भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 12

Swaprava Nath

Slide preparation acknowledgments: Ramsundar Anandanarayanan and Harshvardhan Agarwal

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Contents

Mechanism Design for Single Agent

- Type set $T = [0, \beta]$, Mechanism $M := (f, p)$
- $f : [0, \beta] \to [0, 1]$, $p : [0, \beta] \to \mathbb{R}$
- **Incentive Compatibility** [BIC and DSIC equivalent]

$$tf(t) - p(t) \geqslant tf(s) - p(s), \ \forall t, s \in T$$

- **Individual Rationality** [IR and IIR equivalent]

$$tf(t) - p(t) \geqslant 0, \ \forall t, s \in T$$

- The expected revenue earned by a mechanism M is given by

$$\Pi^M := \int_0^\beta p(t) g(t) dt$$

Optimal Mechanism for Single Agent

Definition (Optimal Mechanism)

An optimal mechanism M^* for a single agent is a mechanism in the class of all IC and IR mechanisms, such that $\Pi^{M^*} \geqslant \Pi^M, \forall M$

Question

What is the structure of an optimal mechanism?

- Consider an IC and IR mechanism $M = (f, p)$
- By the characterization results, we know f is monotone, and

$$p(t) = p(0) + tf(t) - \int_0^t f(x)dx \qquad \text{[IC]}$$

$$p(0) \leqslant 0 \qquad \text{[IR]}$$

- Since we want to maximize the revenue, hence $p(0) = 0$

Optimal Mechanism for Single Agent

- Hence the payment formula is

$$p(t) = tf(t) - \int_0^t f(x)dx$$

- **Note:** In optimal mechanism, payment is completely given once the allocation is fixed
- Hence, we need to optimize only over one variable f
- **Expected revenue:**

$$\begin{aligned}\Pi^f &= \int_0^\beta p(t)g(t)dt \\ &= \int_0^\beta \left(tf(t) - \int_0^t f(x)dx\right) g(t)dt\end{aligned}$$

- Need to maximize this w.r.t. f

The Optimization Problem

Lemma

For any implementable allocation rule f, we have

$$\Pi^f = \int_0^\beta \left(t - \frac{1 - G(t)}{g(t)}\right) g(t) f(t) dt$$

- The following term is also called the virtual valuation of the agent

$$w(t) = \left(t - \frac{1 - G(t)}{g(t)}\right)$$

The Optimization Problem

Proof

$$
\begin{aligned}
\Pi^f &= \left(tf(t) - \int_0^t f(x)dx\right) g(t)dt \\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \int_0^t {\color{red}f(x)dx}\, {\color{blue}g(t)dt} \\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \int_x^\beta {\color{blue}g(t)dt}\, {\color{red}f(x)dx} && \text{[switching the order of integration]} \\
&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \int_t^\beta g(x)dxf(t)dt \\
&= \int_0^\beta (tf(t)g(t) - (1 - G(t)f(t))\, dt \\
&= \int_0^\beta \left(t - \frac{1 - G(t)}{g(t)}\right) g(t)f(t)dt
\end{aligned}
$$

The Modified Optimization Problem

- Hence the optimal mechanism finding mechanism reduces to

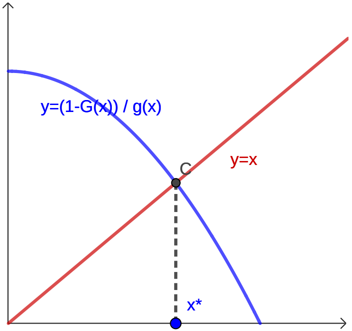
$$\textbf{OPT1:} \quad \max_{f:f \text{ is non-decreasing}} \int_0^\beta \left(t - \frac{1-G(t)}{g(t)} \right) g(t) f(t) dt$$

- **Assumption:** G satisfies the montotone hazard rate condition (MHR), i.e., $\frac{g(x)}{1-G(x)}$ is non-decreasing in x
- Standard distributions like **uniform** and **exponential** statisfy MHR condition

Observation

Fact

If G satisfies MHR condition, there is a soultion to $x = \frac{1-G(x)}{g(x)}$



- Let x^* be a solution of this equation
- Hence, $w(x) = x - \frac{1-G(x)}{g(x)}$ is zero at x^*
- $\implies w(x) \geqslant 0, \ \forall x > x^*$ and $\leqslant 0, \ \forall x < x^*$

Solution to the optimization problem

- The unrestricted solution to OPT1 is therefore

$$f(t) = \begin{cases} 0 & \text{if } t < x^* \\ 1 & \text{if } t > x^* \\ \alpha & \text{if } t = x^*, \alpha \in [0,1] \end{cases} \tag{1}$$

- But this f is non-decreasing, therefore it is the optimal solution of OPT1

Theorem

A mechanism (f, p) under the MHR condition is optimal iff

1. *f is given by Equation (1) where x^* is a solution of $x = \frac{1-G(x)}{g(x)}$, and*
2. *For all $t \in T$, $p(t) = \begin{cases} x^* & \text{if } t \geqslant x^* \\ 0 & \text{otherwise} \end{cases}$*

Contents

Optimal mechanism design for multiple agents

- In this context, we will call a mechanism optimal if it is BIC, IIR, and maximizes revenue
- By previous results, this reduces to:
  1. f_i's are NDE $\forall i \in N$,
  2. $\pi_i(t_i)$ has a specific integral formula and $\pi_i(0) = 0$
- Hence, the expected payment made by agent i is $\int_{T_i} \pi_i(t_i) g_i(t_i)\, dt_i,\ T_i = [0, b_i]$
- This can be simplified to the following in a way similar to the earlier exercise

$$\int_0^{b_i} w_i(t_i) g_i(t_i) \alpha_i(t_i)\, dt_i$$

where, $w_i(t_i) = t_i - \dfrac{1 - G_i(t_i)}{g_i(t_i)}$ (virtual valuation of player i) and,

$$\alpha_i(t_i) = \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i})\, dt_{-i}$$

Optimal mechanism design for multiple agents

- This gives, expected payment made by agent i as

$$\int_T w_i(t_i) f_i(t) g(t)\, dt$$

- The total revenue generated by all players is

$$\sum_{i \in N} \int_T w_i(t_i) f_i(t) g(t)\, dt = \int_T \sum_{i \in N} (w_i(t_i) f_i(t)) g(t)\, dt$$

  where $\sum_{i \in N} (w_i(t_i) f_i(t))$ is the expected total virtual valuation

- Hence, the optimal mechanism problem reduces to

$$\max \int_T \sum_{i \in N} (w_i(t_i) f_i(t)) g(t)\, dt, \text{ s.t. } f \text{ is NDE}$$

Optimal mechanism design for multiple agents

- As before, we try to solve the **unconstrained** optimization problem.

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j),\ \forall j,\ \text{break ties arbitrarily} \\ 0, & \text{otherwise} \end{cases} \quad \textbf{(Sold)} \tag{2}$$

$$f_i(t) = 0, \forall i \in N,\ \text{if } w_i(t_i) < 0,\ \forall i \in N \quad \textbf{(Unsold)}$$

- But it can lead to a case where f is not NDE (for an example, see Roger B Myerson. "Optimal auction design", 1981
- The example is such that the following condition is violated

Definition

A virtual valuation w_i is regular if $\forall s_i, t_i \in T_i$ with $s_i < t_i$, it holds that $w_i(s_i) \leqslant w_i(t_i)$.

- This condition is weaker than MHR condition as MHR implies regularity

Optimal mechanism design for multiple agents

Lemma

Suppose every agent's valuations are regular. The allocation rule of the optimal mechanism is same as the solution of the unconstrained problem.

Proof-sketch:

- The solution is as given in Equation (2)
- Regularity ensures that $w_i(t_i) \geqslant w_i(s_i), \ \forall s_i < t_i$
- Then the optimal allocation also satisfies

$$f_i(t_i, t_{-i}) \geqslant f_i(s_i, t_{-i}), \ \forall t_{-i} \in T_{-i}, \forall s_i < t_i$$

- i.e., f_i is non-decreasing (hence NDE)

