

भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay

CS 6001: Game Theory and Algorithmic Mechanism Design

Week 12 – Optimal Mechanisms, Groves, and dAGVA

Swaprava Nath

October, 2026

Week 12 Roadmap

Lecture	Chunks (20–25 min each)	Quizzes
Lecture 23	1. One Buyer: Revenue Becomes Virtual Value 2. MHR and the Posted Price 3. Many Buyers: Maximize Virtual Surplus 4. Myerson's Optimal Auction	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4
Lecture 24	1. Reserve Prices and (In)efficiency 2. Efficient + DSIC = Groves 3. Groves Cannot Balance the Budget 4. dAGVA, Myerson–Satterthwaite, the Big Picture	Quiz 1 after Chunk 2 Quiz 2 after Chunk 4

Contents

- Lecture 23: Optimal Mechanism Design
- Lecture 24: Efficiency, Groves, and Budget Balance

Chunk 1

One Buyer: Revenue Becomes Virtual Value

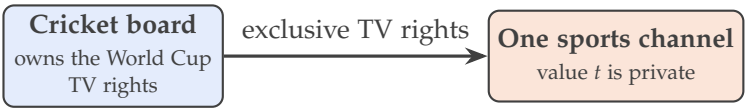
(≈ 20–25 minutes)

World Cup Broadcast Rights

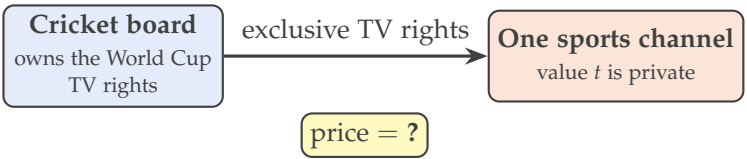
Cricket board
owns the World Cup TV rights

One sports channel
value t is private

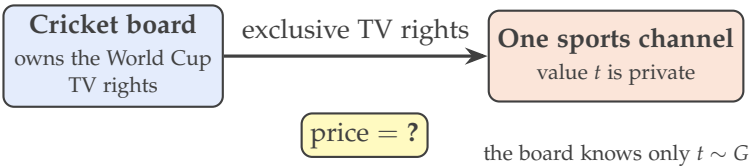
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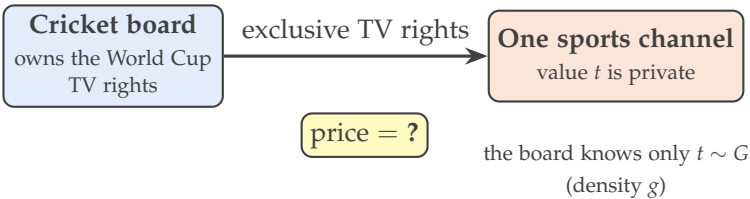
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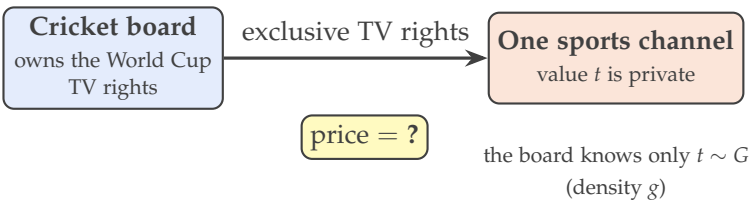


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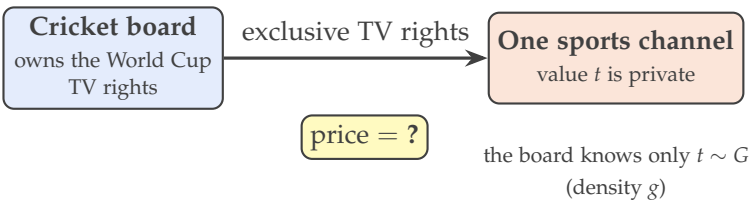
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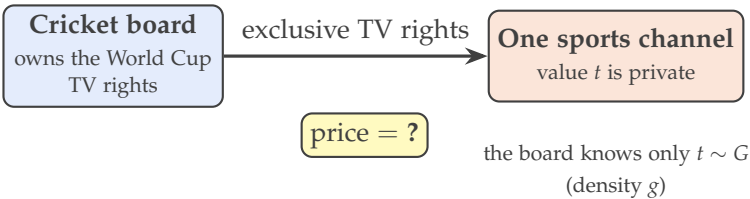
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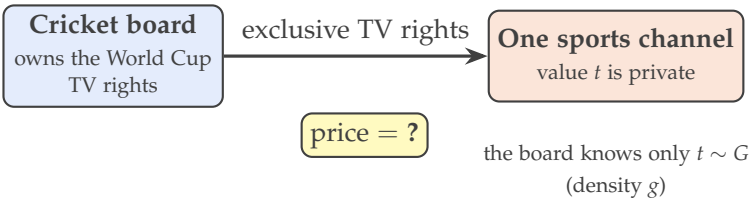
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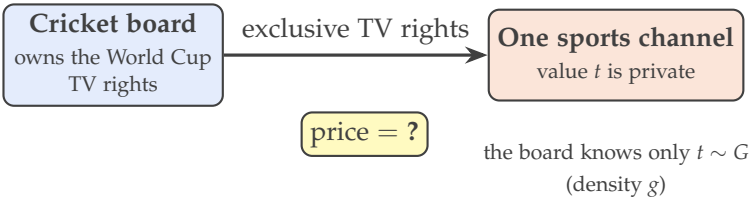
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- **Goal:** earn the most money *on average*.

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Lemma (Myerson 1981 + IIR)

*(f, p) is BIC and IIR **iff** (1) f is NDE, (2) $\pi_i(t_i) = \pi_i(0) + t_i\alpha_i(t_i) - \int_0^{t_i} \alpha_i(x)\, dx$, and (3) $\pi_i(0) \leqslant 0$.*

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- The expected revenue earned by a mechanism M is given by

$$\Pi^M := \int_0^\beta p(t) g(t) dt$$

Optimal Mechanism for Single Agent

Definition (Optimal Mechanism)

An optimal mechanism M^* for a single agent is a mechanism in the class of all IC and IR mechanisms, such that $\Pi^{M^*} \geqslant \Pi^{M}$, $\forall M$

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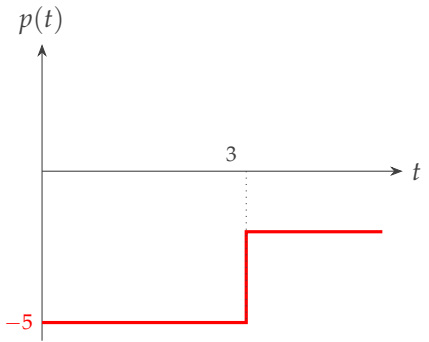
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- Since we want to maximize the revenue, hence $p(0) = 0$

Why $p(0) = 0$? Do Not Pay the Zero Type

- Say an IC mechanism pays the zero type Rs. 5, i.e., $p(0) = -5$.

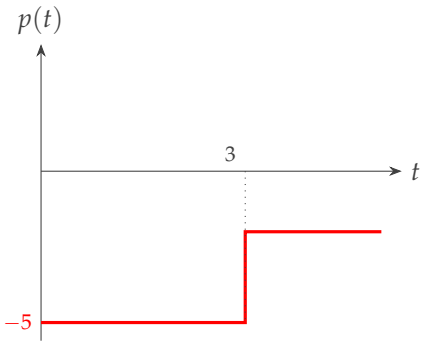


Red: post price 3, and also hand Rs. 5 to every buyer.

A specific example, only to show that the constant $p(0)$ should be 0. In general, $p(t)$ need not be a step function.

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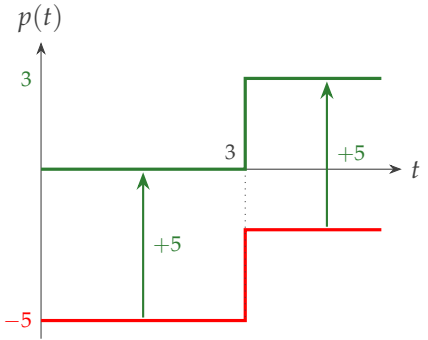


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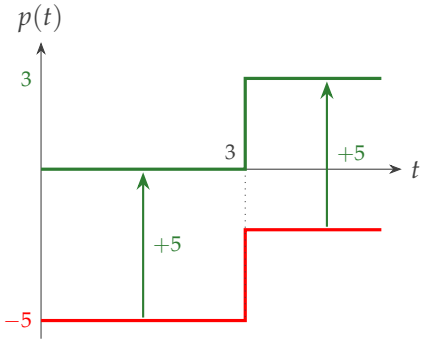


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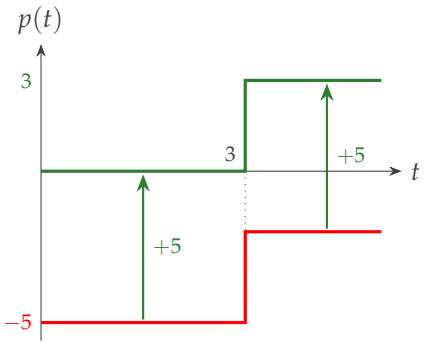


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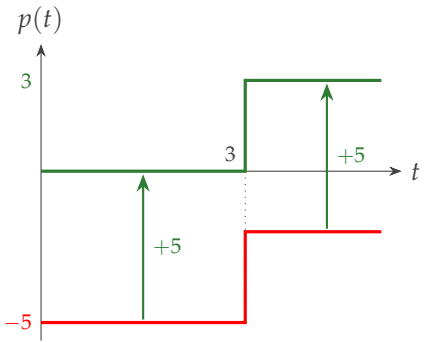


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- Now charge Rs. 5 more to *every* type.
- IC is untouched. A constant shift changes no comparison.
- IR still holds. Utility becomes $\int_0^t f(x)\,dx \geqslant 0$.
- Revenue goes up by Rs. 5. So the optimal mechanism has $p(0) = 0$.



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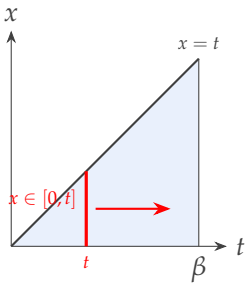
- Need to maximize this w.r.t. f

The Optimization Problem

$$\Pi^f = \int_0^\beta \left(tf(t) - \int_0^t f(x)dx \right) g(t)dt$$
$$= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \int_0^t f(x)dx\, g(t)dt$$

How do we swap the two integrals? Next slide.

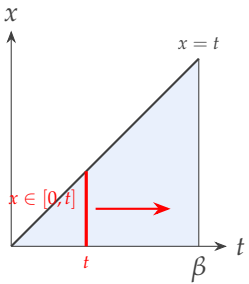
Picture: Switching the Order of Integration



slide t from 0 to β

- The region is a triangle: $0 \leqslant x \leqslant t \leqslant \beta$.

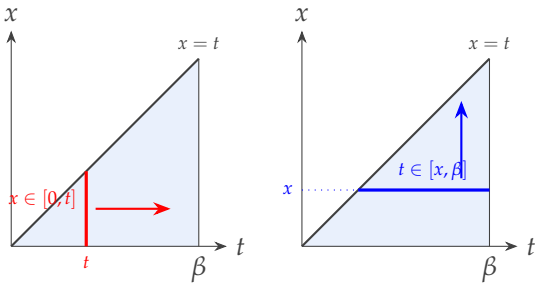
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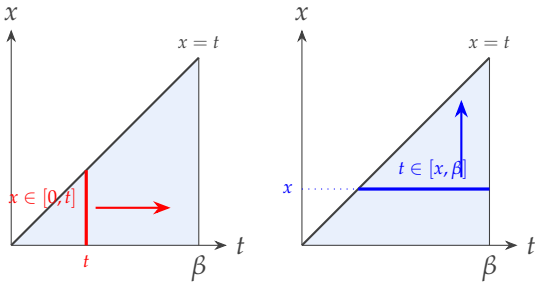
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- Vertical slice: fix t, let x run over $[0, t]$. Sliding t from 0 to β sweeps the triangle.
- Horizontal slice: fix x, let t run over $[x, \beta]$. Sliding x from 0 to β sweeps the *same* triangle.
- The inner integral becomes

$$\int_x^\beta g(t)\,dt = 1 - G(x).$$

The Optimization Problem

$$
\begin{aligned}
\Pi^f &= \int_0^\beta \left(tf(t) - \int_0^t f(x)dx \right) g(t)dt \\
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&= \int_0^\beta tf(t)g(t)dt - \int_0^\beta \int_x^\beta g(t)dt\, f(x)dx
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[switching the order of integration]

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\Pi^f &= \int_0^\beta \left(tf(t) - \int_0^t f(x)dx \right) g(t)dt \\
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&= \int_0^\beta \left(t - \frac{1 - G(t)}{g(t)} \right) g(t)f(t)dt
\end{aligned}
$$

The Optimization Problem

Lemma

For any implementable allocation rule f, we have

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Reading $w(t)$

The seller cannot earn the full value t. Some of it must be left to the buyer to keep her truthful. $w(t)$ is what is left for the seller.

Chunk 2

MHR and the Posted Price

(≈ 20–25 minutes)

The Modified Optimization Problem

- Hence the optimal mechanism finding mechanism reduces to

$$\textbf{OPT1:} \quad \max_{f:f \text{ is non-decreasing}} \int_0^{\beta} \left(t - \frac{1 - G(t)}{g(t)} \right) g(t) f(t) dt$$

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- **Exponential** with mean μ: $\frac{g(x)}{1-G(x)} = \frac{1}{\mu}$, a constant. So $\frac{1-G(x)}{g(x)} = \mu$.

Observation

Fact

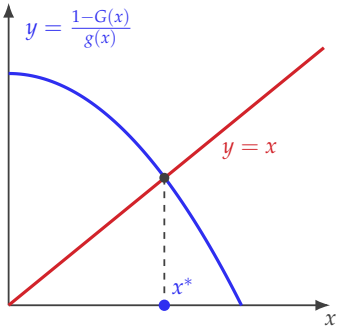
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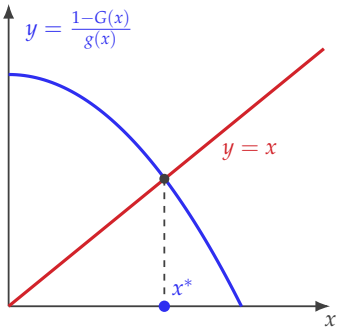


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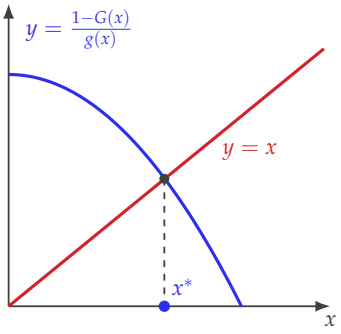


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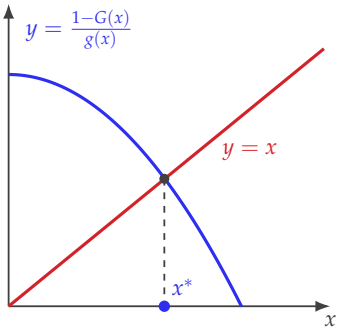


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- The red line $y = x$ rises. The blue curve $\frac{1-G}{g}$ falls. They cross once.



Solution to the optimization problem

- The unrestricted solution to OPT1 is therefore

$$f(t) = \begin{cases} 0 & \text{if } t < x^* \\ 1 & \text{if } t > x^* \\ \alpha & \text{if } t = x^*, \alpha \in [0,1] \end{cases} \tag{1}$$

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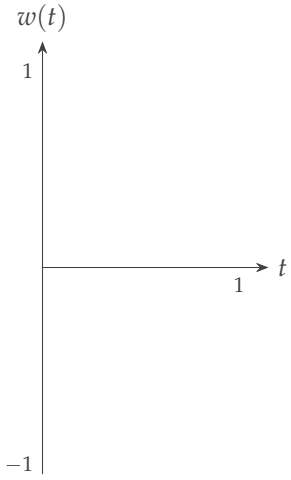
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- In words: **post a take-it-or-leave-it price x^*.**

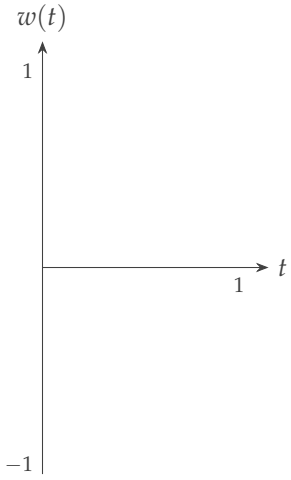
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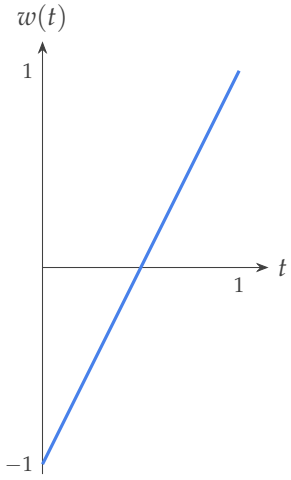
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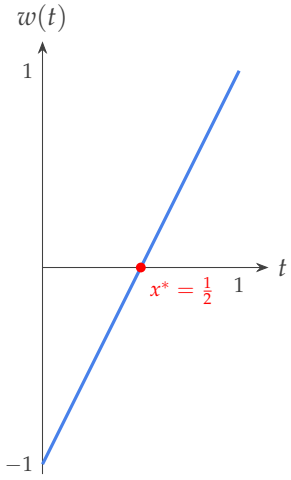
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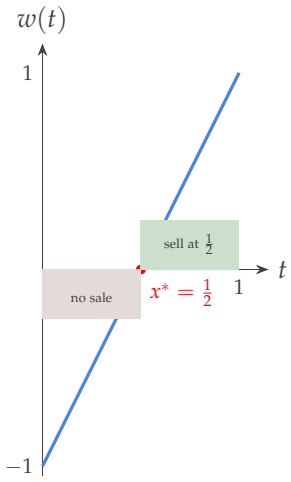
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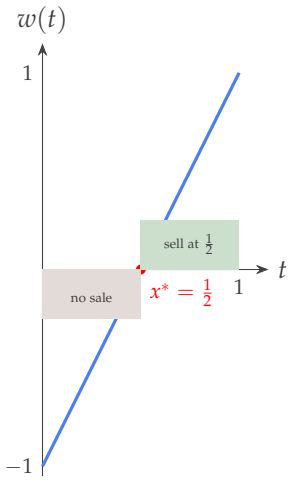
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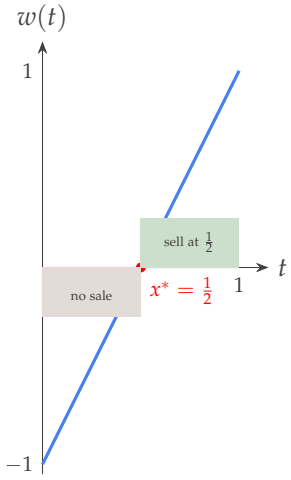
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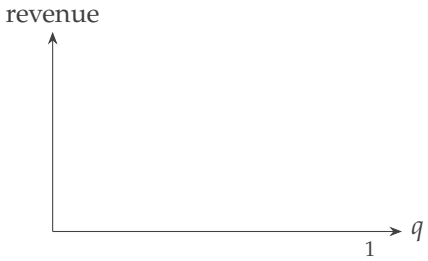
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- **Post price Rs. 500 crore.**
- Sells with probability $\frac{1}{2}$.
- Expected revenue $= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, i.e., Rs. 250 crore.



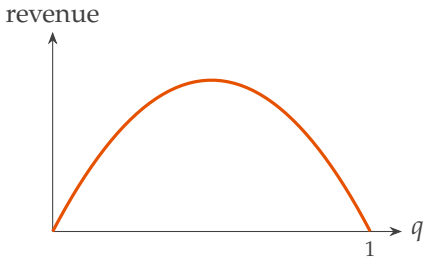
A Shortcut for One Buyer

- With one channel, we could simply guess: *post a price q.*



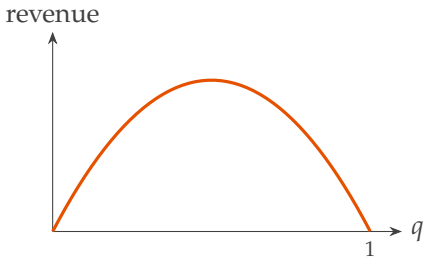
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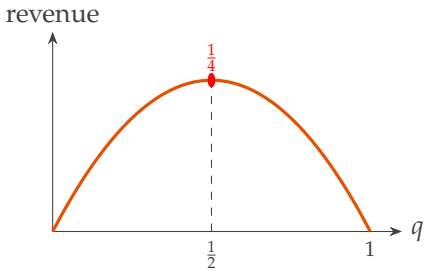
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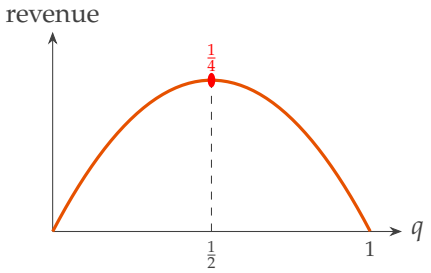
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So why take the long route?

The shortcut only searches over posted prices, and only for one buyer. With two or more channels, the board could run an auction, set reserves, favour one channel, or even randomize. There is nothing simple to guess.

Myerson's route, *revenue = expected virtual surplus*, works for **any** number of buyers. It also proves that, for one buyer, nothing beats the posted price. Many buyers are **coming up next**.

In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.



Chunk 3

Many Buyers: Maximize Virtual Surplus

(≈ 20–25 minutes)

Two Channels Want the Rights

Cricket board
World Cup TV rights

Two Channels Want the Rights



- Back to the board. Now *two* channels can afford the rights: a regional channel (1) and a national network (2).

Two Channels Want the Rights



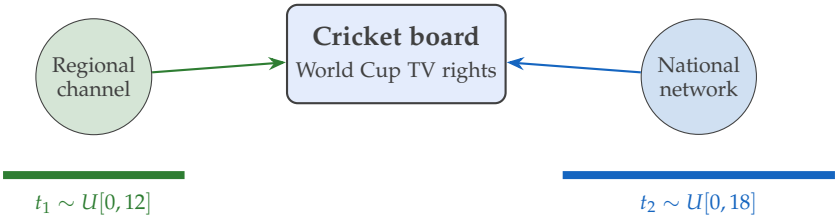
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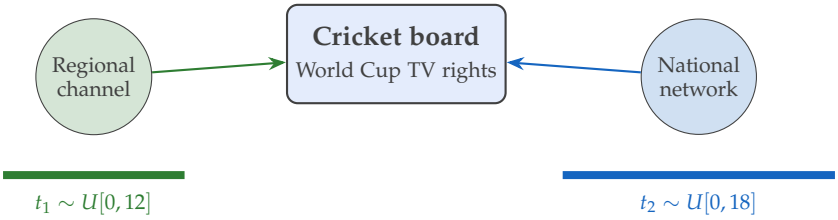
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- Priors are **independent** and *different*. Treat both the same?

Optimal mechanism design for multiple agents



- In this context, we will call a mechanism optimal if it is BIC, IIR, and maximizes revenue

Optimal mechanism design for multiple agents



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- This can be simplified to the following in a way similar to the earlier exercise

$$\int_0^{b_i} w_i(t_i) g_i(t_i) \alpha_i(t_i)\, dt_i$$

where, $w_i(t_i) = t_i - \dfrac{1 - G_i(t_i)}{g_i(t_i)}$ (virtual valuation of player i) and,

$$\alpha_i(t_i) = \int_{T_{-i}} f_i(t_i, t_{-i}) g_{-i}(t_{-i})\, dt_{-i}$$

Optimal mechanism design for multiple agents

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Optimal mechanism design for multiple agents

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  where $\sum_{i \in N} (w_i(t_i) f_i(t))$ is the expected total virtual valuation
- Hence, the optimal mechanism problem reduces to

$$\max \int_T \sum_{i \in N} (w_i(t_i) f_i(t)) g(t)\, dt, \text{ s.t. } f \text{ is NDE}$$

Optimal mechanism design for multiple agents

- As before, we try to solve the **unconstrained** optimization problem.

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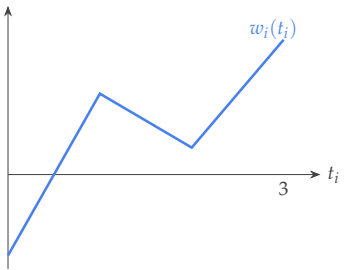
- This condition is weaker than MHR condition as MHR implies regularity

When Regularity Fails, NDE Breaks

Agent j: $t_j \sim U[0,1]$, so $w_j(t_j) = 2t_j - 1$ (regular).

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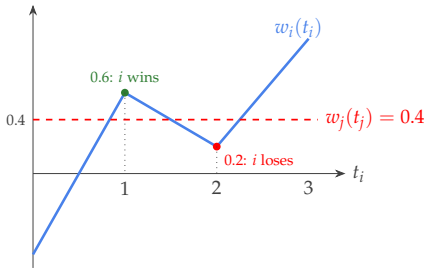
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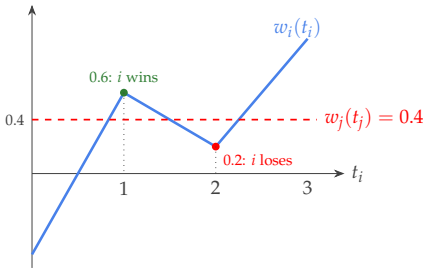


- Fix $t_j = 0.7$, so $w_j = 0.4$. At $t_i = 1$, $w_i = 0.6 > 0.4$: i wins. At $t_i' = 2$, $w_i = 0.2 < 0.4$: i loses.

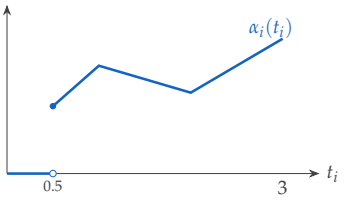
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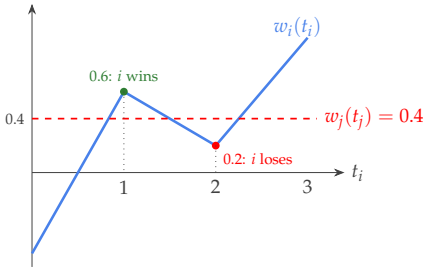


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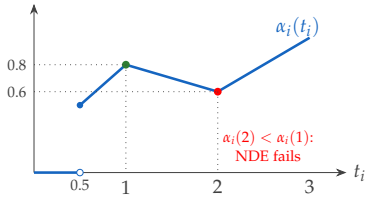
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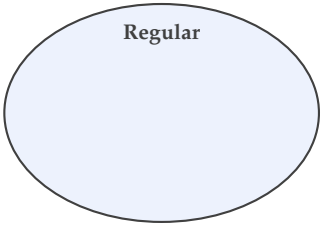
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- $\alpha_i(1) = 0.8 > \alpha_i(2) = 0.6$. So α_i copies the dip in w_i, and NDE fails.

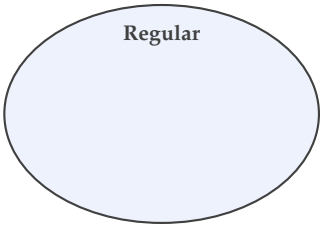
Why MHR $\Rightarrow$ Regular

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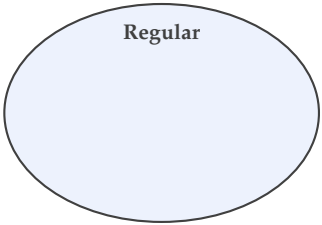
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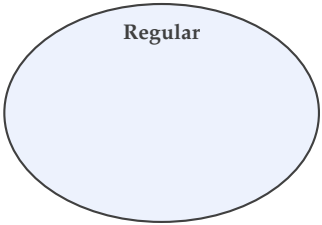
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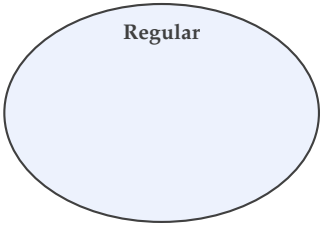
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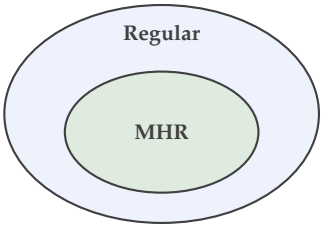
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- t increases. We subtract something that shrinks.
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- The converse fails. Regular is the weaker demand.



Optimal mechanism design for multiple agents



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The punchline

We dropped the constraint. We solved the easy problem. Regularity says the easy answer already obeys the constraint. So it is the answer to the hard problem too.

The solution

- Optimal Mechanism Design Problem

$$\max \int_T \left(\sum_{i \in N} w_i(t_i) f_i(t) \right) g(t)\, dt, \quad \text{such that } f \text{ is NDE}$$

Solution for **regular** w_i's

$$f_i(t) = \begin{cases} 1 & \text{if } w_i(t_i) \geqslant w_j(t_j),\ \forall j, \text{ break ties arbitrarily} \\ 0, & \text{otherwise} \end{cases} \quad \textbf{(Sold)} \tag{3}$$

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- We wanted to find an allocation that is NDE, but found an f that is non-decreasing
- It is also deterministic

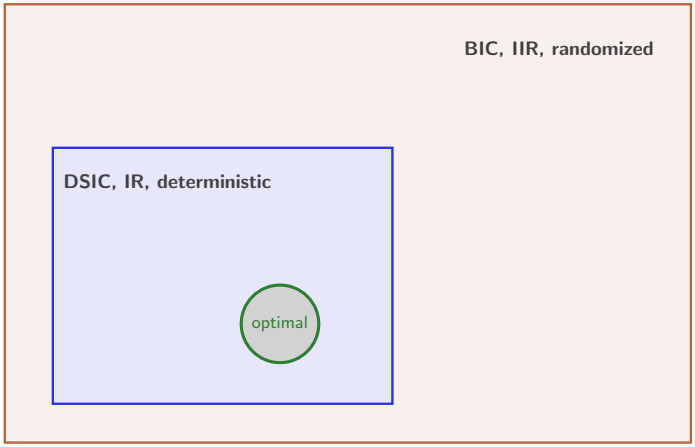


Chunk 4

Myerson's Optimal Auction

(≈ 20–25 minutes)

Optimal Mechanism



Space of mechanisms with regular virtual valuations

Optimal Mechanism: Allocation and Payment

Theorem

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Note: $K_i^*(t_{-i})$ is the minimum of value of t_i where i begins to be the winner

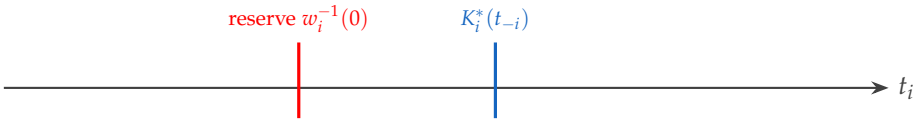
Reading the Payment: Two Bars to Clear



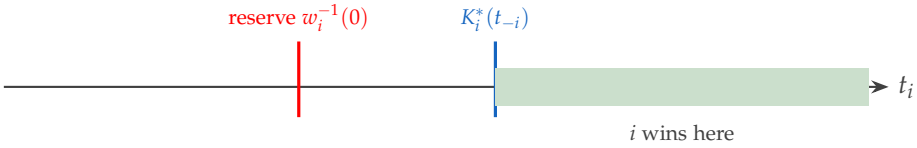
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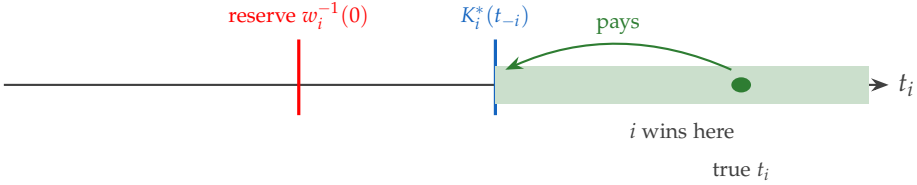
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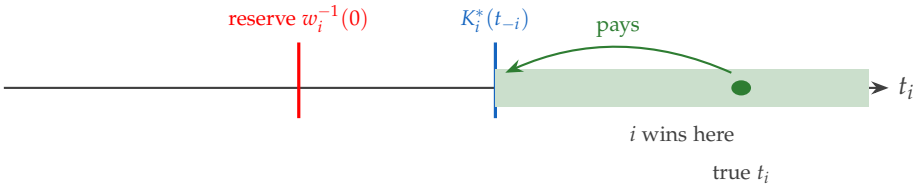
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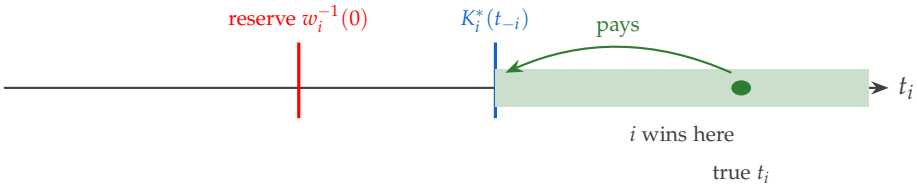


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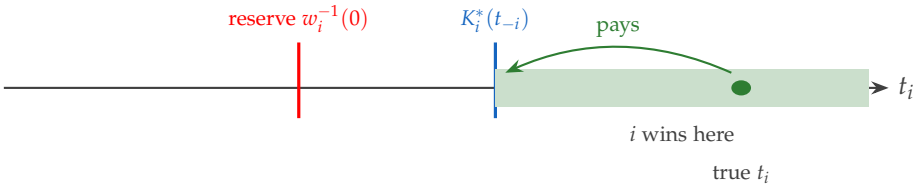
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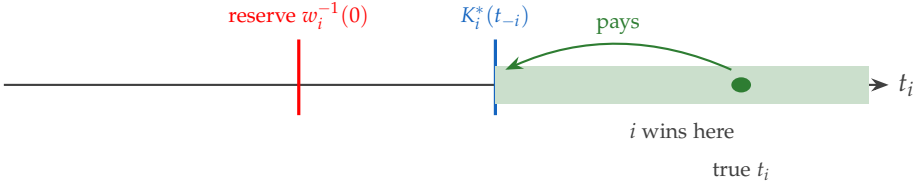
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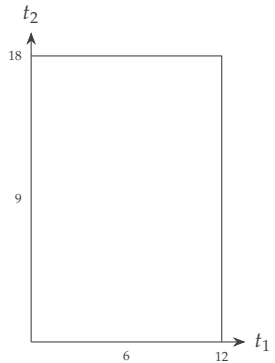
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- Just like Vickrey: your payment does not depend on your own bid.

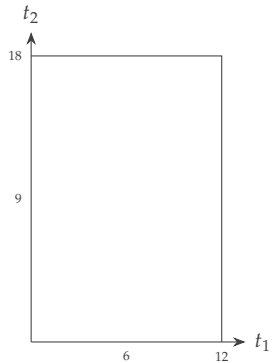
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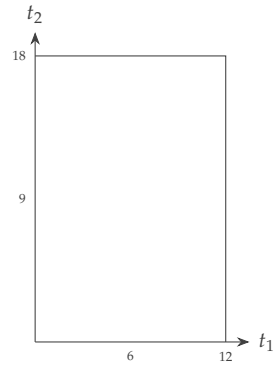
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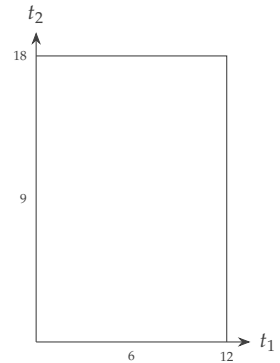
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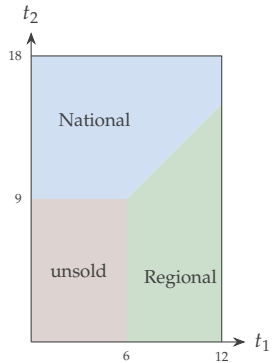
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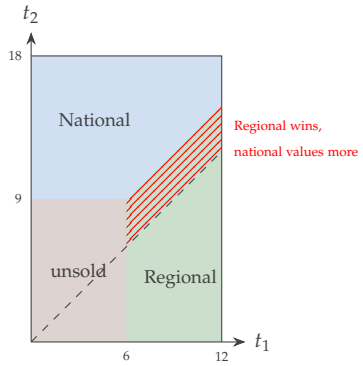
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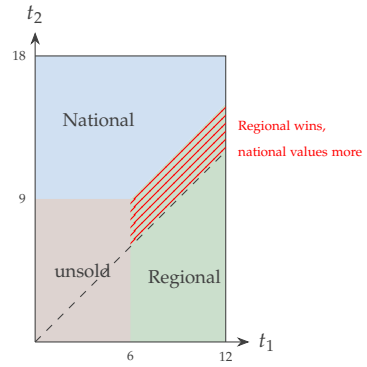
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t_1	t_2	Action	p_1	p_2	why
4	8	unsold	0	0	both below reserve

Check each row

Virtual values:
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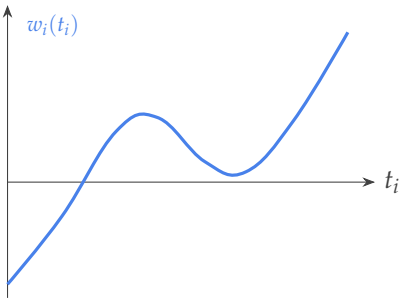
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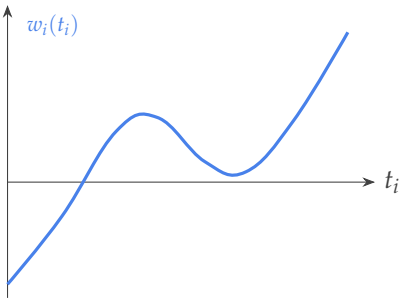
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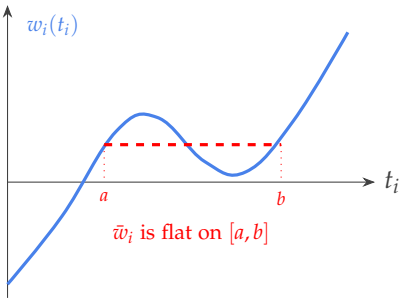
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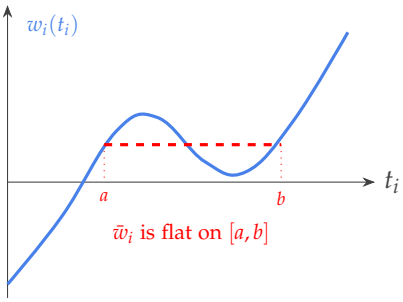
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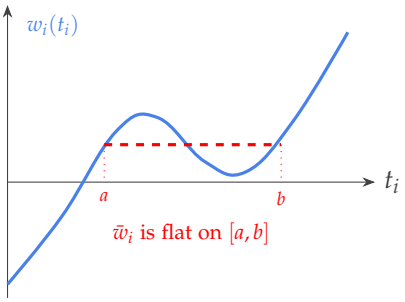
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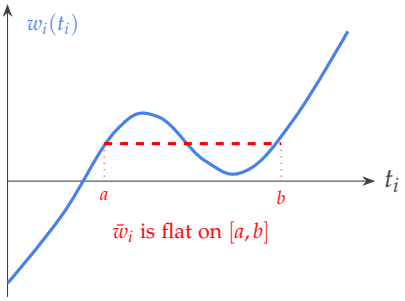
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Read more

R. B. Myerson, "Optimal Auction Design," *Math. of Operations Research*, 6(1), 1981, Sec. 6 (Optimal auctions in the general case).
J. D. Hartline, *Mechanism Design and Approximation*, Sec. 3.3.5 (Virtual Surplus Maximization Subject to Monotonicity).

In-Class Quiz 2

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Lecture 23: What We Built



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- **Next lecture:** what if we insist on efficiency instead?



End of Lecture 23

See you in the next class.

Contents

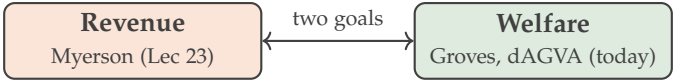


- Lecture 23: Optimal Mechanism Design
- Lecture 24: Efficiency, Groves, and Budget Balance

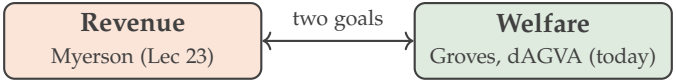
Where We Are

Revenue
Myerson (Lec 23)

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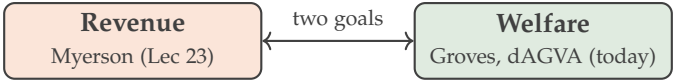

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two goals
Welfare
Groves, dAGVA (today)

Where We Are



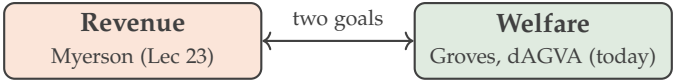
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Where We Are



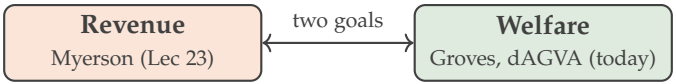
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Where We Are



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Where We Are



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- What does truthfulness force us to pay? And who pays?



Chunk 1

Reserve Prices and (In)efficiency

(≈ 20–25 minutes)

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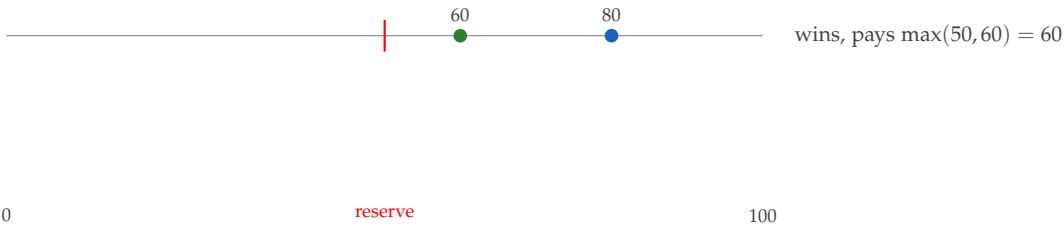
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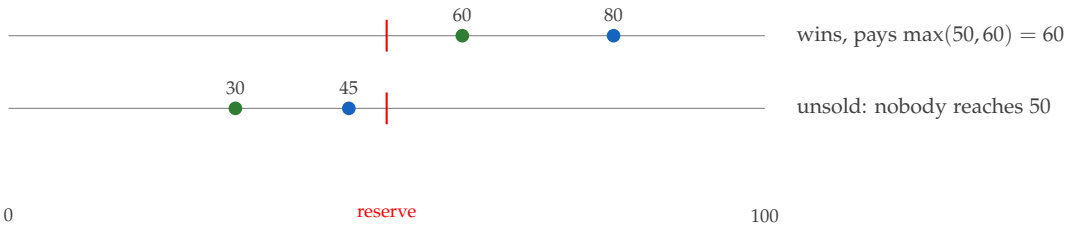
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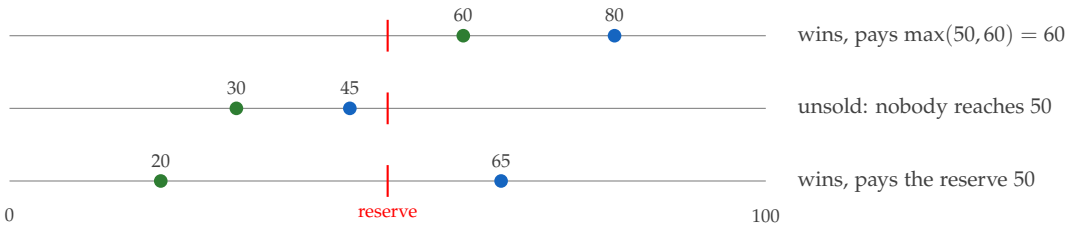
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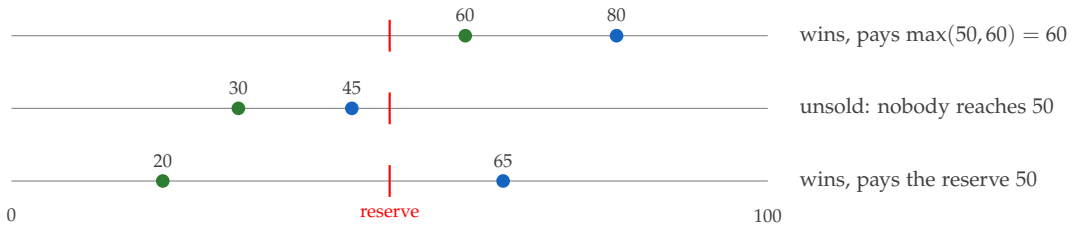
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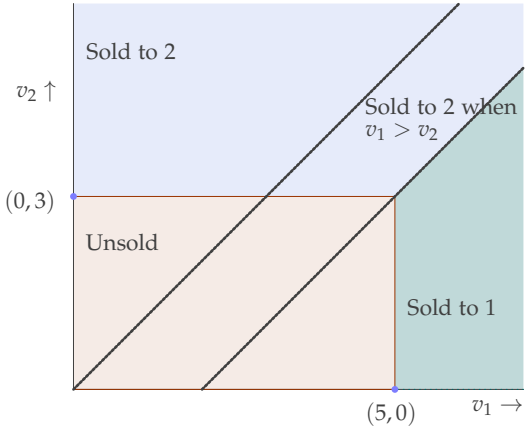


The reserve does the work

Without the reserve, row 3 would pay only 20. The reserve raises it to 50. In row 2 it costs a sale. On average, it pays off.

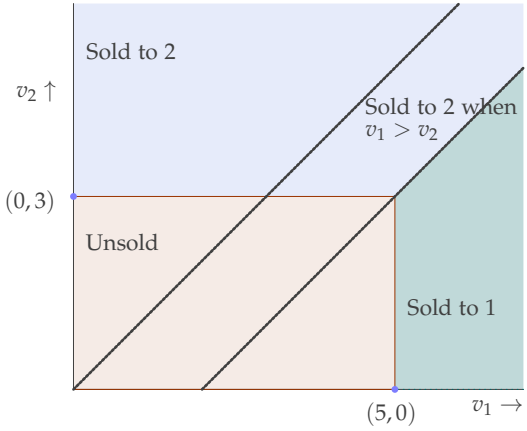
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- **TV vs. streaming:** the board sells the rights to a TV channel (1) or a streaming app (2)



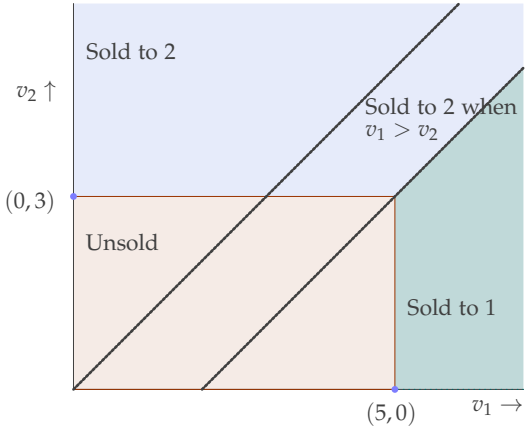
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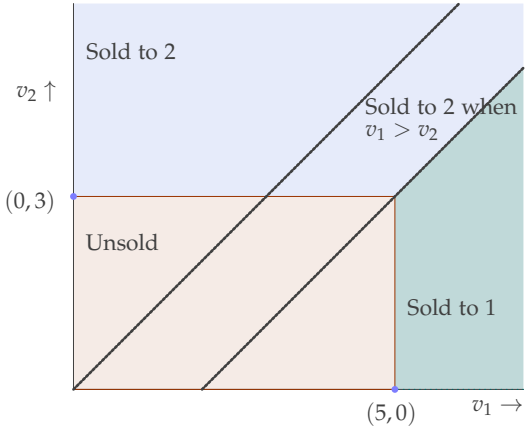
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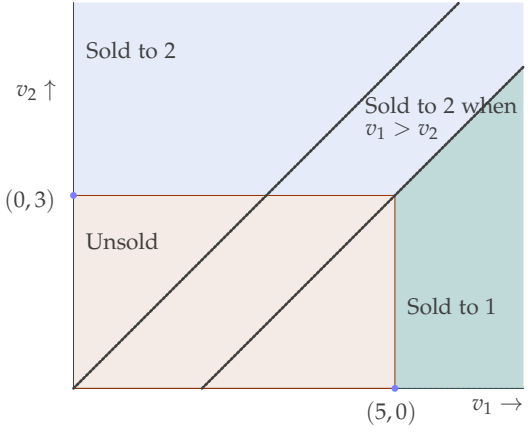
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- The app (2) wins iff $t_2 \geqslant 3$ and $t_2 \geqslant t_1 - 2$
- Unsold is inefficient, also in the region of the plane where 1 has higher valuation but item is sold to 2



Example 3: One Profile, Step by Step

Bids: $t_1 = 7$, $t_2 = 6$.

- $w_1 = 2 \cdot 7 - 10 = 4, \quad w_2 = 2 \cdot 6 - 6 = 6$

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Question

If we want the *efficient* outcome, what must the payments look like?

Chunk 2

Efficient + DSIC = Groves

(≈ 20–25 minutes)

Kick-off Time for the World Cup Final

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Total	**8**	**7**

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- **DSIC**: payments must make such lies useless. Which payments? Next.

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	a	b
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Stadium (i)	$t_i(a)\updownarrow$	1
Streamer	3	1

held fixed only this moves

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  - Fix the valuations of other agents to t_{-i}
  - Fix value of i at alternative b as $t_i(b)$
- $\exists$ some threshold $t_i^*(a)$ s.t.

$$t_i(a) \geqslant t_i^*(a) \Rightarrow \text{outcome } a, \qquad t_i(a) < t_i^*(a) \Rightarrow \text{outcome } b$$

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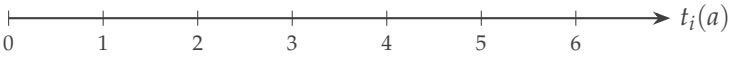
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The Threshold, in Pictures

Take the **stadium** as agent i. Its evening value is $t_i(b) = 1$. The channel and the streamer together have $\sum_{j \neq i} t_j(a) = 4$ and $\sum_{j \neq i} t_j(b) = 6$.

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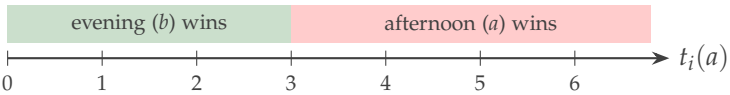


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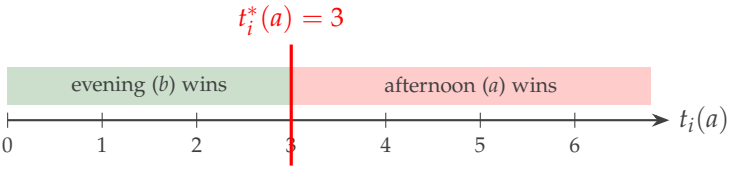
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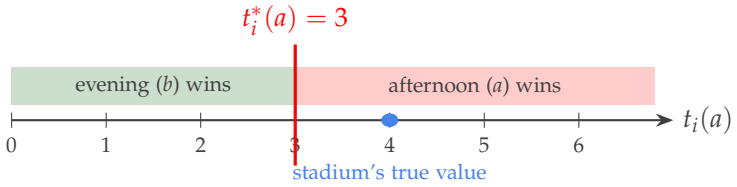
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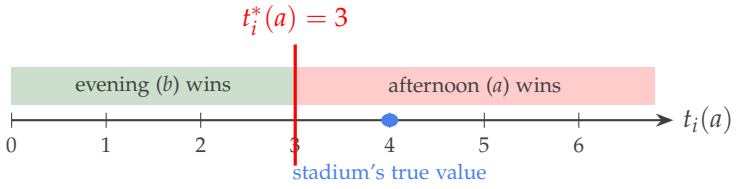
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- At the threshold, both kick-off times give the same total welfare.
- The stadium's true afternoon value is $4 \geqslant 3$, so afternoon is chosen.
- Next: DSIC pins down the payments at this threshold.

Proof sketch (contd.)

- Using DSIC for $t_i^*(a) + \epsilon = t_i(a), \epsilon > 0$ we have,

  $t_i^*(a) + \epsilon - p_{ia} \geqslant t_i(b) - p_{ib}$ (Note: payment for a player has to be the same for an allocation)

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- Similarly, $t_i'(a) = t_i^*(a) - \delta, \delta > 0$ and

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- But $t_i^*(a)$ is the threshold of the efficient outcome, thus,

$$t_i^*(a) + \sum_{j \neq i} t_j(a) = t_i(b) + \sum_{j \neq i} t_j(b) \tag{5}$$

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- From Equations (4) and (5)

$$p_{ia} - p_{ib} = \sum_{j \neq i} t_j(b) - \sum_{j \neq i} t_j(a)$$

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The Groves payment

$$p_i(t) = \underbrace{h_i(t_{-i})}_{\text{any function of the others}} - \underbrace{\sum_{j \neq i} t_j(f(t))}_{\text{others' welfare}}$$

Your utility becomes total welfare minus $h_i(t_{-i})$. So you want the efficient outcome too. If h_i depends on your own report, this breaks.



In-Class Quiz 1

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.



Chunk 3

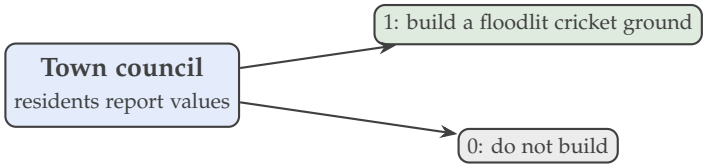
Groves Cannot Balance the Budget

(≈ 20–25 minutes)

A Town Votes on a Cricket Ground

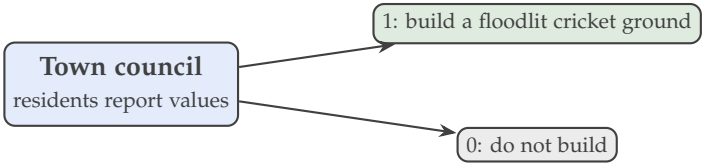
Town council
residents report values

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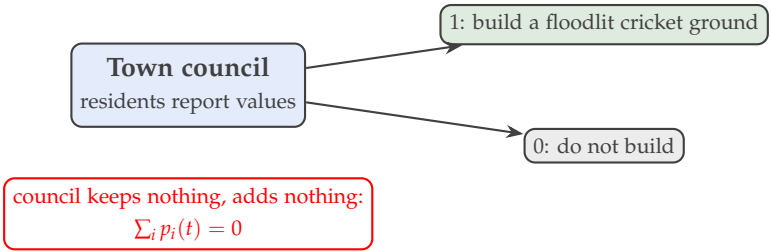
- A town decides: build a public floodlit cricket ground, or not. A go/no-go public project.

A Town Votes on a Cricket Ground



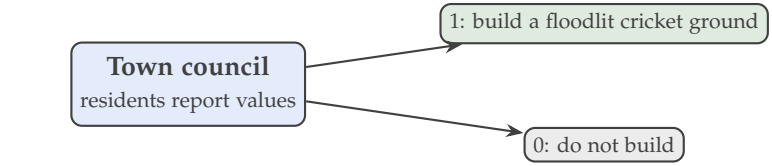
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Corollary

If the valuation space is sufficiently rich, no efficient mechanism can be both DSIC and BB.

Proof sketch of the second theorem

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  $0:$ project is not undertaken $\quad$ $1:$ project is undertaken

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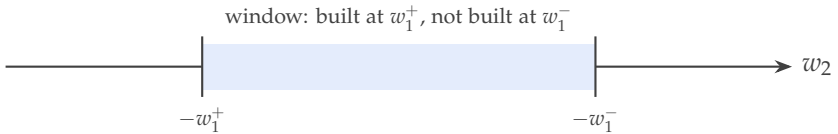
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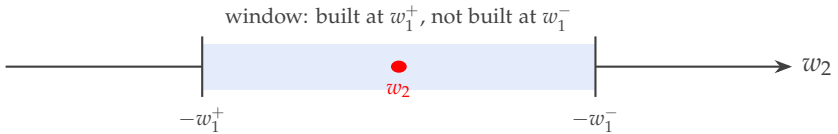
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- Eliminating $h_1(w_2)$, we get $w_2 = h_2(w_1^+) - h_2(w_1^-) - w_1^+$
- The RHS depends only on w_1, hence it is possible to alter w_2 slightly to retain the inequalities, but then the above equality cannot hold.

The Contradiction, in Pictures

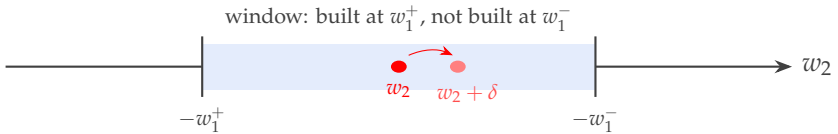


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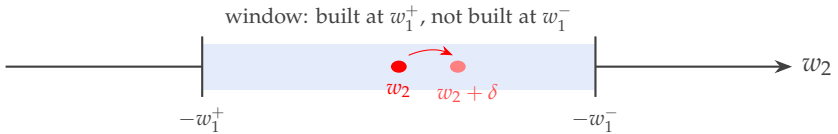
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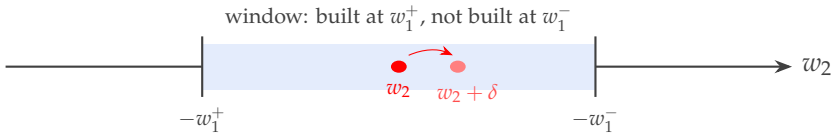
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- Nudge w_2 a little. Both decisions stay the same.
- The same equation must hold again. It cannot. Contradiction.

Chunk 4

dAGVA, Myerson–Satterthwaite, the Big Picture

(≈ 20–25 minutes)

Weakening DSIC for Budget Balance

Back to the cricket ground

The town still wants the efficient decision, and payments that add up to zero: residents pay each other, the council keeps nothing. Groves cannot do this. So we ask for less: truth-telling only *on average* (BIC).

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- Payment is given by (named after d'Aspremont, Gerard-Varet (1979), Arrow (1979)):

$$p_i^{dAGVA}(t) = \frac{1}{n-1} \sum_{j \neq i} \delta_j(t_j) - \delta_i(t_i)$$

dAGVA is BIC

- This payment implements the efficient allocation rule in Bayes Nash equilibrium

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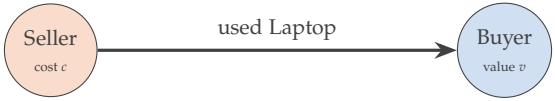
The dAGVA mechanism is efficient, BIC, and BB.

One More Wall: The Used-Laptop Trade

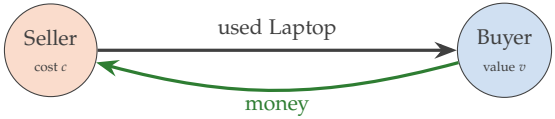




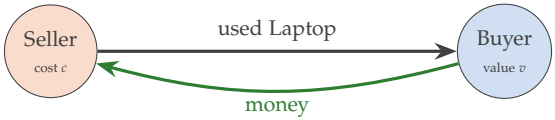
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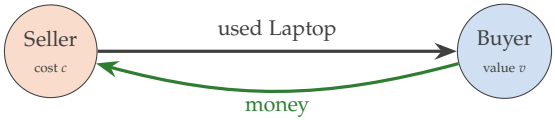


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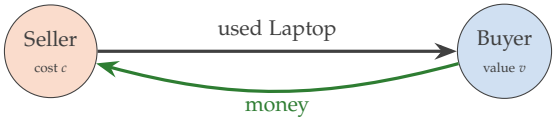
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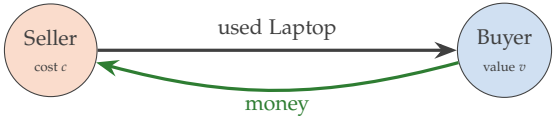
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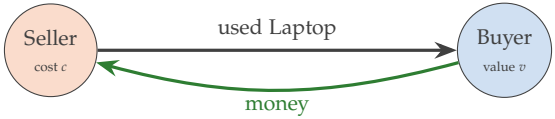


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- If the platform insists on BIC, efficiency, and IIR, payments cannot sum to zero. Someone outside must cover the gap.

Space of Mechanisms



Space of DSIC mechanisms

Figure: Space of Mechanisms 1

Space of Mechanisms



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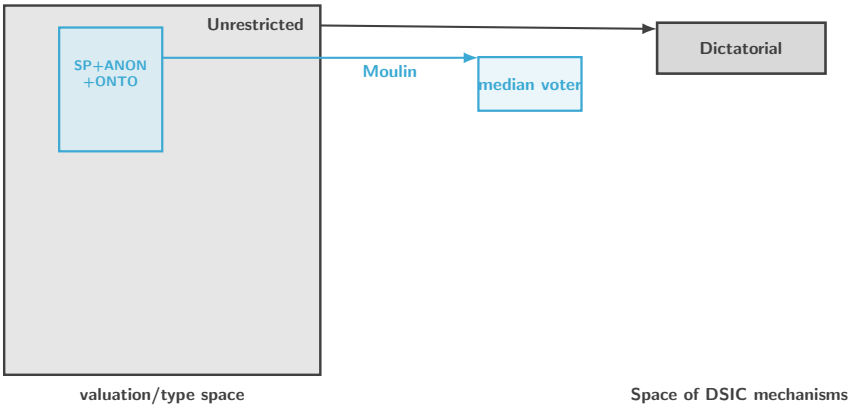


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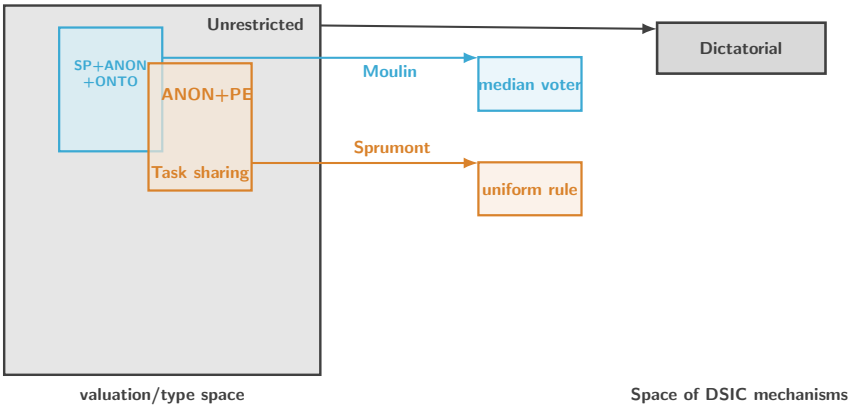


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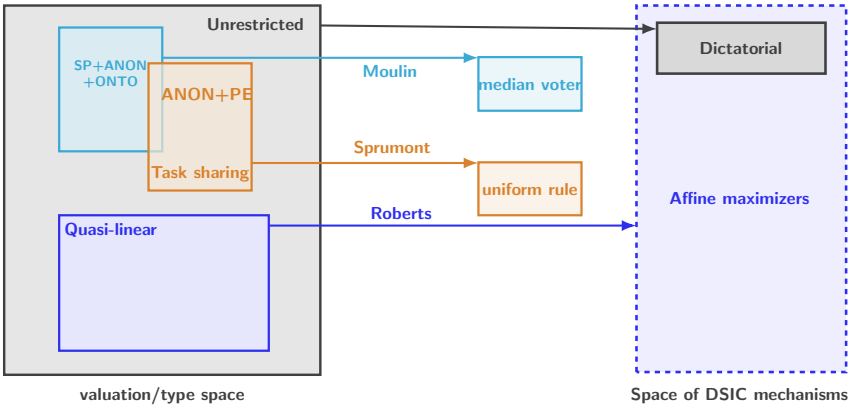


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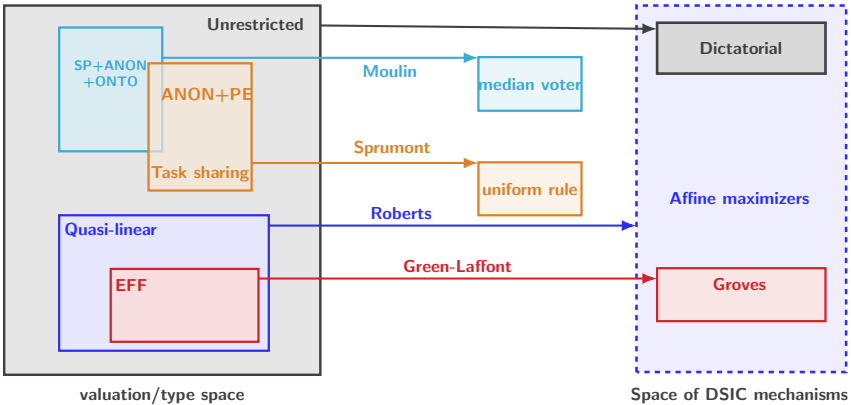


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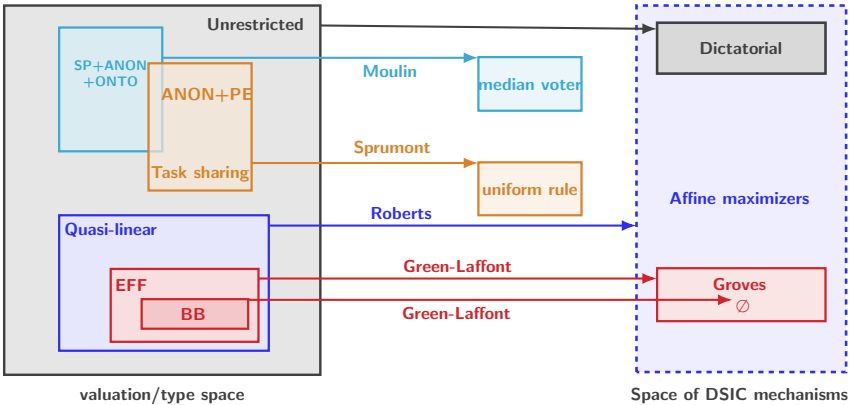


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Space of Mechanisms



Figure: Space of Mechanisms 2

Space of Mechanisms

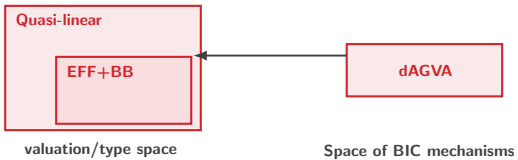


Figure: Space of Mechanisms 2

Space of Mechanisms

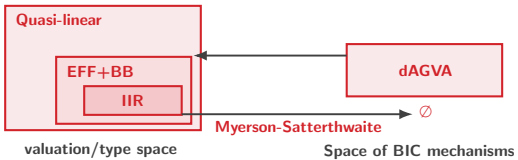


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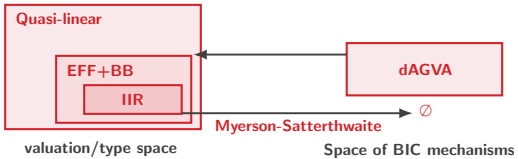


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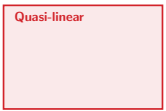


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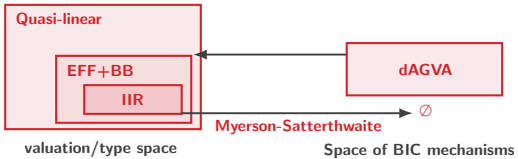


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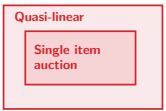


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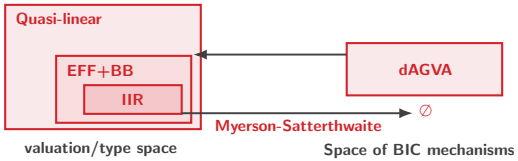


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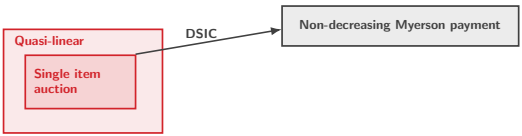


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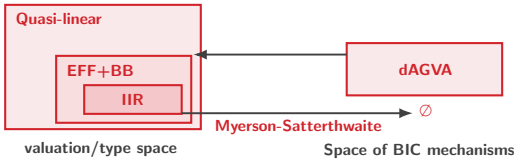


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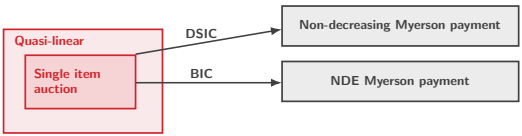


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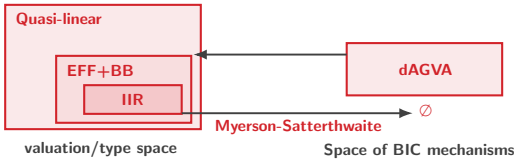


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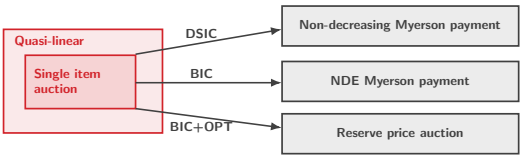


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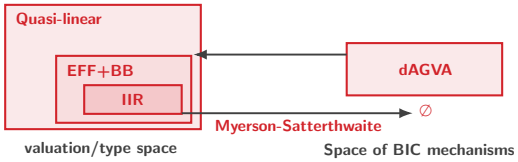


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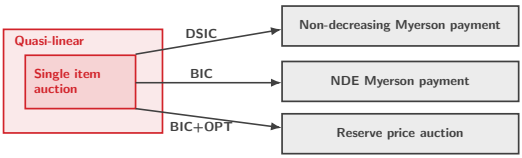


Figure: Space of Mechanisms 3

- DSIC $\Rightarrow$ BIC. Not the reverse.
- Efficient + DSIC = Groves. Efficient + BIC also includes dAGVA.
- Myerson's auctions live in BIC. They trade efficiency for revenue.



In-Class Quiz 2

Open the Google Form now.

Questions refer only to what was just discussed –
you cannot answer them without having been in class.

Lecture 24 and Week 12: What We Built

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End of Lecture 24

All the best for the exams!



भारतीय प्रौद्योगिकी संस्थान मुंबई

Indian Institute of Technology Bombay