भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 12

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Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

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- The expected revenue earned by a mechanism M is given by

$$\Pi^M := \int_0^\beta p(t)g(t)dt$$

Optimal Mechanism for Single Agent

Definition (Optimal Mechanism)

An optimal mechanism M^* for a single agent is a mechanism in the class of all IC and IR mechanisms, such that $\Pi^{M^*} \geqslant \Pi^M, \forall M$

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$$p(t) = p(0) + tf(t) - \int_0^t f(x)dx \qquad \text{[IC]}$$

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- Since we want to maximize the revenue, hence $p(0) = 0$

Optimal Mechanism for Single Agent

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- **Expected revenue:**

$$\begin{aligned}\Pi^f &= \int_0^\beta p(t)g(t)dt \\ &= \int_0^\beta \left(tf(t) - \int_0^t f(x)dx\right) g(t)dt\end{aligned}$$

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- Need to maximize this w.r.t. f

The Optimization Problem

Lemma

For any implementable allocation rule f, we have

$$\Pi^f = \int_0^\beta \left(t - \frac{1 - G(t)}{g(t)} \right) g(t) f(t) dt$$

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- The following term is also called the virtual valuation of the agent

$$w(t) = \left(t - \frac{1 - G(t)}{g(t)} \right)$$

The Optimization Problem

Proof

$$\Pi^f = \left(tf(t) - \int_0^t f(x)dx \right) g(t)dt$$

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&= \int_0^\beta \left(t - \frac{1 - G(t)}{g(t)} \right) g(t)f(t)dt
\end{aligned}$$

The Modified Optimization Problem

- Hence the optimal mechanism finding mechanism reduces to

OPT1:
$$\max_{f:f \text{ is non-decreasing}} \int_0^{\beta} \left(t - \frac{1 - G(t)}{g(t)} \right) g(t) f(t) dt$$

- **Assumption:** G satisfies the montotone hazard rate condition (MHR), i.e., $\frac{g(x)}{1-G(x)}$ is non-decreasing in x
- Standard distributions like **uniform** and **exponential** statisfy MHR condition

Observation

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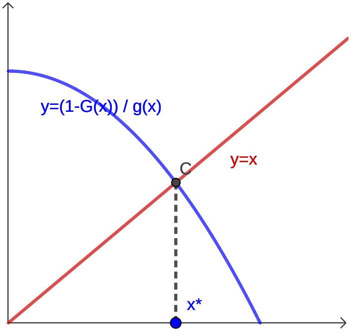
- Let x^* be a solution of this equation
- Hence, $w(x) = x - \frac{1-G(x)}{g(x)}$ is zero at x^*
- $\implies w(x) \geqslant 0, \ \forall x > x^*$ and $\leqslant 0, \ \forall x < x^*$



Solution to the optimization problem

- The unrestricted solution to OPT1 is therefore

$$f(t) = \begin{cases} 0 & \text{if } t < x^* \\ 1 & \text{if } t > x^* \\ \alpha & \text{if } t = x^*, \alpha \in [0,1] \end{cases} \tag{1}$$

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Theorem

A mechanism (f, p) under the MHR condition is optimal iff

1. *f is given by Equation (1) where x^* is a solution of $x = \frac{1-G(x)}{g(x)}$, and*
2. *For all $t \in T$, $p(t) = \begin{cases} x^* & \text{if } t \geqslant x^* \\ 0 & \text{otherwise} \end{cases}$*

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- This can be simplified to the following in a way similar to the earlier exercise

$$\int_0^{b_i} w_i(t_i) g_i(t_i) \alpha_i(t_i)\, dt_i$$

where, $w_i(t_i) = t_i - \dfrac{1 - G_i(t_i)}{g_i(t_i)}$ (virtual valuation of player i) and,

$$\alpha_i(t_i) = \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i})\, dt_{-i}$$

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- Hence, the optimal mechanism problem reduces to

$$\max \int_T \sum_{i \in N} (w_i(t_i) f_i(t)) g(t)\, dt, \text{ s.t. } f \text{ is NDE}$$

Optimal mechanism design for multiple agents

- As before, we try to solve the **unconstrained** optimization problem.

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j),\ \forall j,\ \text{break ties arbitrarily} \\ 0, & \text{otherwise} \end{cases} \quad \textbf{(Sold)}$$
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- This condition is weaker than MHR condition as MHR implies regularity

Optimal mechanism design for multiple agents

Lemma

Suppose every agent's valuations are regular. The allocation rule of the optimal mechanism is same as the solution of the unconstrained problem.

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- Then the optimal allocation also satisfies

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- i.e., f_i is non-decreasing (hence NDE)

The solution

- Optimal Mechanism Design Problem

$$\max \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t) \right) g(t) dt \right), \quad \text{such that } f \text{ is NDE}$$

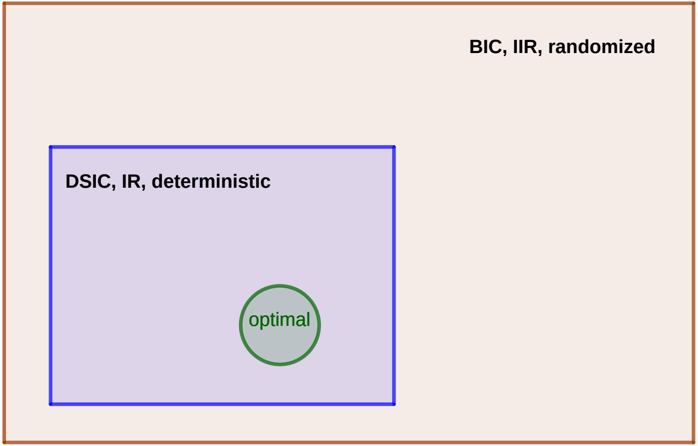
Solution for **regular** w_i's

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j),\ \forall j, \text{ break ties arbitrarily} \\ 0, & \text{otherwise} \end{cases} \quad \textbf{(Sold)} \tag{3}$$

$$f_i(t) = 0, \forall i \in N,\ \text{if } w_i(t_i) < 0,\ \forall i \in N \quad \textbf{(Unsold)}$$

- We wanted to find an allocation that is NDE, but found an f that is non-decreasing
- It is also deterministic

Optimal Mechanism



Space of mechanisms with regular virtual valuations

Optimal Mechanism: Allocation and Payment

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with ties are broken arbitrarily.

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Payments are given by $p_i(t) = \begin{cases} 0 & \text{if } f_i(t) = 0 \\ \max\{w_i^{-1}(0), K_i^*(t_{-i})\} & \text{if } f_i(t) = 1, \end{cases}$

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where $w_i^{-1}(0)$*: the value of* t_i *where* $w_i(t_i) = 0$*, and* $K_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i}) = 1\}$,

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where $w_i^{-1}(0)$: the value of t_i where $w_i(t_i) = 0$, and $K_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i}) = 1\}$, then (f, p) is an optimal mechanism.

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Otherwise, $f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j)\ \forall j \in N \\ 0 & \text{otherwise,} \end{cases}$

with ties are broken arbitrarily.

Payments are given by $p_i(t) = \begin{cases} 0 & \text{if } f_i(t) = 0 \\ \max\{w_i^{-1}(0), K_i^*(t_{-i})\} & \text{if } f_i(t) = 1, \end{cases}$

where $w_i^{-1}(0)$*: the value of* t_i *where* $w_i(t_i) = 0$*, and* $K_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i}) = 1\}$*, then* (f, p) *is an optimal mechanism.*

Note: $K_i^*(t_{-i})$ is the minimum of value of t_i where i begins to be the winner

Contents

Example 1

1. Two buyers : $T_1 = [0, 12]$, $T_2 = [0, 18]$
2. Uniform independent prior
3. $w_1(t_1) = t_1 - \frac{1-G(t)}{g(t)} = t_1 - \frac{1-\frac{t_1}{12}}{\frac{1}{12}} = 2t_1 - 12$
4. $w_2(t_2) = 2t_2 - 18$

t_1	t_2	Action	p_1	p_2
4	8	unsold	0	0
2	12	sold to 2	0	9
6	6	sold to 1	6	0
9	9	sold to 1	6	0
8	15	sold to 2	0	11

Example 2

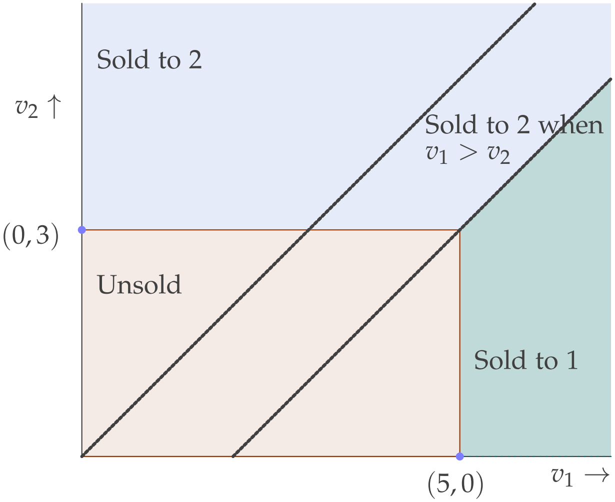
- **Systematic bidders:** the valuations are drawn from the same distribution, $g_i = g$, $T_i = T$, $\forall i \in N$
- **Virtual valuation:** $w_i = w$

$$w(t_i) > w(t_j), \text{ iff } t_i > t_j$$

- The object goes to the highest bidder. Not sold if $w_{-i}(0) > t_i \forall i \in N$.
  $p_i = \max\{w^{-1}(0), \max_{j \neq i} t_j\}$
- Second price auction with a reserve price, and is efficient when the object is sold.

Example 3 : Efficiency and Optimality

- Two buyers : $T_1 = [0, 10]$, $T_2 = [0, 6]$, Uniform independent prior
- $w_1(t_1) = 2t_1 - 10$, $w_2(t_2) = 2t_2 - 6$
- Unsold is inefficient, also in the region of the plane where 1 has higher valuation but item is sold to 2



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  - Fix the valuations of other agents to t_{-i}
  - Fix value of i at alternative b as $t_i(b)$
- $\exists$ some threshold $t_i^*(a)$ s.t.

$$\forall t_i(a) \geqslant t_i^*(a), \; a \text{ is the outcome, and } \forall t_i(a) < t_i^*(a), \; b \text{ is the outcome}$$

Proof sketch (contd.)

- Using DSIC for $t_i^*(a) + \epsilon = t_i(a), \epsilon > 0$ we have,

  $t_i^*(a) + \epsilon - p_{ia} \geqslant t_i(b) - p_{ib}$ (Note: payment for a player has to be the same for an allocation.)

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- But $t_i^*(a)$ is the threshold of the efficient outcome, thus,

$$t_i^*(a) + \sum_{j \neq i} t_j(a) = t_i(b) + \sum_{j \neq i} t_j(b) \tag{5}$$

Proof sketch (contd.)

- From Equations (4) and (5)

$$p_{ia} - p_{ib} = \sum_{j \neq i} t_j(b) - \sum_{j \neq i} t_j(a)$$

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- Hence, the payment has to be of the form $p_{ix} = h_i(t_{-i}) - \sum_{j \neq i} t_j(x)$

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Corollary

If the valuation space is sufficiently rich, no efficient mechanism can be both DSIC and BB.

Proof sketch of the second theorem

- Consider two alternatives $\{0, 1\}$ s.t.

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- Eliminating $h_1(w_2)$, we get $w_2 = h_2(w_1^+) - h_2(w_1^-) - w_1^+$
- The RHS depends only on w_1, hence it is possible to alter w_2 slightly to retain the inequalities, but then the above equality cannot hold.

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&\geqslant \mathbb{E}_{t_{-i}|t_i} \sum_{j \in N} t_j(a^*(t_i', t_{-i})) - \mathbb{E}_{t_{-i}|t_i} \left[\frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) \right]
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&= \mathbb{E}_{t_{-i}|t_i} \left[t_i(a^*(t'_i, t_{-i})) - p_i^{dAGVA}(t'_i, t_{-i}) \right]
\end{aligned}$$

Budget Balance?

- To show budget balance, consider

$$\sum_{i \in N} p_i^{dAGVA}(t) = \frac{1}{n-1} \sum_{i \in N} \sum_{j \neq i} \delta_j(t_j) - \sum_{i \in N} \delta_i(t_i)$$
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Theorem

The dAGVA mechanism is efficient, BIC, and BB.

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Theorem

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Theorem (Myerson, Satterthwaite (1983))

In a bilateral trade (that involves two types of agents: seller and buyer) no mechanism can be simultaneously BIC, efficient, IIR and budget balanced.

Space of Mechanisms

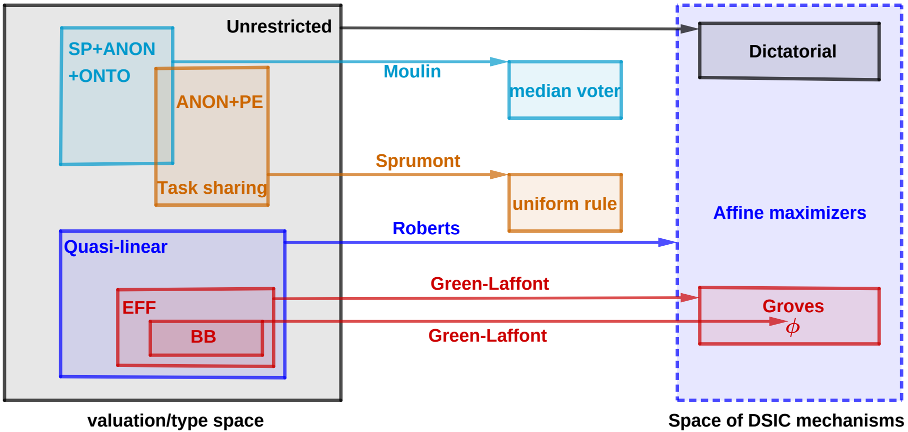

Figure: Space of Mechanisms 1

Space of Mechanisms

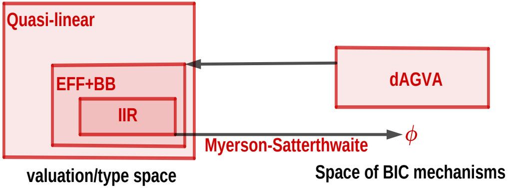

Figure: Space of Mechanisms 2

Space of Mechanisms

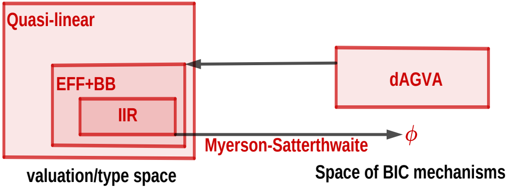

Figure: Space of Mechanisms 2

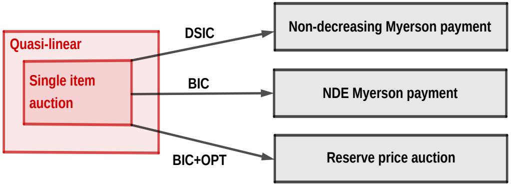

Figure: Space of Mechanisms 3

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay