

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 10 – The VCG Mechanism and Its Applications

Swaprava Nath

October, 2026

ज्ञानम् परमम् ध्येयम्

Knowledge is the supreme goal

Week 10 Roadmap

Lecture	Chunks (20–25 min each)	Quizzes
Lecture 19	1. Recap and the VCG Mechanism 2. VCG Examples: Second Price and Pivotal Agents 3. VCG in Combinatorial Allocations 4. Individual Rationality	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4
Lecture 20	1. Internet Advertising and Position Auctions 2. Advertiser Value, Slots, and VCG Payments 3. Pros of VCG, Privacy, and Collusion 4. Frugality, Revenue, and Budget Balance	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4

Contents



- Lecture 19: The VCG Mechanism and Combinatorial Allocations
- Lecture 20: Internet Advertising and the Limits of VCG



Chunk 1

Recap and the VCG Mechanism

(≈ 20–25 minutes)

Where We Are: Money Enters the Picture

Until now, a mechanism could only pick an outcome. Now it can also **move money**.

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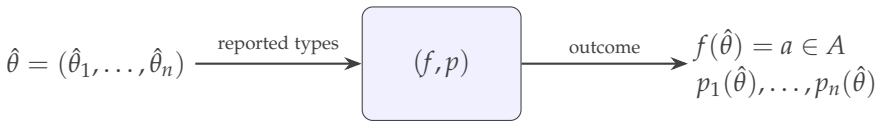

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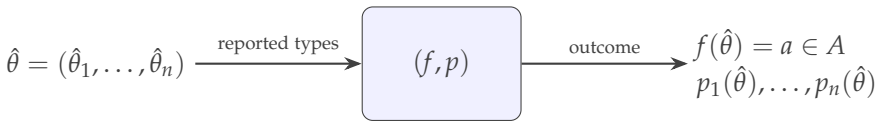

Definition (Social choice function, decomposed)

In the quasi-linear domain the SCF is $F = (f, p_1, \ldots, p_n)$, a mechanism (f, p) with

- **Allocation rule** $f : \Theta_1 \times \cdots \times \Theta_n \to A$, and
- **Payment function** $p_i : \Theta_1 \times \cdots \times \Theta_n \to \mathbb{R}$.

Recap: Utility, Efficiency, and Groves

- **Quasi-linear utility**: $u_i(a, p_i, \theta_i) = v_i(a, \theta_i) - p_i(\theta)$

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- **Groves**: (f^{AE}, p^G) with $p_i^G(\theta) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)$ is **DSIC for any h_i**

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Reading the sign

Your lab's shared GPU cluster: a scheduler picks who gets the next training slot (a) and settles a chargeback (p_i). Printing $p_3(\theta) = -6$ on the invoice means user 3 is *credited* 6, not billed.

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Question

Groves leaves h_i free. Which h_i should we pick? That is today's mechanism.

Recap: Properties of a Payment Rule

A gig-work platform matches couriers to orders and settles transfers $p_i(\theta)$.

No deficit	$\sum_{i \in N} p_i(\theta) \geqslant 0 \;\; \forall \theta$	*needs no external funds*
No subsidy	$p_i(\theta) \geqslant 0 \;\; \forall i, \forall \theta$	*no agent ever receives money*
Budget balance	$\sum_{i \in N} p_i(\theta) = 0 \;\; \forall \theta$	*money only moves between agents*

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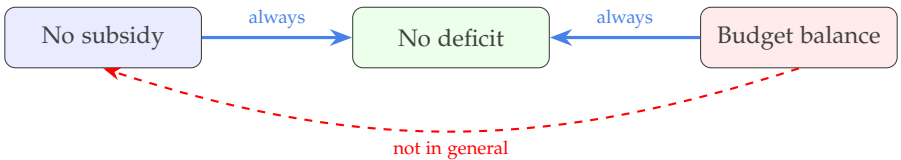

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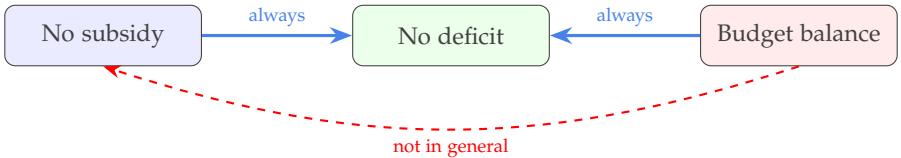

Answer

No subsidy $\implies$ **no deficit** and **budget balance** $\implies$ **no deficit**. No other implication holds in general: a budget-balanced rule can still pay some agents.

The VCG Mechanism

Definition (VCG, the pivotal mechanism, V'61, C'71, G'73)

The **Vickrey-Clarke-Groves** mechanism is the Groves mechanism with

$$h_i(\theta_{-i}) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j), \qquad \text{giving} \qquad \boxed{p_i^{VCG}(\theta) = \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) - \sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)}$$

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- **Utility** $= \sum_{j \in N} v_j(f^{AE}(\theta), \theta_j) - \max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j) =$ ***i*'s marginal contribution to social welfare**



Chunk 2

VCG Examples: Second Price and Pivotal Agents

(≈ 20–25 minutes)

Example 1: One Ticket, Four Bidders

Four classmates bid in sealed envelopes for one front-row ticket to the campus concert.
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VCG payment of the winner:

$$p_1^{VCG} = \max_{a \in A} \sum_{j \neq 1} v_j(a, \theta_j) - \sum_{j \neq 1} v_j(f^{AE}(\theta), \theta_j) = 9 - 0 = 9$$

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The highest bid wins and pays 9. Which number is 9? Have we met this auction before?

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Answer

9 is the *second highest* bid. VCG on a single object *is* the **second price (Vickrey) auction**.

Example 2: What Does “Pivotal” Mean?

Three student bodies vote on one shared campus facility. Reported valuations:

	Football	Library	Food Joint
A	0	70	50
B	95	10	50
C	10	50	50

Reported valuations

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Reported valuations

Social welfare: Football $= 105$, Library $= 130$, **Food Joint** $= 150$. VCG allocation: **Food Joint**.

Example 2: Who Actually Pays?

The Food Joint wins (welfare 150). Remove one agent at a time:

- **Remove A.** Others' best is Football, $95 + 10 = 105$. At the Food Joint they get $50 + 50 = 100$.
  A pays $105 - 100 = 5$

	Football	Library	Food Joint
A	0	70	50
B	95	10	50
C	10	50	50
Σ_{-A}	105	60	100

A removed

red: others' best without the agent blue: others at the Food Joint

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  B pays $120 - 100 = 20$

	Football	Library	Food Joint
A	0	70	50
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C	10	50	50
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- **Remove C.** Others' best is the Food Joint itself, $50 + 50 = 100$.
  C pays $100 - 100 = 0$ $\longleftarrow$ **non-pivotal**

	Football	Library	Food Joint
A	0	70	50
B	95	10	50
C	10	50	50
Σ_{-C}	95	80	100

C removed

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Answer

Only agents whose presence *changes the outcome* pay: the **pivotal** ones.



In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.



Chunk 3

VCG in Combinatorial Allocations

(≈ 20–25 minutes)

Example 3: Two Ad Inventory Units

Two bidders compete for two ad inventory units $\{1, 2\}$, value = type, $v_i(a, \theta_i) = \theta_i(a)$.

	$\emptyset$	$\{1\}$	$\{2\}$	$\{1\&2\}$
θ_1	0	8	6	12
θ_2	0	9	4	14

Valuations over bundles

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Valuations over bundles

Efficient allocation a^*: $\{1\} \to$ player 2 and $\{2\} \to$ player 1 (welfare $9 + 6 = 15$, beating 12 and 14).

Example 3: The Payments

Efficient allocation a^*: $\{1\} \to$ player 2, $\{2\} \to$ player 1, with welfare $9 + 6 = 15$.

$$p_i^{VCG}(\theta_1, \theta_2) = \max_{a \in A} \sum_{j \neq i} \theta_j(a) - \sum_{j \neq i} \theta_j(a^*)$$

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$$p_2^{VCG} = 12 - 6 = 6, \qquad \text{payoff } = 9 - 6 = 3$$

Both payoffs are non-negative. Is that a coincidence? We settle it in Chunk 4.

The Combinatorial Allocation Setting, Formally



- $M = \{1, \ldots, m\}$: set of objects

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- An **allocation** is a partition of the objects:

$$a = (a_0, a_1, \ldots, a_n), \quad a_i \in \Omega, \quad a_i \cap a_j = \emptyset \ \forall i \neq j, \quad \cup_{i=0}^{n} a_i = M$$

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Why this matters right now

This is the model behind every bundled-resource sale you will meet: spectrum licences, cloud reservations, and the ad-slot packages a demand-side platform buys in one shot.

Getting Nothing Costs Nothing



Claim

In the allocation of goods, the VCG payment for an agent who gets no object in the efficient allocation is zero.

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Proof. Let

$$a^* \in \arg\max_{a \in A} \sum_{j \in N} \theta_j(a), \quad a_i^* = \emptyset; \qquad a_{-i}^* \in \arg\max_{b \in A_{-i}} \sum_{j \in N \setminus \{i\}} \theta_j(b), \qquad p_i^{VCG}(\theta) = \sum_{j \neq i} \theta_j(a_{-i}^*) - \sum_{j \neq i} \theta_j(a^*)$$

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- $p_i^{VCG} \geqslant 0$. Agent i gets nothing in a^*, so a^* also splits the objects among $N \setminus \{i\}$ alone: $a^* \in A_{-i}$. As a_{-i}^* is the best of A_{-i} for the others, $\sum_{j \neq i} \theta_j(a^*) \leqslant \sum_{j \neq i} \theta_j(a_{-i}^*)$.
  In general a^* need not be available without i; here it is, because i gets nothing.

Getting Nothing Costs Nothing

Claim

In the allocation of goods, the VCG payment for an agent who gets no object in the efficient allocation is zero.

Proof. Let

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$$p_i^{VCG}(\theta) = \sum_{j \in N} \theta_j(a^*_{-i}) - \sum_{j \in N} \theta_j(a^*) \leqslant 0 \quad \text{since } a^*_{-i} \in A_{-i} \subseteq A \text{ and } a^* \text{ maximizes over } A$$

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- Squeezed between $\geqslant 0$ and $\leqslant 0$: $\boxed{p_i^{VCG}(\theta) = 0}$

Chunk 4

Individual Rationality

(≈ 20–25 minutes)

Will Anyone Show Up?

DSIC says: once you are in, tell the truth. It says nothing about whether you should enter.

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A direct mechanism (f, p) is **ex-post IR** if for every $\theta \in \Theta$ and every agent,

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- Participation never makes an agent worse off, so agents have an incentive to show up
- VCG is *not* ex-post IR in every domain – two conditions make it so

Two Conditions That Make VCG IR

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For all $i \in N$, $A_{-i} \subseteq A$. *addition of agents weakly increases choices*

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Definition (No negative externality)

For all $i \in N$, $\theta \in \Theta$, $v_i(a^*_{-i}(\theta_{-i}), \theta_i) \geqslant 0$.

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the efficient allocation without an agent yields non-negative value to that agent

Our combinatorial allocation domain satisfies both

- **Choice set monotonicity**: any split of the objects among $N \setminus \{i\}$ is also a split among N in which i gets $\emptyset$. So $A_{-i} \subseteq A$
- **No negative externality**: in a^*_{-i} agent i gets nothing, so $v_i(a^*_{-i}, \theta_i) = \theta_i(\emptyset) = 0 \geqslant 0$
- *Two ad units*: drop player 1 and the best split hands $\{1, 2\}$ to player 2. That split is in A, with player 1 holding $\emptyset$ and valuing it at 0

VCG Is Individually Rational

Claim

If the allocations satisfy choice set monotonicity and the valuations have no negative externality, then the VCG mechanism is individually rational.

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Corollary

VCG is ex-post IR for combinatorial allocations. *That settles the two non-negative payoffs we saw earlier.*

In-Class Quiz 2

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Lecture 19: What We Built

- **VCG** is the Groves mechanism with $h_i(\theta_{-i}) = \max_a \sum_{j \neq i} v_j(a, \theta_j)$: each agent pays the **externality** it imposes on the others

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- **Next:** VCG at work in Internet ad auctions, and where it breaks



End of Lecture 19

See you next class.

Contents



- Lecture 19: The VCG Mechanism and Combinatorial Allocations
- Lecture 20: Internet Advertising and the Limits of VCG



Chunk 1

Internet Advertising and Position Auctions

(≈ 20–25 minutes)

Recap: Lecture 19 in One Frame

$$p_i^{VCG}(\theta) = \underbrace{\max_{a \in A} \sum_{j \neq i} v_j(a, \theta_j)}_{\text{others' best welfare without } i} - \underbrace{\sum_{j \neq i} v_j(f^{AE}(\theta), \theta_j)}_{\text{others' welfare with } i}$$

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Today: the largest real-world home of these ideas, the sponsored-search ad auction – and then the cracks.

Application Domain: Internet Advertising



Why Internet advertising beats a newspaper column:

1. **User data**: advertisers learn which buyers are interested in which products, so targeting is possible

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The scale of it

Every one of these three advantages is really a statement about *information*: the platform knows more, measures more, and acts faster than any newspaper ever could. That is exactly the setting where a mechanism, rather than a posted price, becomes necessary.

Types of Advertisements on the Internet



Three kinds of ads:

- **Sponsored search ads**: advertisers bid on the keywords a user types. *Stop words* prevent a car ad from firing on the animal 'jaguar'; a *cap* limits how often one user sees the same ad

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Ads are complex – modern Internet advertising is handled via **ad exchanges**, which also let small businesses customize.

Position Auctions



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Position Auctions: How the Mechanism Evolved



Evolution of the mechanism:

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- **Today**: rank by the product of the **probability of a click** and the **bid value** – risk is shared

Chunk 2

Advertiser Value, Slots, and VCG Payments

(≈ 20–25 minutes)

Advertiser Value

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- Hence

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What Is Private, and What Is Not

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Why the split matters

The engine does not have to guess how much a click is worth to you – it only has to measure how often your ad gets clicked. The mechanism's job is to make you report v_i honestly through b_i.

Allocation of Slots

- Let $a = (a_1, \ldots, a_n)$ be the allocation, where a_i is the slot given to i. Agent i bids b_i, so the **reported** value is

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- **Efficient** allocation: $a^* \in \arg\max_{a \in A} \sum_{i \in N} \hat{v}_i(a)$
- **Observe:** a is efficient *if and only if* it is a **rank-by-expected-revenue** mechanism, ranking by $\hat{\rho}_i \cdot b_i$

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- **Efficient** allocation: $a^* \in \arg\max_{a \in A} \sum_{i \in N} \hat{v}_i(a)$
- **Observe:** a is efficient *if and only if* it is a **rank-by-expected-revenue** mechanism, ranking by $\hat{\rho}_i \cdot b_i$
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Note: actual industrial implementations differ; here we discuss the principle of the computation.

Why Sorting Is Efficient: An Exchange Argument

Write $w_i = \hat{\rho}_i b_i$ (agent effect) and recall $p_1 = 1 > p_2 > \cdots > p_m > 0$. The value of allocation a is $V(a) = \sum_{i \in N} p_{a_i} w_i$.

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A name to remember

This is the **rearrangement inequality**: a sum of products is largest when both sequences are sorted the same way. Computing f^{AE} here is just *sorting* by $\hat{\rho}_i b_i$.

VCG Payments in a Position Auction

Given bids $(b_1, \ldots, b_n)$, WLOG order them so that $\hat{\rho}_1 b_1 \geqslant \hat{\rho}_2 b_2 \geqslant \cdots \geqslant \hat{\rho}_n b_n$.

	with i**:** a^*	
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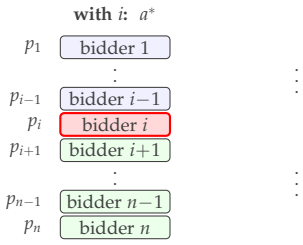

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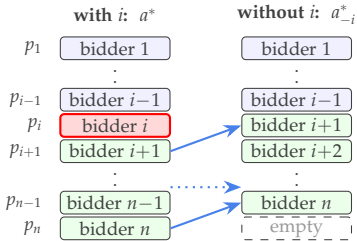

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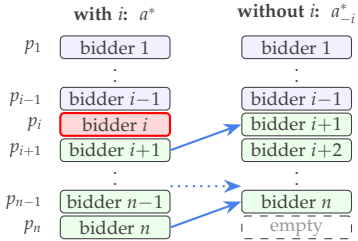

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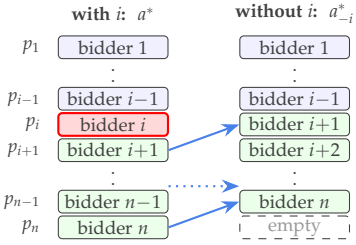

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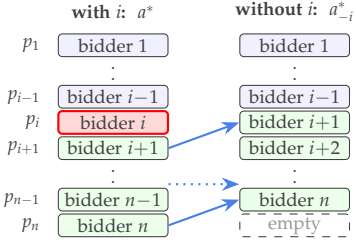

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- p_i^{VCG} **= the sum of these gains**: the harm i does by pushing everyone below down a slot

$$p_i^{VCG} = \sum_{j \neq i} \hat{v}_j(a^*_{-i}) - \sum_{j \neq i} \hat{v}_j(a^*) = \sum_{j=i}^{n-1} p_j(\hat{\rho}_{j+1} b_{j+1}) - \sum_{j=i}^{n-1} p_{j+1}(\hat{\rho}_{j+1} b_{j+1})$$

$$= \sum_{j=i}^{n-1} (p_j - p_{j+1})(\hat{\rho}_{j+1} b_{j+1}), \quad \forall i = 1, \ldots, n-1, \qquad \text{and} \qquad p_n^{VCG}(b) = 0$$

Reading the Position-Auction Payment

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- The last bidder displaces nobody, so pays nothing – the "gets nothing extra, pays nothing" logic again
- To convert this total expected payment into a pay-per-click price: $\dfrac{1}{p_i \hat{\rho}_i}\, p_i^{VCG}(b)$

In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Chunk 3

Pros of VCG, Privacy, and Collusion

(≈ 20–25 minutes)

Pros of the VCG Mechanism



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So why is the world not entirely VCG?

Because every one of these guarantees is about a *single*, *honest*, *isolated* run of the mechanism. The rest of this lecture is about what happens when agents talk to each other, when an irrelevant cost moves, or when one more bidder walks in.

Con 1: Privacy and Transparency



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A live concern

This is precisely why on-chain auction designs worry about the sequencer, and why ad exchanges are audited: the mechanism is only as honest as the party computing $\max_a \sum_{j \neq i} v_j(a, \theta_j)$.

Con 2: Susceptibility to Collusion

Consider a public good decision between A and B. VCG is **not group strategyproof**.

Truthful reports

	A	B	Payment
1	200	0	150
2	100	0	50
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- Individually, neither can gain by lying. *Together*, they can.

Chunk 4

Cons of VCG: Frugality, Revenue, Budget

(≈ 20–25 minutes)

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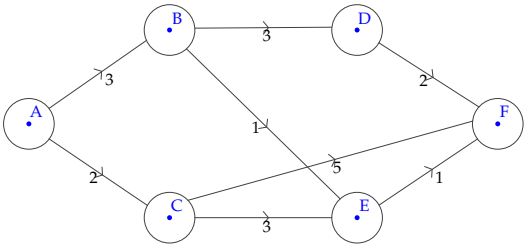

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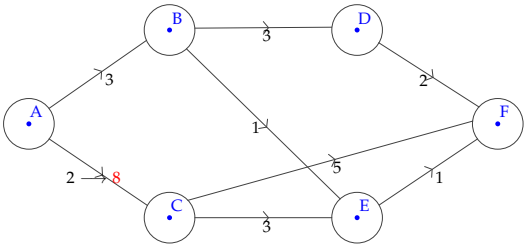
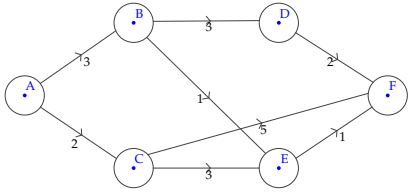


A last-mile delivery network. Source A, destination F. **Each edge is an agent**; its cost is a *negative* valuation.

Con 3: Not Frugal (contd.)

- This is a cost setup, so values are negative



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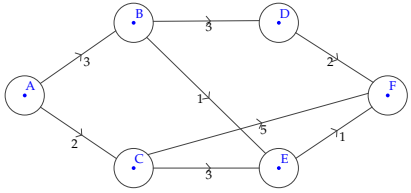

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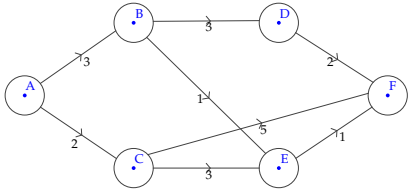

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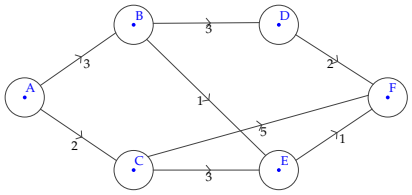

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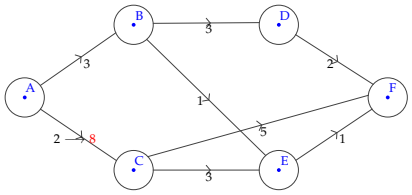

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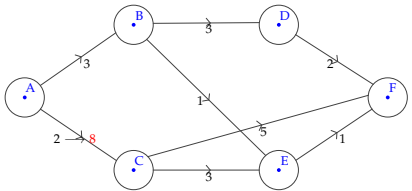

The edge $A \to C$ is not on the efficient path and never was. Its cost changed, and the payment to $A \to B$ more than doubled. VCG payments are hostage to **irrelevant alternatives**.

Con 4: Revenue Monotonicity Violation

Revenue should weakly *increase* with the number of players. Under VCG it need not.

	F	M	Payment
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- One more bidder, and the auctioneer's revenue collapses from 90 to 20

Con 5: Not Budget Balanced



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- This money cannot be redistributed among the same players: that would change their payoffs and break DSIC

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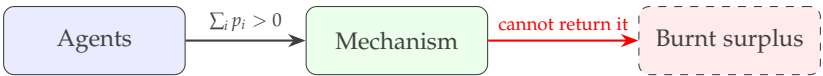

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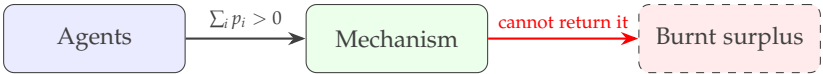

Theorem (Green–Laffont)

In a general quasi-linear setting, no mechanism can be allocatively efficient, DSIC, and budget balanced at the same time.

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These are certain limitations that are good to know for effective use of VCG; however, it is the most widely used mechanism in the literature.

In-Class Quiz 2

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Lecture 20 and Week 10: What We Built

- Position auctions: $v_{ij} = p_j(\rho_i v_i)$, efficiency = rank by expected revenue $\hat{\rho}_i b_i$, so allocation is just *sorting*

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- **Next week:** onward from VCG – what we can still do once we give one of these up



End of Lecture 20

See you next class.



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Indian Institute of Technology Bombay