भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 3 – Matrix Games and Mixed Strategy Nash Equilibrium

Swaprava Nath

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ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Week 3 Roadmap

Lecture	Chunks (20–25 min each)	Quizzes
Lecture 5	1. Matrix Games & Saddle Points 2. Saddle Point = PSNE 3. When Does a PSNE Exist? 4. Mixed Strategies	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4
Lecture 6	1. MSNE 2. Support & Characterization 3. Proof of Characterization 4. Algorithm, Dominance, Existence	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4

Contents

- Lecture 5: Matrix Games, Saddle Points, and Security Values
- Lecture 6: Mixed Strategy Nash Equilibrium

Chunk 1

Matrix Games, Best Replies, and Saddle Points

($\approx$ 20–25 minutes)

Matrix Games: Two-Player Zero-Sum Games

A special class with certain nice **security** and **stability** properties

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Where these show up

One side's gain is exactly the other's loss: a red-team attacker racing a blue-team detector, or a penalty shooter against a goalkeeper.

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Why called **matrix** game?

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Where these show up

One side's gain is exactly the other's loss: a red-team attacker racing a blue-team detector, or a penalty shooter against a goalkeeper.

Question

Why called **matrix** game?

Answer

Possible to represent the game with only one matrix considering the utilities of player 1; player 2's utilities are negative of this matrix

The Ghost and the Sentinel

Ghost probes ingress channel L or R; **Sentinel** watches one channel. A match means Sentinel catches Ghost; a mismatch means Ghost slips through.

		Sentinel	
		L	R
Ghost	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

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$$\Longrightarrow \qquad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} =: u$$

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*Sentinel's **maxmin** value is the negative of the **minmax** value of this matrix*

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Scaling Up: Three Attack Vectors

Now suppose the exercise gets bigger. Ghost has three moves: T (Trojan payload), M (Man-in-the-middle), B (Botnet flood). Sentinel has three defenses: L (Log monitoring), C (Content filtering), R (Rate limiting). Payoffs are Ghost's threat-bounty score.

Ghost \ Sentinel	L	C	R
T	$3, -3$	$-5, 5$	$-2, 2$
M	$1, -1$	$4, -4$	$1, -1$
B	$6, -6$	$-3, 3$	$-5, 5$

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Two Duels Side by Side

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What are the PSNEs for the above games?

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Answer

Left (two-channel exercise): no PSNE; **Right (three-vector exercise):** (M,R)

Saddle Point

Saddle point of a matrix

The value is simultaneously the maximum in its column and minimum in its row i.e., maximum for player 1 and minimum for player 2

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Question

What are the saddle points for the previous two duels?

Saddle Point, Continued

Ghost	L	R
L	-1	1
R	1	-1

Ghost	L	C	R
T	3	-5	-2
M	1	4	1
B	6	-3	-5

Saddle Point, Continued

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Answer

For the two-channel exercise: no saddle point, for the three-vector exercise: (M,R)

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Answer

For the two-channel exercise: no saddle point, for the three-vector exercise: (M,R)

Theorem

In a matrix game with utility matrix u, (s_1^*, s_2^*) is a saddle point iff it is a PSNE.

Chunk 2

Saddle Point = PSNE, and Security Values

($\approx$ 20–25 minutes)

Saddle Point and PSNE

Proof.

Consider (s_1^*, s_2^*) to be a saddle point. By definition of saddle point, this happens iff $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$ and $u(s_1^*, s_2^*) \leqslant u(s_1^*, s_2), \forall s_2 \in S_2$. Since, $u \equiv u_1 \equiv -u_2$, the above is equivalent to $u_1(s_1^*, s_2^*) \geqslant u_1(s_1, s_2^*), \forall s_1 \in S_1$ and $u_2(s_1^*, s_2^*) \geqslant u_2(s_1^*, s_2), \forall s_2 \in S_2 \Leftrightarrow (s_1^*, s_2^*)$ is a PSNE. $\square$

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Consider **maxmin** and **minmax** values

$$\underline{v} = \max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2) \qquad \textbf{maxmin}$$

$$\overline{v} = \min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2) \qquad \textbf{minmax}$$

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Question

How are the maxmin and minmax values related?

Relationship of the Security Values

Lemma

For matrix games $\overline{v} \geqslant \underline{v}$.

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$$u(s_1, s_2) \geqslant \min_{t_2 \in S_2} u(s_1, t_2), \ \forall s_1, s_2, \qquad \text{definition of min}$$

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□

Reading it in the exercise

Sentinel's minimum highest value (minmax) never falls below Ghost's maximum lowest utility (maxmin).

Revisiting the Two Duels: No PSNE

Ghost \	L	R	maxmin
L	-1	1	-1
R	1	-1	-1
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PSNE does not exist – Ghost and Sentinel must keep guessing

Revisiting the Two Duels: PSNE Exists

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T	3	−5	−2	−5
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PSNE exists – with three vectors on the table, Sentinel's rate-limiting defense (R) settles the exercise at (M,R)

In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Chunk 3

When Does a Matrix Game Have a PSNE?

($\approx$ 20–25 minutes)

PSNE Theorem

Define the following strategies

$$s_1^* \in \arg\max_{s_1 \in S_1} \min_{s_2 \in S_2} u(s_1, s_2), \qquad \textbf{maxmin strategy of player 1}$$

$$s_2^* \in \arg\min_{s_2 \in S_2} \max_{s_1 \in S_1} u(s_1, s_2), \qquad \textbf{minmax strategy of player 2}$$

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Theorem

A game has a PSNE (equivalently, a saddle point) if and only if $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$, where s_1^* and s_2^* are maxmin and minmax strategies for players 1 and 2 respectively.

Corollary: (s_1^*, s_2^*) is a PSNE

Proof of the PSNE Theorem

Proof

($\implies$) let (s_1^*, s_2^*) is a PSNE $\implies$ $\overline{v} = \underline{v} = u(s_1^*, s_2^*)$ and s_1^* and s_2^* are maxmin and minmax

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Since (s_1^*, s_2^*) is a PSNE, $u(s_1^*, s_2^*) \geqslant u(s_1, s_2^*), \forall s_1 \in S_1$.

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\implies u(s_1^*, s_2^*) &\geqslant \max_{t_1 \in S_1} u(t_1, s_2^*) \\
&\geqslant \min_{t_2 \in S_2} \max_{t_1 \in S_1} u(t_1, t_2), \text{ since } s_2^* \text{ is a specific strategy} \\
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$$\begin{aligned}
u(s_1^*, s_2^*) \geqslant \overline{v} \geqslant \underline{v} &\geqslant u(s_1^*, s_2^*) \\
\implies u(s_1^*, s_2^*) &= \overline{v} = \underline{v}
\end{aligned}$$

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\implies u(s_1^*, s_2^*) = \overline{v} &= \underline{v}
\end{aligned}$$

Also implies that the maxmin for player 1 and minmax for player 2 are s_1^* and s_2^* respectively.

Proof of the PSNE Theorem (contd.)

Proof (contd.)

($\Longleftarrow$) i.e. $\overline{v} = \underline{v} = u(s_1^*, s_2^*) = v$ (say) and s_1^* and s_2^* are maxmin and minmax $\implies$ (s_1^*, s_2^*) is a PSNE

Proof of the PSNE Theorem (contd.)

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\begin{aligned}
u(s_1^*, s_2) &\geqslant \min_{t_2 \in S_2} u(s_1^*, t_2), \text{ by definition of min} \\
&= \max_{t_1 \in S_1} \min_{t_2 \in S_2} u(t_1, t_2), \text{ since } s_1^* \text{ is the maxmin strategy for player 1} \\
&= v \text{ (given)}
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Similarly, we can show $u(s_1, s_2^*) \leqslant v, \ \forall s_1 \in S_1$

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($\Longleftarrow$) i.e. $\overline{v} = \underline{v} = u(s_1^*, s_2^*) = v$ (say) and s_1^* and s_2^* are maxmin and minmax $\implies$ (s_1^*, s_2^*) is a PSNE

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Similarly, we can show $u(s_1, s_2^*) \leqslant v, \ \forall s_1 \in S_1$
But $v = u(s_1^*, s_2^*)$. Substitute and get that (s_1^*, s_2^*) is a PSNE.

Chunk 4

Mixed Strategies

($\approx$ 20–25 minutes)

Mixed Strategies

Mixed strategy: *probability distribution* over the set of strategies of that player.

		Sentinel	
		L	R
Ghost	L	$-1, 1$	$1, -1$
	R	$1, -1$	$-1, 1$

Mixed Strategies

Mixed strategy: *probability distribution* over the set of strategies of that player.

		Sentinel	
		L	R
Ghost	$\frac{2}{3}$ L	$-1, 1$	$1, -1$
	$\frac{1}{3}$ R	$1, -1$	$-1, 1$

Mixed Strategies

Mixed strategy: *probability distribution* over the set of strategies of that player.

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- Mixed strategy set of Ghost: $\Delta S_1 = \Delta\{L, R\}$, $(\frac{2}{3}, \frac{1}{3}) \in \Delta S_1$

Mixed Strategies, Notation

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- We are *overloading* u_i to denote the utility at *pure* and *mixed* strategies
- Utility at a mixed strategy is the **expectation** of the utilities at pure strategies; all the rules of expectation hold, e.g., linearity, conditional expectation, etc.

Computing Expected Utility

Ghost probes L with probability 2/3 and R with probability 1/3; Sentinel watches L with probability 4/5 and R with probability 1/5.

		Sentinel	
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In-Class Quiz 2

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- Matrix (two-player zero-sum) games; saddle point $\Leftrightarrow$ PSNE

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- **Next:** Mixed Strategy Nash Equilibrium and how to actually find one

End of Lecture 5

See you next class.

Contents

- Lecture 5: Matrix Games, Saddle Points, and Security Values
- Lecture 6: Mixed Strategy Nash Equilibrium

Chunk 1

Mixed Strategy Nash Equilibrium

($\approx$ 20–25 minutes)

Mixed Strategy Nash Equilibrium

Definition (Mixed Strategy Nash Equilibrium)

A **mixed strategy Nash equilibrium (MSNE)** is a mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$, s.t.

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geqslant u_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i \text{ and } \forall i \in N.$$

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Relation between **PSNE** and **MSNE**?

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Question

Relation between **PSNE** and **MSNE**?

Answer

PSNE $\implies$ MSNE

An Alternative Definition

Theorem

A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$*, is an* ***MSNE*** *if and only if* $\forall s_i \in S_i$ *and* $\forall i \in N$

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Proof.

$(\Rightarrow)$: The pure strategy s_i is a special case of the mixed strategy, the mixed strategy with s_i having probability 1. Inequality holds by definition of MSNE

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($\Leftarrow$) Pick an arbitrary mixed strategy σ_i of player i

$$u_i(\sigma_i, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, \sigma_{-i}^*)$$

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Checking a Candidate: Is This an MSNE?

Question

Does mixing $(2/3, 1/3)$ against $(4/5, 1/5)$ give an **MSNE**?

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Checking a Candidate: What Balances the Duel?

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Is the balanced mixed strategy profile an **MSNE**?

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Checking a Candidate: What Balances the Duel?

Question

Is the balanced mixed strategy profile an **MSNE**?

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- Sentinel's expected utility will increase if some probability is transferred from R to L
- $\Rightarrow$ the $((2/3, 1/3), (4/5, 1/5))$ profile is **not** an MSNE
- Some balance in the utilities is needed
- **Does there exist any improving mixed strategy for the balanced $(1/2, 1/2)$ profile?**

Chunk 2

Support and the Characterization Theorem

($\approx$ 20–25 minutes)

Support of a Mixed Strategy

Definition (Support of mixed strategy/probability distribution)

For mixed strategy σ_i, the subset of strategy set of i on which σ_i has a positive mass is called the **support** of σ_i and is denoted by $\delta(\sigma_i)$. Formally, $\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$.

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Using the definition of support, here is a characterization of MSNE

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A mixed strategy profile $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE iff[a] $\forall i \in N$

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Solving the Ghost–Sentinel Duel: Ruling Out Cases

Back to the two-channel exercise

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Case 1: support profile $(\{L\}, \{L\})$: for Ghost , $s_1' = R$ – violates condition 2

Case 2: support profile $(\{L, R\}, \{L\})$ – symmetric for the other case

For Ghost, the expected utility has to be the same for L and R - not possible – violates condition 1

Solving the Ghost–Sentinel Duel: The Balanced Case

Case 3: support profile $(\{L, R\}, \{L, R\})$: condition 2 is vacuously satisfied

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For condition 1, let Ghost chooses L w.p. p and Sentinel chooses L w.p. q

For Ghost:

$$u_1(L, (q, 1-q)) = u_1(R, (q, 1-q)) \;\Rightarrow (-1)q + 1 \cdot (1-q) = 1 \cdot q + (-1)(1-q) \;\Rightarrow q = \frac{1}{2}$$

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MSNE =

$$\left(\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$$

Exercise 1

Back to Riya and Karan's Chess-or-Football game night from Week 2 – now find the MSNE.

		Friend 2	
		F	C
Friend 1	F	2, 1	0, 0
	C	0, 0	1, 2

Exercise 2

Now suppose Friend 2 gets a third option, D (a documentary neither prefers, but at least it is not a total mismatch):

		Friend 2		
		F	C	D
Friend 1	F	2, 1	0, 0	1, 1
	C	0, 0	1, 2	2, 0

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Chunk 3

Proof of the Characterization Theorem

($\approx$ 25–30 minutes)

MSNE Characterization Theorem

Theorem

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2. $u_i(s_i, \sigma^*_{-i}) \geqslant u_i(s'_i, \sigma^*_{-i}), \forall s_i \subseteq \delta(\sigma^*_i), s'_i \notin \delta(\sigma^*_i)$.

Observations:

- $\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i})$
  maximizing w.r.t. a distribution $\Leftrightarrow$ whole probability mass at max

MSNE Characterization Theorem

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- If $(\sigma^*_i, \sigma^*_{-i})$ is an MSNE, then

$$\max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma^*_{-i}) = \max_{s_i \in S_i} u_i(s_i, \sigma^*_{-i}) = \max_{s_i \in \delta(\sigma^*_i)} u_i(s_i, \sigma^*_{-i})$$

  the maximizer must lie in $\delta(\sigma^*_i)$ – if not, then put all probability mass on that $s'_i \notin \delta(\sigma^*_i)$ that has the maximum value of the utility – $(\sigma^*_i, \sigma^*_{-i})$ is not a MSNE

Proof of MSNE Characterization Theorem

($\Rightarrow$) Given $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \max_{\sigma_i \in \Delta S_i} u_i(\sigma_i, \sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) = \max_{s_i \in \delta(\sigma_i^*)} u_i(s_i, \sigma_{-i}^*) \tag{1}$$

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By definition of expected utility

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in S_i} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*) \tag{2}$$

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Equations (1) and (2) are equal, i.e., max is equal to positive weighted average – can happen only when all values are same: **proves condition 1**

Proof (contd.)

For **condition 2**: Suppose for contradiction, there exists $s_i \in \delta(\sigma_i^*)$ and $s_i' \notin \delta(\sigma_i^*)$ s.t. $u_i(s_i, \sigma_{-i}^*) < u_i(s_i', \sigma_{-i}^*)$

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This completes the proof of the necessary direction.

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($\Leftarrow$) Given the 2 conditions of the theorem, need to show that $(\sigma_i^*, \sigma_{-i}^*)$ is an MSNE

$$\begin{aligned} &\text{Let} \quad u_i(s_i, \sigma_{-i}^*) = m_i(\sigma_{-i}^*), \forall s_i \in \delta(\sigma_i^*) && \textbf{condition 1} \\ &\text{Note} \quad m_i(\sigma_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, \sigma_{-i}^*) && \textbf{condition 2} \end{aligned}$$

Proof (contd.)

$$u_i(\sigma_i^*, \sigma_{-i}^*) = \sum_{s_i \in \delta(\sigma_i^*)} \sigma_i^*(s_i) u_i(s_i, \sigma_{-i}^*), \qquad \textbf{by definition of } \delta(\sigma_i^*)$$

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$$= m_i(\sigma_{-i}^*) \qquad \textbf{previous conclusion}$$

Proof (contd.)

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\end{aligned}
$$

This proves the sufficient direction. The result yields an algorithmic way to find MSNE

Chunk 4

Algorithm, Dominance, and Existence of MSNE

($\approx$ 25–30 minutes)

MSNE Characterization Theorem to Algorithm

Consider a NFG $G = \langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$

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The total number of supports of $S_1 \times S_2 \times S_3 \cdots \times S_n$ is
$K = (2^{|S_1|} - 1) \times (2^{|S_2|} - 1) \times \cdots \times (2^{|S_n|} - 1)$

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For every support profile $X_1 \times X_2 \times \cdots X_n$, where $X_i \subseteq S_i$, solve the following feasibility program

Program

$$w_i = \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in X_i, \forall i \in N$$

$$w_i \geqslant \sum_{s_{-i} \in S_{-i}} (\prod_{j \neq i} \sigma_j(s_j)) \cdot u_i(s_i, s_{-i}), \forall s_i \in S_i \setminus X_i, \forall i \in N$$

$$\sigma_j(s_j) \geqslant 0, \forall s_j \in S_j, \forall j \in N, \qquad \sum_{s_j \in X_j} \sigma_j(s_j) = 1, \forall j \in N$$

Remarks on the Algorithm

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- For general game, there is no poly-time algorithm
- Problem of finding an MSNE is PPAD-complete [Polynomial Parity Argument on Directed graphs] [1]

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MSNE and Dominance

A gig-platform driver picks a shift type: Standard rides (T), Pooled rides (M), or Premium rides (B). The platform's dispatch algorithm runs in mode L or R. The previous algorithm can be applied to a smaller set of strategies by removing the dominated strategies.

Is there a dominated strategy in this game? Domination can be via mixed strategies too

		Platform	
		L	R
Driver	T	4, 1	2, 5
	M	1, 3	6, 2
	B	2, 2	3, 3

MSNE and Dominance

Theorem

If a pure strategy s_i is strictly dominated by a mixed strategy $\sigma_i \in \Delta S_i$, then in every MSNE of the game, s_i is chosen with probability zero.

So, We can remove such strategies without loss of equilibrium

Back to the driver

Neither T nor M strictly dominates B in pure strategies, but a 50-50 mix of T and M beats B against both L and R – B can be dropped before running the algorithm.

Existence of MSNE

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Why this matters

No matter how adversarial the setup – Ghost against Sentinel, a driver against a dispatch algorithm, or two agents locked in an arms race – some equilibrium is guaranteed to exist once randomization is allowed. This is the theoretical bedrock beneath training procedures like GANs and adversarial ML, which search for exactly such an equilibrium.

Proof requires a few tools and a result from real analysis. Proof is given in the lecture notes.

Existence of MSNE: Background

Some background for understanding the proof.

- A set $S \subseteq \mathbb{R}^n$ is **convex** if $\forall x, y \in S$ and $\forall \lambda \in [0, 1]$, $\lambda x + (1 - \lambda) y \in S$.

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- A set $S \subseteq \mathbb{R}^n$ is **bounded** if $\exists x_0 \in \mathbb{R}^n$ and $R \in (0,\infty)$ s.t. $\forall x \in S, ||x - x_0||_2 < R$.

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A result from real analysis (proof omitted):

Brouwer's fixed point theorem

If $S \subseteq \mathbb{R}^n$ is **convex** and **compact** and $T : S \to S$, is **continuous** then T has a fixed point, i.e., $\exists\, x^* \in S$ s.t. $T(x^*) = x^*$.

In-Class Quiz 2

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Lecture 6 and Week 3: What We Built

- MSNE: no unilateral profitable deviation, even to a pure strategy

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- Nash's theorem (1951): every finite game has an MSNE, via Brouwer's fixed point theorem
- **Next week:** onward to mechanism design

End of Lecture 6

See you next class.

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay