



भारतीय प्रौद्योगिकी संस्थान मुंबई
Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 2 – Dominance, Nash Equilibrium, Security, and Matrix Games

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Week 2 Roadmap



Lecture	Chunks (20–25 min each)	Quizzes
Lecture 3	1. Domination & Dominant Strategies 2. Auctions, Non-existence, Nash Equilibrium 3. Best Response & the SDSE–WDSE–PSNE Chain 4. Elimination Order & a Peek at Risk	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4
Lecture 4	1. Maxmin Strategies & Robust ML 2. Maxmin vs. Dominance and PSNE 3. Elimination's Effect on Security & Equilibrium 4. Matrix Games, Saddle Points, the Cricket Duel	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4



► Lecture 3: Dominance and Nash Equilibrium

► Lecture 4: Security, Elimination, and Matrix Games



Chunk 1

Domination and Dominant Strategies

($\approx$ 20–25 minutes)

Recap: Strategic Form



- $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$, and we write $u_i(s_i, s_{-i})$

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- We have games, but so far no theory of what a rational player *does*
- Today: two solution concepts – **domination** and **Nash equilibrium**

A Pricing Duel: Two Streaming Platforms



Two platforms simultaneously set a monthly price: Low, Mid, or High (payoffs: Rs. lakh profit/month).

Why this is not a race to the bottom

The two platforms carry *different* content libraries, so a subscriber does not simply flee to whichever is one rupee cheaper. Demand shifts **gradually** with price, not all-or-nothing – so profit (margin $\times$ share) can peak at a middling price rather than falling monotonically as price rises.



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		RegionalPlay		
		Low	Mid	High
StreamCo	Low	0,0	5,1	6,0
	Mid	1,5	6,6	8,2
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Question

Compare row High to row Mid, for every column. What do you notice?

Dominated Strategy



Definition (Strictly Dominated Strategy)

$s'_i \in S_i$ is **strictly dominated** if $\exists s_i \in S_i$ such that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ for **every** $s_{-i} \in S_{-i}$.

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Definition (Weakly Dominated Strategy)

s'_i is **weakly dominated** by s_i if $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \forall s_{-i}$, with strict inequality for **some** $\tilde{s}_{-i}$.

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Back to the pricing duel

High gives StreamCo (0, 2, 4) against RegionalPlay's three prices – strictly worse than Mid's (1, 6, 8) in every column. High is strictly dominated by Mid, for both platforms by symmetry.

Dominated Strategy



		Player 2		
		L	C	R
Player 1	U	1,0	1,3	3,2
	D	-1,6	0,5	3,3

Dominated Strategy



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		L	C	R
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Strictly / Weakly dominated strategy?

Dominated Strategy



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Strictly / Weakly dominated strategy?

R is strictly dominated (by C) while D is weakly dominated (by U)

Dominant Strategy and Dominant Strategy Equilibrium



Definition (Dominant Strategy)

s_i is (strictly/weakly) **dominant** for i if it (strictly/weakly) dominates every other strategy in S_i .

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Definition (SDSE / WDSE)

$(s_1^*, \dots, s_n^*)$ is a strict (weak) **dominant strategy equilibrium** if every s_i^* is a strict (weak) dominant strategy.

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Question

Is there a dominant strategy or dominant strategy equilibrium in the pricing-duel game?

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Answer

Yes – Mid strictly dominates both High and Low for both players. (Mid, Mid) is the SDSE.

Example: Dominant Strategy Equilibrium



Example of **dominant strategy equilibrium**

		Player 2	
		D	E
Player 1	A	5, 5	0, 5
	B	5, 0	1, 1
	C	4, 0	1, 1

Example: Dominant Strategy Equilibrium



Example of **dominant strategy equilibrium**

		Player 2	
		D	E
Player 1	A	5,5	0,5
	B	5,0	1,1
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Question

What kind of equilibrium in this game?



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Chunk 2

A Weakly Dominant Strategy: Bidding for GPU Credits

($\approx$ 20–25 minutes)

One GPU Block, Two Labmates



Riya and Karan each want an **indivisible** block of GPU-hours this week; only one can have it. Private values v_1, v_2 . Each submits a sealed bid $b_i \in [0, M]$.

Second-price rule

Higher bid wins (ties to Riya); **winner pays the other player's bid.**

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$$u_1(b_1, b_2) = \begin{cases} v_1 - b_2 & b_1 \geq b_2 \\ 0 & \text{else} \end{cases}$$

$$u_2(b_1, b_2) = \begin{cases} v_2 - b_1 & b_1 < b_2 \\ 0 & \text{else} \end{cases}$$

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Question

Should Riya ever bid above her true value v_1 ? Below it?

Truthful Bidding Weakly Dominates



Theorem (Vickrey 1961)

Bidding $b_i = v_i$ weakly dominates every other bid.



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- WLOG $b_1 < v_1$: if $\max b_{-1} \leq b_1$, both reports give same utility; same conclusion if $\max b_{-1} \geq v_1$. We assume that Player 1 wins if there is a tie.



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- So v_1 is never worse, sometimes strictly better $\Rightarrow$ weakly dominant.

Why this matters

This is the seed of the entire second half of this course (mechanism design, VCG). Spot GPU markets and rendering marketplaces that use second-price-style rules inherit exactly this property.

Dominant Strategies Are Not Guaranteed to Exist



Two friends must independently pick tonight's game – they only have fun together.

		Friend 2	
		Chess	Football
Friend 1	Chess	2, 1	0, 0
	Football	0, 0	1, 2

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Does either friend have a dominant strategy here?

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Question

Does either friend have a dominant strategy here?

Answer

No. What's best for Friend 1 depends entirely on Friend 2's choice, and vice versa.



In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
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Chunk 3

Nash Equilibrium and Best Response

($\approx$ 20–25 minutes)

Nash Equilibrium (Nash 1951)



No player gains by a unilateral deviation



Nash Equilibrium (Nash 1951)

No player gains by a unilateral deviation

Definition (Nash Equilibrium)

A strategy profile (s_i^*, s_{-i}^*) is a pure strategy Nash equilibrium (PSNE) if $\forall i \in N$ and $\forall s_i \in S_i$

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		Friend 2	
		C	F
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Chess or Football Game

Best Response and PSNE



Definition (Best Response)

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Best Response and PSNE



Definition (Best Response)

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(s_i^*, s_{-i}^*) is a PSNE if $s_i^* \in B_i(s_{-i}^*) \forall i \in N$: **no player gains from a unilateral deviation.**

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In the sport-selection game, are (Chess, Chess) and (Football, Football) both PSNE?



Best Response and PSNE

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Question

In the sport-selection game, are (Chess, Chess) and (Football, Football) both PSNE?

Answer

Yes – neither friend wants to switch alone from either. Two equilibria, unequal payoffs. Which one do real people converge to, without talking? (Schelling's *focal points* – outside our scope, but worth knowing the name.)

The Chain: SDSE $\implies$ WDSE $\implies$ PSNE



Theorem

Every SDSE is a WDSE, and every WDSE is a PSNE.

The Chain: $\text{SDSE} \implies \text{WDSE} \implies \text{PSNE}$



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Question

Does the converse hold – is every PSNE a dominant strategy equilibrium?



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Question

Does the converse hold – is every PSNE a dominant strategy equilibrium?

Answer

No – the sports-selection game has two PSNE but no dominant strategy of either strength.



Chunk 4

Iterated Elimination: A Word of Caution

(≈ 15 – 20 minutes)

How to find equilibrium?



- Rational players do not play **dominated strategies**

How to find equilibrium?



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- To obtain rational outcomes eliminate dominated strategies

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- For strictly dominated strategies the order of elimination does not matter (**homework**)

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		L	C	R
Player 1	T	1,2	2,3	0,3
	M	2,2	2,1	3,2
	B	2,1	0,0	1,0

How to find equilibrium?



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- Order T, R, B, C $\rightarrow$ (M,L) : (2,2)

How to find equilibrium?



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- Order T, R, B, C $\rightarrow$ (M, L) : (2,2)
- Order B, L, C, T $\rightarrow$ (M, R) : (3,2)

When Trusting the Opponent Is Risky



Nash equilibrium assumes every player will actually play their part of the equilibrium.

When Trusting the Opponent Is Risky



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Question

What should a player do if they cannot be sure the other player will play their Nash-equilibrium action?

When Trusting the Opponent Is Risky



Nash equilibrium assumes every player will actually play their part of the equilibrium.

Question

What should a player do if they cannot be sure the other player will play their Nash-equilibrium action?

A different notion of rationality

Forget equilibrium for a moment. What can a player **guarantee** for themselves, no matter what the other player does? That is the **maxmin value** – next lecture.



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Lecture 3: What We Built



- Strict/weak domination, dominant strategies, SDSE/WDSE

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- SDSE $\Rightarrow$ WDSE $\Rightarrow$ PSNE, converse false
- Elimination order can change which outcome survives
- **Next:** making “worst-case guarantee” precise – maxmin strategies



End of Lecture 3

See you next class.



- ▶ Lecture 3: Dominance and Nash Equilibrium
- ▶ Lecture 4: Security, Elimination, and Matrix Games



Chunk 1

Maxmin Strategies and Risk Aversion

($\approx$ 20–25 minutes)

Risk Aversion of Players



		Player 2	
		L	R
Player 1	T	2, 1	1, -20
	M	3, 0	-10, 1
	B	-100, 2	3, 3

Risk Aversion of Players



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Question

What if the other player does not pick an equilibrium action (Nash)?

Risk Aversion of Players



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Question

What if the other player does not pick an equilibrium action (Nash)?

Picking T is less risky for Player 1.

Max-min Strategy



Definition

The worst case optimal choice is **max-min strategy**

$$u_i(s_i, s_{-i})$$



Max-min Strategy

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$$\min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$$

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Note: $s_{-i}^{\min}(s_i) \in \arg \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})$ is indeed a function of s_i ; as s_i changes the minimizer keeps on changing



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Max-min value (utility at the max-min strategy) of player i is given by

$$\begin{aligned} \underline{v}_i &= \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \\ u_i(s_i^{\max \min}, t_{-i}) &\geq \underline{v}_i, \quad \forall t_{-i} \in S_{-i} \end{aligned}$$

Maxmin Value and Maxmin Strategy



		Player 2	
		L	R
Player 1	T	2, 1	1, -20
	M	3, 0	-10, 1
	B	-100, 2	3, 3

Maxmin Value and Maxmin Strategy



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Maxmin strategy and value for Player 1

- Worst case per row: T gives 1, M gives -10, B gives -100.

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Application: Robust optimization

Adversarial training of ML models (Madry et al., 2018) optimizes exactly this worst-case objective, not an expected one.



Chunk 2

Maxmin vs. Dominance, Maxmin vs. PSNE

($\approx$ 30–35 minutes)

Dominant Strategies Are Maxmin Strategies



Theorem

If s_i^ is dominant for i , it is also a maxmin strategy for i .*

Dominant Strategies Are Maxmin Strategies



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Proof.

Dominance gives $u_i(s_i^*, s_{-i}) \geq u_i(s'_i, s_{-i}) \forall s_{-i}, \forall s'_i$.



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Let $s_{-i}^{\min}(s'_i)$ be the worst reply to s'_i . Plug it in:





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So s_i^* 's worst case is at least every other strategy's worst case: $s_i^* \in \arg \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$. $\square$



PSNE Payoffs Never Fall Below the Maxmin Value

Theorem (MSZ Thm. 4.29)

Every PSNE s^ satisfies $u_i(s^*) \geq \underline{v}_i, \forall i \in N$.*



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$$u_i(s_i, s_{-i}^*) \geq \min_{s_{-i}} u_i(s_i, s_{-i}), \quad \forall s_i \quad (\text{def. of min})$$



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Question

Does this mean equilibrium is always at least as good as playing safe? What is it silent about?



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Question

Does this mean equilibrium is always at least as good as playing safe? What is it silent about?

Answer

It only says the equilibrium payoff cannot be *worse* than the guarantee – it says nothing about whether the opponent will actually cooperate with that equilibrium (recall the adaptive attacker).



In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.



Chunk 3

Elimination: Effects on Security and on Equilibrium

($\approx$ 20–25 minutes)

Iterated elimination of dominated strategies



Iterated elimination of dominated strategies



The story so far

- Dominance cannot explain all outcomes; games may not have dominant strategies

Iterated elimination of dominated strategies



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- PSNE: unilateral deviation; gives **stability**

Iterated elimination of dominated strategies



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- Maxmin: rationality for risk-aversion; gives **security**

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- Dominance cannot explain all outcomes; games may not have dominant strategies
- PSNE: unilateral deviation; gives **stability**
- Maxmin: rationality for risk-aversion; gives **security**

Question

What happens to stability and security when some strategies are eliminated?

Iterated elimination of dominated strategies (contd.)



		Player 2		
		L	C	R
Player 1	T	1,2	2,3	0,3
	M	2,2	2,1	3,2
	B	2,1	0,0	1,0

Iterated elimination of dominated strategies (contd.)



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- Order T, R, B, C $\rightarrow (M, L) : (2, 2)$

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- Order B, L, C, T $\rightarrow (M, R) : (3, 2)$

Iterated elimination of dominated strategies (contd.)



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Iterated elimination of dominated strategies (contd.)



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- Order T, R, B, C $\rightarrow (M, L) : (2, 2)$
- Order B, L, C, T $\rightarrow (M, R) : (3, 2)$

Question

Does it change the maxmin value?

Iterated elimination of dominated strategies (contd.)



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Consider in the above example: elimination of dominated strategy B for player 1

Iterated elimination of dominated strategies (contd.)



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Maxmin values	Player 1	Player 2
Before		
After		

Iterated elimination of dominated strategies (contd.)



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Maxmin values	Player 1	Player 2
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Consider in the above example: elimination of dominated strategy B for player 1

Maxmin values	Player 1	Player 2
Before	2	0
After	2	2

Maxmin value is not affected for the player whose **dominated strategy** is removed

A Result for Iterated Elimination



Theorem

Consider an NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, and let $s'_j \in S_j$ be a dominated strategy. Let G' be the residual game after removing s'_j . Then, the maxmin value of j in G' is equal to her maxmin value in G .



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Intuition



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- Maxmin is the 'max' of all 'min's



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Intuition

- Maxmin is the 'max' of all 'min's
- Elimination affects one 'min'
- But that does not affect the 'max' since the strategy was dominated



Maxmin value of player j in G

$$\underline{v}_j = \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$



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Suppose t_j dominates s'_j in G , $t_j \in S_j \setminus \{s'_j\}$, then,

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$\underline{v}_j$ [maxmin value of j in G]

$$= \max_{s_j \in S_j} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j})$$

$$= \max \left\{ \max_{s_j \in S_j \setminus \{s'_j\}} \min_{s_{-j} \in S_{-j}} u_j(s_j, s_{-j}), \min_{s_{-j} \in S_{-j}} u_j(s'_j, s_{-j}) \right\}$$



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Question

What happens to existing equilibrium after iterated elimination?



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Theorem

Consider G and $\hat{G}$ are games before and after elimination of a strategy (not necessarily dominated). If s^ is a PSNE in G and survives in $\hat{G}$, then s^* is a PSNE in $\hat{G}$ too.*



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What happens to existing equilibrium after iterated elimination?

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Consider G and $\hat{G}$ are games before and after elimination of a strategy (not necessarily dominated). If s^ is a PSNE in G and survives in $\hat{G}$, then s^* is a PSNE in $\hat{G}$ too.*

Intuition

PSNE was a maxima of utility of i among the strategies of i . Removing other strategies does not affect maximality.

Proof: exercise.

Can new equilibrium be generated?



Theorem

Consider NFG G . Let $\hat{s}_j$ be a weakly dominated strategy of j . If $\hat{G}$ is obtained from G eliminating $\hat{s}_j$, then every PSNE of $\hat{G}$ is a PSNE of G .

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No new PSNE if the eliminated strategy is dominated

But old PSNEs could be killed: saw in the previous example



In the game $\hat{G}$, modified strategy sets are $\hat{S}_j = S_j \setminus \{\hat{s}_j\}$, $\hat{S}_i = S_i, \forall i \neq j$



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Need to show: no profitable deviation for any player in G . For $i \neq j$, this is immediate since no strategy is removed.



In the game $\hat{G}$, modified strategy sets are $\hat{S}_j = S_j \setminus \{\hat{s}_j\}$, $\hat{S}_i = S_i, \forall i \neq j$

Need to show: if $s^* = (s_j^*, s_{-j}^*)$ is a PSNE in $\hat{G}$, it is a PSNE in G .

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Since s^* is a PSNE in $\hat{G}$ and $t_j \in \hat{S}_j$

$$u_j(s_j^*, s_{-j}^*) \geq u_j(t_j, s_{-j}^*) \geq u_j(\hat{s}_j, s_{-j}^*)$$



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- Elimination of weakly dominated strategy may reduce the set of PSNEs, but never adds new
- The maxmin values of the player whose strictly or weakly dominated strategies are removed remain unaffected



Chunk 4

Matrix Games

($\approx 10\text{--}15$ minutes)

Matrix games: *two player zero-sum* games



A special class with certain nice **security** and **stability** properties



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Definition (Two player zero-sum games)

A NFG $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$ with $N = \{1, 2\}$ and $u_1 + u_2 \equiv 0$



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Why called **matrix** game?



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Question

Why called **matrix** game?

Answer

Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

Example: Penalty shoot game



		Player 2	
		L	R
Player 1	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

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$\Rightarrow$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

		L	R	<i>maxmin</i>
<i>Player 1</i>	<i>L</i>	-1	1	-1
	<i>R</i>	1	-1	-1
	<i>minmax</i>	1	1	



Example: Penalty shoot game

Player 1

	Player 2	
	L	R
L	-1, 1	1, -1
R	1, -1	-1, 1

 $\Rightarrow$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

Player 2's **maxmin** value is the negative of the **minmax** value of this matrix

		L	R	maxmin
Player 1	L	-1	1	-1
	R	1	-1	-1
	minmax	1	1	



In-Class Quiz 2

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Lecture 4 and Week 2: What We Built



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- Penalty shoot game: read the maxmin and minmax values straight off the matrix – check for yourself whether they agree
- **Next week:** mixed strategies, MSNE characterization, and Nash's existence theorem



End of Lecture 4

See you next class.



भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay