

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 9 – The Median Voter Theorem, Task Sharing, and Mechanisms with Money

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October, 2026

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Week 9 Roadmap

Lecture	Chunks (20–25 min each)	Quizzes
Lecture 17	1. Building the Phantom Peaks 2. Case 1: The Median Is a Phantom 3. Case 2: The Median Is a Team's Peak 4. Sharing a Divisible Task	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4
Lecture 18	1. Fair and Truthful Splits 2. The Uniform Rule 3. Money Enters: Quasi-Linear Preferences 4. DSIC, Efficiency, and Groves Payments	Quiz 3 after Chunk 2 Quiz 4 after Chunk 4

Contents



- Lecture 17: The Median Voter Theorem and Task Sharing
- Lecture 18: The Uniform Rule and Mechanisms with Money



Chunk 1

Building the Phantom Peaks

(≈ 20 minutes)

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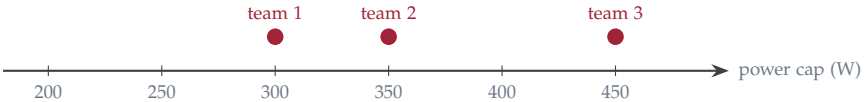

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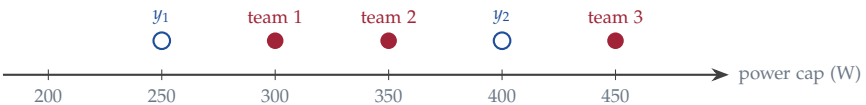

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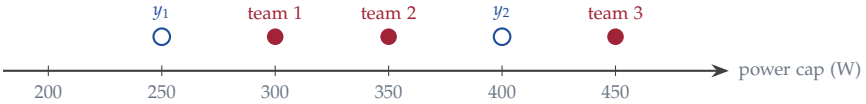

A **median voter SCF** adds $n-1$ fixed **phantom peaks** B, and picks the median of all $2n-1$ points.

Theorem (Moulin, 1980)

A strategyproof SCF f is ONTO and anonymous if and only if it is a median voter SCF.

The Easy Direction ($\Longleftarrow$)

Let $f(P) = \text{median}(B, \text{peaks}(P))$. Why is it SP, ANON, and ONTO?

- **SP:** we proved it last week. To move the median, a team must report a peak on the *other* side of it. That only pushes the outcome away from its true peak.

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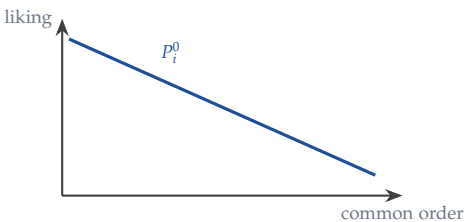
The real work

The other direction. Given *any* f that is SP, ANON, ONTO, we must *find* phantoms that make f a median rule.

Two Extreme Reports

- P_i^0: team i's peak is the **leftmost** alternative. It wants the lowest cap.



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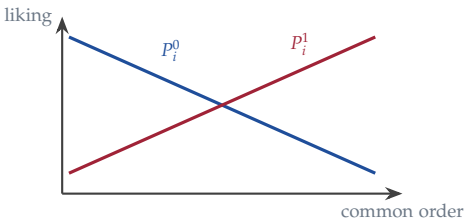

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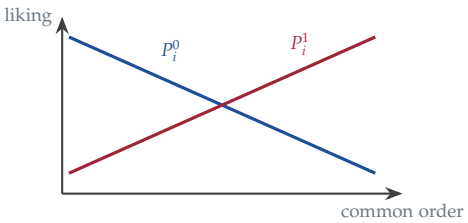

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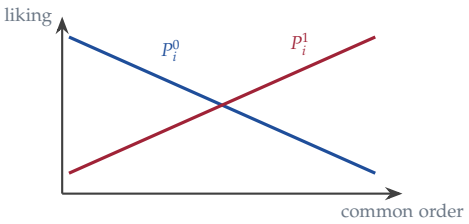
- Under P_i^0, smaller is always better. Under P_i^1, larger is always better.



Constructing the Phantom Peaks

Use f itself. Send $n-j$ teams to the far left and j teams to the far right:

$$y_j = f\big(\underbrace{P_1^0, \ldots, P_{n-j}^0}_{n-j \text{ leftmost}}, \underbrace{P_{n-j+1}^1, \ldots, P_n^1}_{j \text{ rightmost}}\big), \quad j = 1, \ldots, n-1.$$

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By **anonymity**, only the *count* j matters, not which teams move.

The Phantoms Come Out Sorted



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Now define the candidate rule: $g(P) = \text{median}(y_1, \ldots, y_{n-1}, p_1, \ldots, p_n)$. Goal: $f \equiv g$.



Chunk 2

Case 1: The Median Is a Phantom

(≈ 20 minutes)

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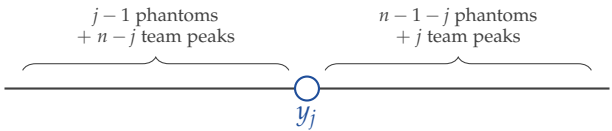
Case 1, counting. The median has $n - 1$ points on each side.



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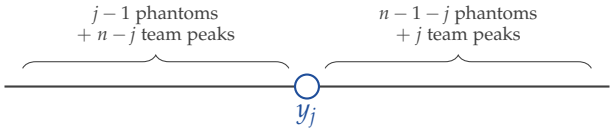

So $p_1 \leqslant \cdots \leqslant p_{n-j} \leqslant y_j \leqslant p_{n-j+1} \leqslant \cdots \leqslant p_n$.

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- Hence $b = y_j$.

Repeat for teams $2, \ldots, n-j$. Each has $p_k \leqslant y_j$, so the same two lines work:

$$f(P_1, \ldots, P_{n-j}, P_{n-j+1}^1, \ldots, P_n^1) = y_j.$$

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Now swap team n from P_n^1 to its true P_n:

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In words

Each team's lie was pinned between two SP inequalities. Nobody could move the outcome. So the phantom y_j survives all the way to the true profile.



In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.



Chunk 3

Case 2: The Median Is a Team's Peak

(≈ 25 minutes)

Case 2: Two Teams Are Enough to See It

Take $N = \{1, 2\}$. One phantom y_1. So $g(P) = \text{median}(p_1, p_2, y_1)$.

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Lemma

If P and P' have the same peaks, $P_i(1) = P'_i(1)$ for $i = 1, 2$, then $f(P) = f(P')$.

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- WLOG $x < a < y$. Recall SP + ONTO $\iff$ SP + PE. PE needs both outcomes in $[a, b]$. But $x < a$. Contradiction.

Case 2: Two Teams Are Enough to See It

Take $N = \{1, 2\}$. One phantom y_1. So $g(P) = \text{median}(p_1, p_2, y_1)$.

First, a lemma. It says only the peaks matter.

Lemma

If P and P' have the same peaks, $P_i(1) = P'_i(1)$ for $i = 1, 2$, then $f(P) = f(P')$.

Proof. Let $a = P_1(1) = P'_1(1)$, $b = P_2(1) = P'_2(1)$. Let $x = f(P_1, P_2)$, $y = f(P'_1, P_2)$.

- WLOG $a < b$ (else pick Player 2 for the following arguments)
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- Same argument for $(P'_1, P_2) \to (P'_1, P'_2)$. □

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Profile P: team 1 peaks at a, team 2 at b. WLOG the median of $\{a, b, y_1\}$ is a.

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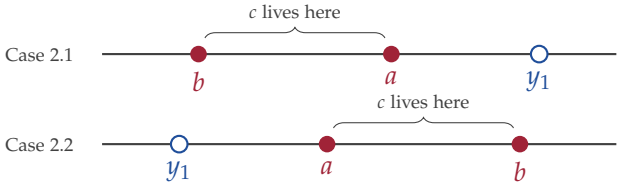

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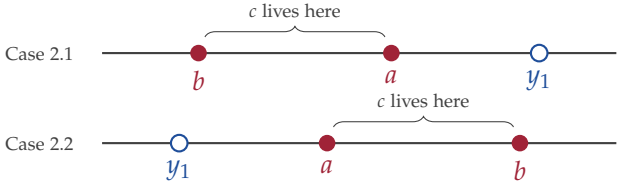

The trick: make team 1 *want* the phantom, then show it can grab it.

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By PE, $c < a$. So c and y_1 sit on opposite sides of a.

- Build P_1' with the same peak a, but ranking y_1 above c. Allowed, since they are on different sides of a.

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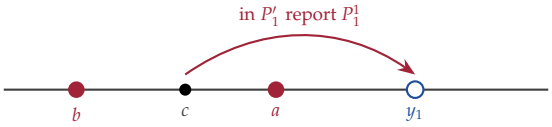

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So $f(P) = a = g(P)$ in Case 2 as well. For two teams, f *is* the median rule with phantom y_1. For general n, the same ideas go through with more bookkeeping. See **Moulin (1980), *On strategy-proofness and single peakedness***.

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Big picture: single-peakedness + SP + ONTO + ANON leaves exactly one family of rules: **median voter**. GS said "only dictators". Here, a whole spectrum survives.



Chunk 4

Sharing a Divisible Task

(≈ 20 minutes)

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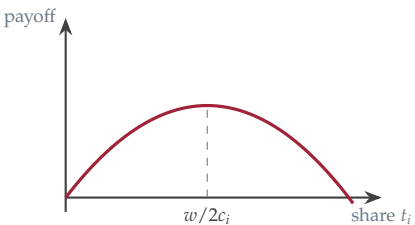

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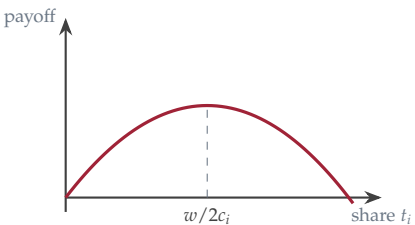

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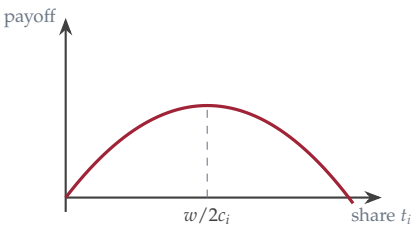

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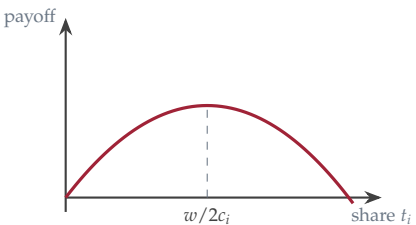
- It falls on both sides. **Single-peaked** in the share.



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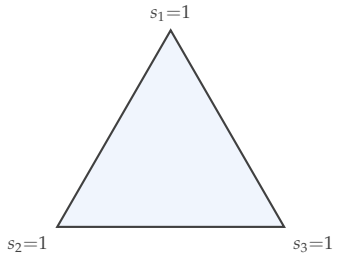

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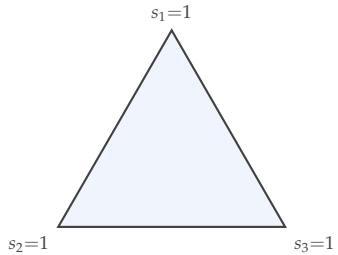

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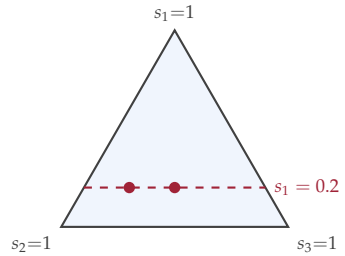

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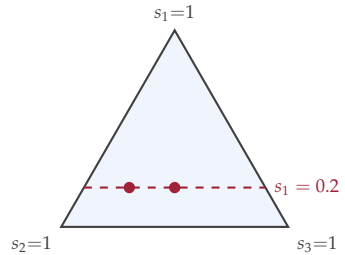

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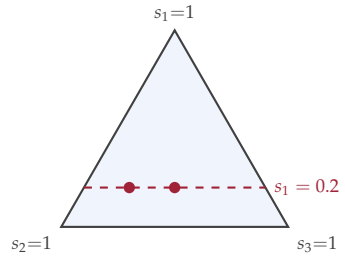

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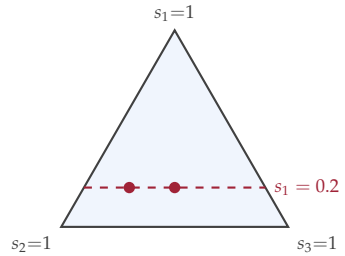
- So Moulin's theorem does not apply directly. We need new rules.



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SCFs and Pareto Efficiency for Shares

- The **domain of task allocation**: T. Allocations: A.
- **SCF**: $f : T^n \to A$, with $f(P) = (f_1(P), \ldots, f_n(P))$, $f_i(P) \in [0, 1]$, $\sum_i f_i(P) = 1$.
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In words: you cannot make one rider happier without making another rider sadder.

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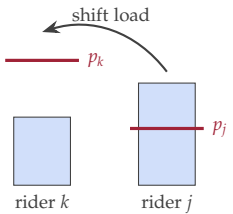

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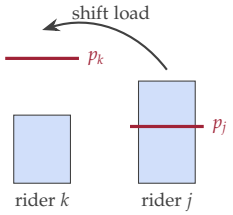

Rule of thumb

PE $\iff$ every rider stays on the *same side* of her peak.

In-Class Quiz 2

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Lecture 17: What We Built

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- Task sharing: single-peaked in one's own share. PE means same side of the peak.
- **Next lecture:** which rule is PE, ANON, and SP for shares?



End of Lecture 17

See you next class.

Contents

- Lecture 17: The Median Voter Theorem and Task Sharing
- Lecture 18: The Uniform Rule and Mechanisms with Money



Chunk 1

Fair and Truthful Splits

(≈ 20 minutes)

Anonymity for Shares



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Definition (Anonymity)

An SCF f is *anonymous* (ANON) if for every agent permutation $\sigma : N \to N$, the task shares get permuted accordingly, i.e.,

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The rider who asked for 0.7 gets the same load, whatever her name.

Strategyproofness for Shares

Manipulability: f is **manipulable** if $\exists i \in N$, a profile P, and a report P_i' with

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Equivalent form. Only rider i's own share matters to her:

$$f(P_i, P_{-i}) \; P_i \, f(P_i', P_{-i}) \quad \text{OR} \quad f_i(P_i, P_{-i}) = f_i(P_i', P_{-i}),$$

$$\forall P_i, P_i' \in T, \; \forall i \in N, \; \forall P_{-i} \in T^{n-1}.$$

Candidate 1: Serial Dictatorship

Definition (Serial Dictatorship)

Fix an order of the riders in advance. Each rider in turn gets her peak share, or whatever is left. If $\sum_{i \in N} p_i < 1$, the last rider takes the leftover.

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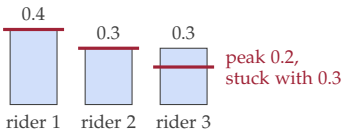

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Answer

PE and SP: early riders get their peak, later ones cannot change what is left. Not ANON: the order decides who is stuck. Quite unfair to the last rider.

Candidate 2: Proportional

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Every rider gets c times her peak, where $c \sum_{i \in N} p_i = 1$.

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The pattern

If $\sum_j p_j < 1$, under-report to shed load. If $\sum_j p_j > 1$, over-report to grab load.



Chunk 2

The Uniform Rule

(≈ 25 minutes)

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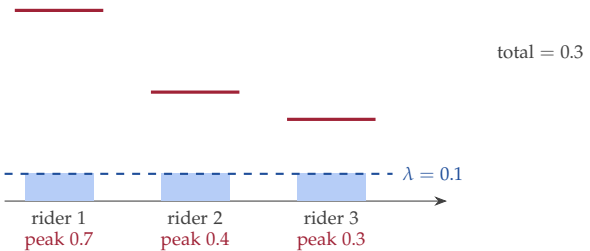

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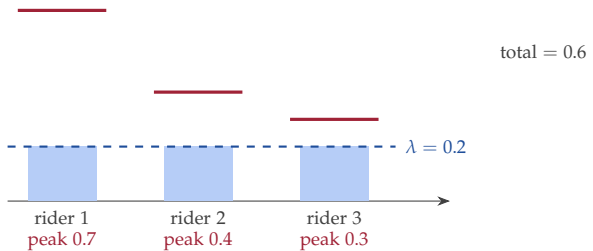

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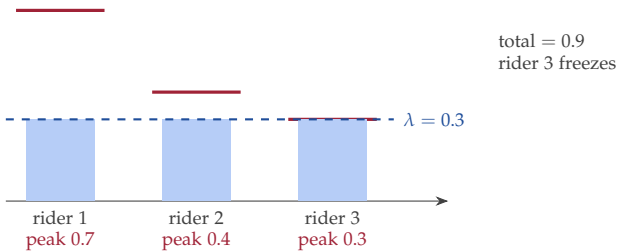

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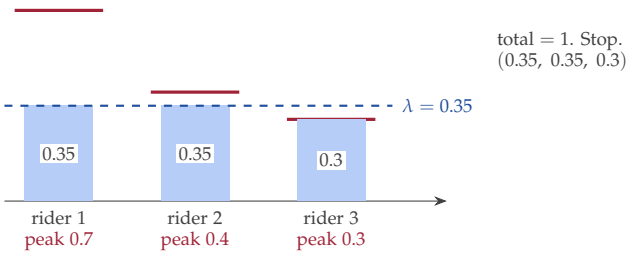

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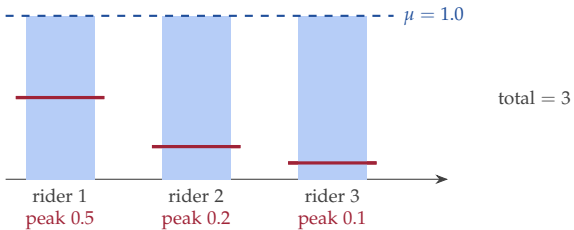

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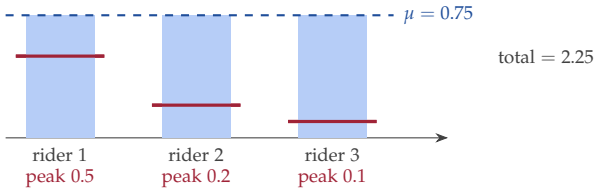

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total $= 1.5$
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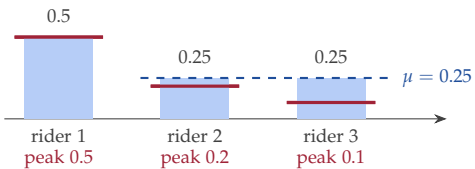

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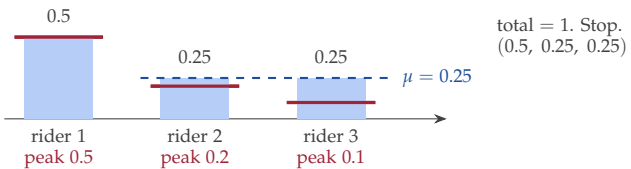

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Proportional would give $(0.625,\ 0.25,\ 0.125)$. Rider 1 would be pushed above her peak. Here she gets exactly 0.5.

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- **PE**: in each case *everyone* gets $= p_i$, $\geqslant p_i$, or $\leqslant p_i$. Same side of the peak, so PE.

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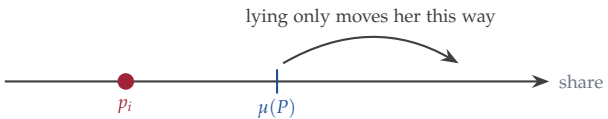

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Theorem (Sprumont 1991)

An SCF in the task allocation domain is SP, PE, and ANON iff it is the uniform rule.

See **Sprumont (1991): The division problem with single-peaked preferences**.

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Uniform rule: SP, PE, ANON, EF, and fast.

In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.



Chunk 3

Money Enters: Quasi-Linear Preferences

(≈ 20 minutes)

Mechanism Design with Transfers

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  4. **Partitioning objects.** A set S of cloud VMs.
     $a = \{(S_1, \cdots, S_n) : S_i \subseteq S,\ \forall i \in N, S_i \cap S_j = \emptyset,\ \forall i \neq j\}$

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- Such types form the **quasi-linear domain**.

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This restriction opens the door to many non-dictatorial mechanisms.

Allocation Rules

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- **Allocation rule** $f : \Theta_1 \times \cdots \times \Theta_n \to A$
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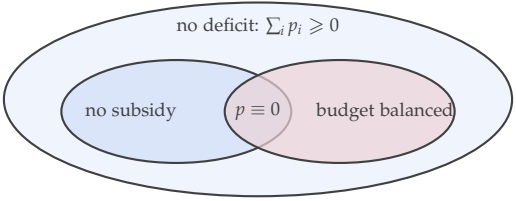

both no subsidy and budget balance imply no deficit



Chunk 4

DSIC, Efficiency, and Groves Payments

(≈ 25 minutes)

DSIC

Definition (DSIC)

A mechanism (f, p) is **dominant strategy incentive compatible (DSIC)** if

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Example. $N = \{1, 2\}$, $\Theta_1 = \Theta_2 = \{\theta^H, \theta^L\}$. For player 1, $\forall \theta_2 \in \Theta_2$:

$$v_1(f(\theta^H, \theta_2), \theta^H) - p_1(\theta^H, \theta_2) \geqslant v_1(f(\theta^L, \theta_2), \theta^H) - p_1(\theta^L, \theta_2)$$
$$v_1(f(\theta^L, \theta_2), \theta^L) - p_1(\theta^L, \theta_2) \geqslant v_1(f(\theta^H, \theta_2), \theta^L) - p_1(\theta^H, \theta_2)$$

Two more for player 2. Every true type beats every lie, against every report of the other.

Properties of the Payment

- Say (f, p) is DSIC. Consider another payment:

$$q_i(\theta_i, \theta_{-i}) = p_i(\theta_i, \theta_{-i}) + h_i(\theta_{-i}), \quad \forall \theta, \forall i \in N$$

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- Is (f, q) DSIC? Yes. $h_i(\tilde{\theta}_{-i})$ appears on both sides and cancels:

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- **A fact [exercise]**: if $f(\theta) = f(\tilde{\theta}_i, \theta_{-i}) = a$ and p implements f, then $p_i(\theta) = p_i(\tilde{\theta}_i, \theta_{-i})$.

Pareto Optimality in the Quasi-Linear Domain



Definition (Pareto Optimal)

$(f, (p_1, \ldots, p_n))$ is **Pareto optimal** if for every $\theta \in \Theta$, there is no allocation $b \neq f(\theta)$ and payments $(\pi_1, \ldots, \pi_n)$ such that

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Theorem

*A mechanism $(f, (p_1, \cdots, p_n))$ is **Pareto optimal** iff it is allocatively efficient.*

Proof: PO $\iff$ AE

A mechanism $(f, (p_1, \cdots, p_n))$ is **Pareto optimal** iff it is allocatively efficient.

($\Longleftarrow$) Show $\neg$PO $\implies$ $\neg$AE.

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Takeaway

Split the extra welfare δ equally. Money makes efficiency a question about f alone.

Groves Payments

Can we implement f^{AE} truthfully? Try:

$$p_i^G(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(f^{AE}(\theta_i, \theta_{-i}), \theta_j), \quad h_i : \Theta_{-i} \to \mathbb{R} \text{ arbitrary.}$$

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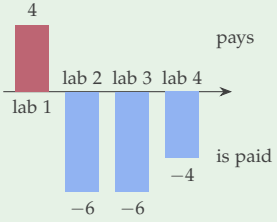

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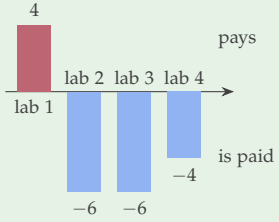

Only the winner pays. The others get paid. **Surprisingly, this is truthful.**

Groves Mechanisms Are Truthful



Theorem

Groves mechanisms are DSIC.

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\end{aligned}$$

- Player i was arbitrary. Hence the claim. □

Why it works: i's utility is total welfare minus $h_i(\tilde{\theta}_{-i})$, a term she cannot touch. Her goal is f^{AE}'s goal.

In-Class Quiz 2

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

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- **Next week:** the VCG mechanism, a special Groves mechanism.



End of Lecture 18

See you next class.



भारतीय प्रौद्योगिकी संस्थान मुंबई

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