The solution

- Optimal Mechanism Design Problem

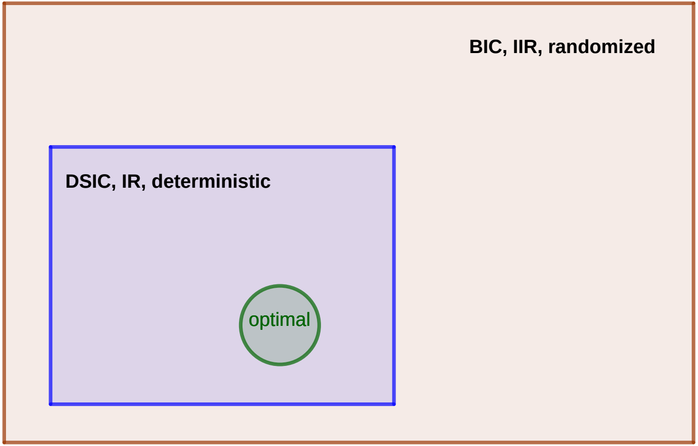
$$\max \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t) \right) g(t) dt \right), \quad \text{such that } f \text{ is NDE}$$

Solution for **regular** w_i's

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j), \ \forall j, \text{ break ties arbitrarily} \\ 0, & \text{otherwise} \end{cases} \quad \textbf{(Sold)}$$
$$f_i(t) = 0, \forall i \in N, \text{ if } w_i(t_i) < 0, \ \forall i \in N \quad \textbf{(Unsold)} \tag{3}$$

- We wanted to find an allocation that is NDE, but found an f that is non-decreasing
- It is also deterministic

Optimal Mechanism



Space of mechanisms with regular virtual valuations

Optimal Mechanism: Allocation and Payment

Theorem

Suppose every agent's valuation is regular. Then, for every type profile t, if $w_i(t_i) < 0, \forall i \in N$, $f_i(t) = 0, \forall i \in N$.

Otherwise, $f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j) \ \forall j \in N \\ 0 & \text{otherwise,} \end{cases}$

with ties are broken arbitrarily.

Payments are given by $p_i(t) = \begin{cases} 0 & \text{if } f_i(t) = 0 \\ \max\{w_i^{-1}(0), K_i^*(t_{-i})\} & \text{if } f_i(t) = 1, \end{cases}$

where $w_i^{-1}(0)$: the value of t_i where $w_i(t_i) = 0$, and $K_i^*(t_{-i}) = \inf\{t_i : f_i(t_i, t_{-i}) = 1\}$, then (f, p) is an optimal mechanism.

Note: $K_i^*(t_{-i})$ is the minimum of value of t_i where i begins to be the winner

Contents

Example 1

1. Two buyers : $T_1 = [0, 12]$, $T_2 = [0, 18]$
2. Uniform independent prior
3. $w_1(t_1) = t_1 - \frac{1-G(t)}{g(t)} = t_1 - \frac{1-\frac{t_1}{12}}{\frac{1}{12}} = 2t_1 - 12$
4. $w_2(t_2) = 2t_2 - 18$

t_1	t_2	Action	p_1	p_2
4	8	unsold	0	0
2	12	sold to 2	0	9
6	6	sold to 1	6	0
9	9	sold to 1	6	0
8	15	sold to 2	0	11

Example 2

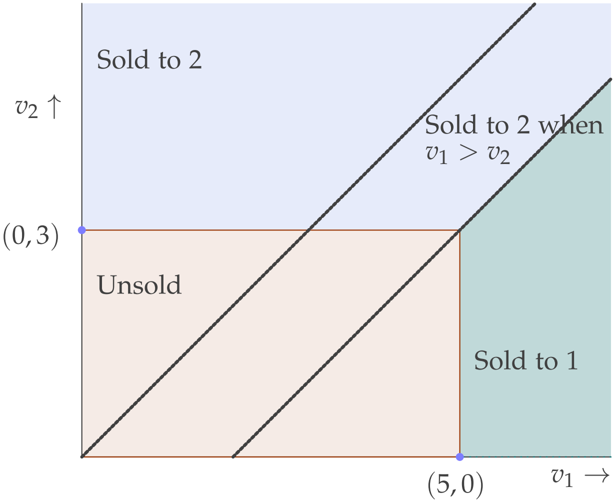
- **Systematic bidders:** the valuations are drawn from the same distribution, $g_i = g$, $T_i = T$, $\forall i \in N$
- **Virtual valuation:** $w_i = w$

$$w(t_i) > w(t_j), \text{ iff } t_i > t_j$$

- The object goes to the highest bidder. Not sold if $w_{-i}(0) > t_i \forall i \in N$.
  $p_i = \max\{w^{-1}(0), \max_{j \neq i} t_j\}$
- Second price auction with a reserve price, and is efficient when the object is sold.

Example 3 : Efficiency and Optimality

- Two buyers : $T_1 = [0, 10]$, $T_2 = [0, 6]$, Uniform independent prior
- $w_1(t_1) = 2t_1 - 10$, $w_2(t_2) = 2t_2 - 6$
- Unsold is inefficient, also in the region of the plane where 1 has higher valuation but item is sold to 2



Contents

Efficiency and Groves Mechanism

- Uniqueness of Groves for efficiency $f^{eff}(t) \in \arg\max_{a \in A} \sum_{i \in N} t_i(a)$

Theorem (Green and Laffont (1979), Holmström (1979))

If the type space is 'sufficiently rich', every efficient and DSIC mechanism is a Groves mechanism.

- **Proof-sketch:** Two alternatives $A = \{a, b\}$ with respective welfare of $\sum_{i \in N} t_i(a)$ and $\sum_{i \in N} t_i(b)$
- $\sum_{i \in N} t_i(a) \geqslant \sum_{i \in N} t_i(b)$ then a is chosen.
  - Fix the valuations of other agents to t_{-i}
  - Fix value of i at alternative b as $t_i(b)$
- $\exists$ some threshold $t_i^*(a)$ s.t.

$$\forall t_i(a) \geqslant t_i^*(a), \;\; a \text{ is the outcome, and } \forall t_i(a) < t_i^*(a), \;\; b \text{ is the outcome}$$

Proof sketch (contd.)

- Using DSIC for $t_i^*(a) + \epsilon = t_i(a), \epsilon > 0$ we have,

  $t_i^*(a) + \epsilon - p_{ia} \geqslant t_i(b) - p_{ib}$ (Note: payment for a player has to be the same for an allocation.)

- Similarly, $t_i'(a) = t_i^*(a) - \delta, \delta > 0$ and

$$t_i(b) - p_{ib} \geqslant t_i^*(a) - \delta - p_{ia}$$

- Since, ϵ, δ are arbitrary , then

$$t_i^*(a) - p_{ia} = t_i(b) - p_{ib} \tag{4}$$

- But $t_i^*(a)$ is the threshold of the efficient outcome, thus,

$$t_i^*(a) + \sum_{j \neq i} t_j(a) = t_i(b) + \sum_{j \neq i} t_j(b) \tag{5}$$

Proof sketch (contd.)

- From Equations (4) and (5)

$$p_{ia} - p_{ib} = \sum_{j \neq i} t_j(b) - \sum_{j \neq i} t_j(a)$$

- Hence, the payment has to be of the form $p_{ix} = h_i(t_{-i}) - \sum_{j \neq i} t_j(x)$

Efficiency and Budget Balance

Theorem (Green and Laffont (1979), Holmström (1979))

If the type space is 'sufficiently rich', every efficient and DSIC mechanism is a Groves mechanism.

Theorem (Green and Laffont (1979))

No Groves mechanism is budget balanced, i.e., $\nexists\, p_i^G$ *s.t.,* $\sum_{i \in N} p_i^G(t) = 0,\ \forall t \in T.$

Corollary

If the valuation space is sufficiently rich, no efficient mechanism can be both DSIC and BB.

Proof sketch of the second theorem

- Consider two alternatives $\{0, 1\}$ s.t.

$$0: \text{ project is not undertaken} \qquad 1: \text{ project is undertaken}$$

  and at outcome 0, every agent has zero value.
- Suppose, $\exists\, h_i, \forall i \in N$ s.t. $\sum_{i \in N} p_i(t) = 0$
- Consider two types w_1^+, w_1^- for player 1, and one type w_2 for player 2 s.t.

$$w_1^+ + w_2 > 0: \text{ project is built} \qquad w_1^- + w_2 < 0: \text{ project is not built}$$

- Budget balance at type profile (w_1^+, w_2) gives $h_1(w_2) - w_2 + h_2(w_1^+) - w_1^+ = 0$ and at type profile (w_1^-, w_2) gives $h_1(w_2) + h_2(w_1^-) = 0$
- Eliminating $h_1(w_2)$, we get $w_2 = h_2(w_1^+) - h_2(w_1^-) - w_1^+$
- The RHS depends only on w_1, hence it is possible to alter w_2 slightly to retain the inequalities, but then the above equality cannot hold.

Weakening DSIC for Budget Balance

- Allocation is still the efficient one $a^*(t) \in \arg\max_{a \in A} \sum_{i \in N} t_i(a)$
- Payment in this setting is also defined via a prior $\delta_i(t_i) = \mathbb{E}_{t_{-i}|t_i} \sum_{j \neq i} t_j(a^*(t))$
- Payment is given by (named after d'Aspremont, Gerard-Varet (1979), Arrow (1979)):

$$p_i^{dAGVA}(t) = \frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) - \delta_i(t_i)$$

- This payment implements the efficient allocation rule in Bayes Nash equilibrium

$$\begin{aligned}
\mathbb{E}_{t_{-i}|t_i}[t_i(a^*(t)) - p_i^{dAGVA}(t)] &= \mathbb{E}_{t_{-i}|t_i} \sum_{j \in N} t_j(a^*(t)) - \mathbb{E}_{t_{-i}|t_i} \left[\frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) \right] \\
&\geqslant \mathbb{E}_{t_{-i}|t_i} \sum_{j \in N} t_j(a^*(t_i', t_{-i})) - \mathbb{E}_{t_{-i}|t_i} \left[\frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) \right] \\
&= \mathbb{E}_{t_{-i}|t_i} \left[t_i(a^*(t_i', t_{-i})) - p_i^{dAGVA}(t_i', t_{-i}) \right]
\end{aligned}$$

Budget Balance?

- To show budget balance, consider

$$\sum_{i \in N} p_i^{dAGVA}(t) = \frac{1}{n-1} \sum_{i \in N} \sum_{j \neq i} \delta_j(t_j) - \sum_{i \in N} \delta_i(t_i)$$
$$= \frac{n-1}{n-1} \sum_{j \in N} \delta_j(t_j) - \sum_{i \in N} \delta_i(t_i) = 0$$

Theorem

The dAGVA mechanism is efficient, BIC, and BB.

- However, **dAGVA** is not IIR

Theorem (Myerson, Satterthwaite (1983))

In a bilateral trade (that involves two types of agents: seller and buyer) no mechanism can be simultaneously BIC, efficient, IIR and budget balanced.

Space of Mechanisms

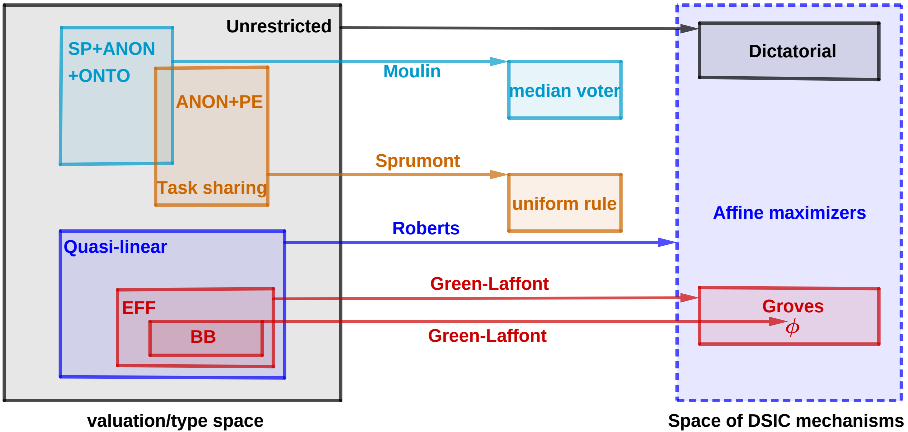


Figure: Space of Mechanisms 1

Space of Mechanisms

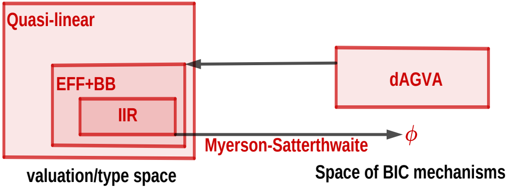


Figure: Space of Mechanisms 2

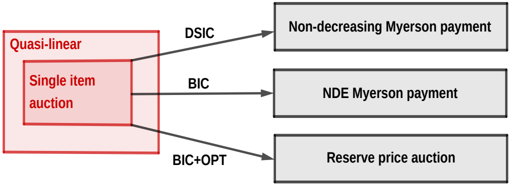


Figure: Space of Mechanisms 3

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay