seller and buyer) no mechanism can be simultaneously BIC, efficient, IIR and budget balanced.*

# Space of Mechanisms

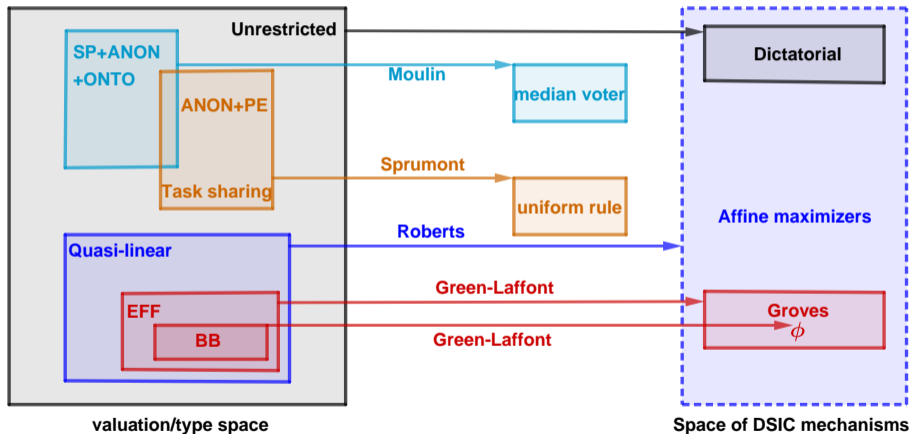


Figure: Space of Mechanisms 1

# Space of Mechanisms

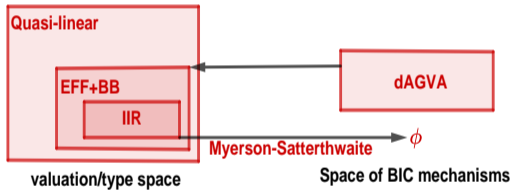


Figure: Space of Mechanisms 2

# Space of Mechanisms

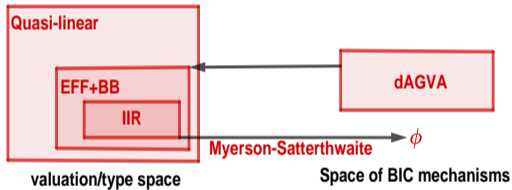


Figure: Space of Mechanisms 2

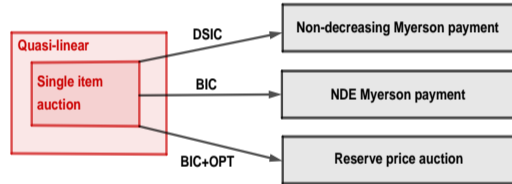


Figure: Space of Mechanisms 3



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