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THE DIVISION PROBLEM WITH SINGLE-PEAKED PREFERENCES: A CHARACTERIZATION OF THE UNIFORM ALLOCATION RULE

BY YVES SPRUMONT [1]

1. INTRODUCTION

IMAGINE THAT A GROUP OF AGENTS participate in some production process. Each is to contribute some quantity of a homogenous input, say labor, to the business. The technology is given and the total amount of work that is needed is fixed. All agents have agreed that everyone would receive a quantity of output proportional to his work effort. In such a framework, preferences over the participation levels in the business are rather naturally single-peaked: each agent has an optimal share around which his utility decreases monotonically (this is a direct consequence of the assumption that preferences are strictly convex in the labor-output space). The optimal shares, however, may not be compatible: they might add up to more or less than one. How should we determine everyone's share in the business? This is what we call the "division problem with single-peaked preferences."

A slightly different version of the same problem is encountered in the literature on so-called fixed-price equilibria. Consider a two-good exchange economy. Suppose that at the current relative price, which is rigid, the net demands do not add up to zero. How should the transactions be determined? This problem is a little different from the previous one since (i) an agent's net demand may be any real number (while his "optimal share" had to be between zero and one) and (ii) the net trades must add up to zero (while the shares had to sum up to one).

However, if we impose the requirement that no agent "on the short side of the market" be rationed, we are back to the problem of dividing some fixed amount (demand or supply) among several agents (sellers or buyers) whose claims exceed that amount. The single-peakedness assumption is again quite natural: if an agent has strictly convex preferences, then his preferences over those bundles which belong to his budget line are single-peaked.

This paper is concerned with allocation rules for the division problem with single-peaked preferences. An allocation rule is a mapping that associates with each vector of (single-peaked) preferences some division of the amount to be shared.

Our first concern is that our rule be strategy-proof: no agent should have an incentive to misreport his preferences, no matter what the others do. Strategy-proofness is the strongest decentralizability property that a rule could possess; indeed every agent need only know his own preferences to compute his best choice. As such, it is very desirable. In the fixed-price literature for instance, the main proofs of existence of a general equilibrium with quantity rationing rely on it. Drèze's model (Drèze (1975)) cannot handle manipulable rationing schemes and the concept of "effective demand", central in Benassy's approach (Benassy (1982)), is satisfactory only in the case of nonmanipulable schemes. Without strategy-proofness, one is forced to an approach in terms of Nash equilibria, which involves substantial difficulties: see Grandmont (1977) for a discussion.

It is well known that strategy-proofness is hard to meet. A fundamental result in social choice theory, established by Gibbard (1973) and Satterthwaite (1975), states that every strategy-proof voting scheme must be dictatorial—provided that there are more than two alternatives at hand. Reasonable strategy-proof rules do exist, however, if appropriate

[1] I wish to thank H. Moulin for the many helpful discussions I had with him. I also benefited from comments by G. Weinrich and two referees.

restrictions are imposed on the preferences. One such restriction is precisely single-peakedness. When preferences are single-peaked, the Condorcet winner voting rule is strategy-proof. Conversely, every strategy-proof, efficient and anonymous voting scheme must be a mere variant of Condorcet's rule: see Moulin (1980) for details.

Several authors have shown that impossibility results similar to the Gibbard-Satterthwaite theorem hold true in specific economic environments. See Hurwicz (1972) and Dasgupta, Hammond, and Maskin (1979) for the case of an exchange economy with private goods and Satterthwaite and Sonnenschein (1981) for the case where public goods and production are introduced in the model.

Yet, the possibility of constructing "nice" strategy-proof allocation mechanisms in such environments when preferences are single-peaked, has not been much explored. Writers in the fixed-price literature are aware that strategy-proof rationing schemes do exist. The two focal examples are the "uniform rationing scheme" and the "queuing scheme" (see, e.g., Benassy (1982)). What characterizes these two rules, however, and what other rules—if any—are strategy-proof, is a question that has received little attention.

Besides strategy-proofness, we would like our rule to possess two additional properties: it should be efficient and fair.

In the present context, efficiency simply requires that if the optimal shares sum up to more (less) than one, then no agent should get more (less) than his optimal share. In models of exchange with quantity rationing, this corresponds to the familiar requirement that no agent on the short side of the market be rationed.

Fairness, on the other hand, certainly demands that our rule be anonymous. In addition, one might want that no agent ever prefers someone else's share to his own: this is the well-known property of "envy-freeness" first introduced by Foley (1967).

Our result is that the properties of strategy-proofness, efficiency, and anonymity together characterize a unique allocation rule. This rule is nothing but the adaptation of the uniform rationing scheme to the division problem: it gives to everyone his preferred share in the business, within the limits of an upper bound and a lower bound determined by the feasibility condition that the shares add up to one. Alternatively, the anonymity axiom may be replaced by envy-freeness.

This characterization theorem is proved in Section 3 (with a proof of the variant in the Appendix), after the formal model has been presented in Section 2. Concluding comments are gathered in Section 4.

2. THE MODEL

Given is $N=\{1,\dots,n\}$, a finite set of agents who must share one unit of some perfectly divisible good. The preferences of every agent $i \in N$ are represented by a complete preordering of $[0,1]$ denoted R_i. For all $x, y \in [0,1]$, xR_iy means that consuming a quantity x of the good is, from i's viewpoint, at least as good as consuming a quantity y. Strict preference will be denoted by P_i, indifference by I_i. We assume that i's preferences are continuous, that is, for each $x \in [0,1]$, $\{y \in [0,1] | yR_ix\}$ and $\{y \in [0,1] | xR_iy\}$ are closed sets.

Preferences are further restricted to be single-peaked and "strictly decreasing around their peak." That is to say, for each $i \in N$, R_i satisfies the following condition:

$$(1)\qquad \begin{cases} \text{There exists } x^* \in [0,1] \text{ such that for all } y, z \in [0,1]: \\ x^* < y < z \Rightarrow x^* P_i y P_i z, \\ z < y < x^* \Rightarrow x^* P_i y P_i z. \end{cases}$$

We call x^* the peak of R_i. To emphasize the dependence upon the preference preordering, we write $x^*(R_i)$. We let S denote the set of continuous preorderings of

$[0,1]$ satisfying (1). For any $x^* \in [0,1]$, $S(x^*)$ stands for the subset of those preferences in S whose peak is x^*. The symbol $R = (R_i)_{i \in N}$ denotes the vector of announced preferences, while R_{-j} stands for $(R_i)_{i \in N \setminus j} (j \in N)$.

An allocation rule associates a vector of shares with each vector of preferences. It is thus a function ϕ: $S^N \to [0,1]^N$ satisfying:

FEASIBILITY: *For all* $R \in S^N$, $\sum_{i \in N} \phi_i(R) = 1$.

Our three basic axioms are the following:

EFFICIENCY: *For all* $R \in S^N$,

$$\left\{ \sum_{i \in N} x^*(R_i) \leqslant 1 \right\} \Rightarrow \{ \phi_i(R) \geqslant x^*(R_i) \text{ for all } i \in N \},$$

$$\left\{ \sum_{i \in N} x^*(R_i) \geqslant 1 \right\} \Rightarrow \{ \phi_i(R) \leqslant x^*(R_i) \text{ for all } i \in N \}.$$

ANONYMITY: *For all permutations* π *of* N, *all* $R \in S^N$, $\phi_i(R^\pi) = \phi_{\pi(i)}(R)$, *where* $R^\pi = (R_{\pi(i)})_{i \in N}$.

STRATEGY-PROOFNESS: *For all* $i \in N, R \in S^N, R_i' \in S$, $\phi_i(R_i, R_{-i}) R_i \phi_i(R_i', R_{-i})$.

3. THE RESULTS

The axioms listed above characterize a unique rule that we shall call the uniform allocation rule (the term is borrowed from the fixed-price literature). This rule gives to each agent his preferred share in the business, as long as it falls within certain bounds which are the same for everyone and chosen so as to satisfy the feasibility condition.

DEFINITION: *The uniform allocation rule* ϕ^*: $S^N \to [0,1]^N$ is defined as follows: for all $i \in N$,

$$\phi_i^*(R) = \begin{cases} \min\{x^*(R_i), \lambda(R)\} & \text{if } \sum_{i \in N} x^*(R_i) \geqslant 1, \\ \max\{x^*(R_i), \mu(R)\} & \text{if } \sum_{i \in N} x^*(R_i) \leqslant 1, \end{cases}$$

where $\lambda(R)$ solves the equation $\sum_{i \in N} \min\{x^*(R_i), \lambda(R)\} = 1$ and $\mu(R)$ solves the equation $\sum_{i \in N} \max\{x^*(R_i), \mu(R)\} = 1$.

Our main result can be stated as follows:

THEOREM: *The allocation rule* ϕ: $S^N \to [0,1]^N$ *is efficient, anonymous, and strategy-proof if and only if* $\phi = \phi^*$.

The proof relies on two preliminary results that will be presented as lemmas. Our first lemma will show that every efficient and strategy-proof allocation rule possesses a certain property of continuity with respect to the announced preferences. In order to define this property in a rigorous way, we need a few definitions.

Consider a preference preordering $R_i \in S$ and some $x \in [0,1]$. Define the equivalent of x under R_i, denoted $e_{R_i}(x)$, as its "closest substitute on the other side of the peak of

R_i." Formally, letting $Y_{R_i}(x) = \{y \in [0,1] | y \geqslant x^*(R_i)$ if $x \leqslant x^*(R_i)$ and $y \leqslant x^*(R_i)$ if $x \geqslant x^*(R_i)\}$, $e_{R_i}(x)$ is given by the following two conditions: (i) $e_{R_i}(x) \in Y_{R_i}(x)$; (ii) there is no $y \in Y_{R_i}(x)$ such that $e_{R_i}(x) P_i y R_i x$. For each $R_i \in S$ and $x \in [0,1]$, it is clear that $e_{R_i}(x)$ exists and is unique. Moreover, any two preferences $R_1, R_2 \in S$ are the same if and only if $e_{R_1}(x) = e_{R_2}(x)$ for all $x \in [0,1]$. It is therefore meaningful to define the distance between two preferences $R_1, R_2 \in S$ as follows:

$$d(R_1, R_2) = \max_{x \in [0,1]} \left| e_{R_1}(x) - e_{R_2}(x) \right|.$$

The assumption of continuity of the preferences ensures that e_{R_1} and e_{R_2} are continuous functions, so that $d(R_1, R_2)$ is well defined.

A function f: $S \to [0,1]$ will be called continuous at $R \in S$ if and only if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $0 < d(R, R') < \delta$, $R' \in S$, then $|f(R) - f(R')| < \varepsilon$. The function f is continuous if and only if it is continuous at every $R \in S$.

Consider an allocation rule ϕ: $S^N \to [0,1]^N$. For all $i \in N$, $R_{-i} \in S^{N \setminus i}$, define the function $\phi_i^{R_{-i}}$: $S \to [0,1]$ by $\phi_i^{R_{-i}}(R_i) = \phi_i(R)$ for all $R_i \in S$. The property that we are interested in can be stated as follows:

Continuity:

(2) *For all* $i \in N$, $R_i \in S^{N \setminus i}$, *the function* $\phi_i^{R_{-i}}$ *is continuous*.

We are now ready to prove the following lemma.

Lemma 1: *If an allocation rule* ϕ: $S^N \to [0,1]^N$ *is efficient and strategy-proof, then it satisfies continuity*.

Proof: (a) We first show that every efficient and strategy-proof allocation rule satisfies the following property:

For all $i \in N$, $R_{-i} \in S^{N \setminus i}$ and $R_i, R_i' \in S$,

(3) $$\{x^*(R_i) = x^*(R_i')\} \Rightarrow \{\phi_i(R_i, R_{-i}) = \phi_i(R_i', R_{-i})\}.$$

Indeed, suppose that ϕ is an efficient allocation rule violating (3): $\exists i \in N$, $R_{-i} \in S^{N \setminus i}$ and $R_i, R_i' \in S$ such that $x^*(R_i) = x^*(R_i')$ and, say:

(4) $$\phi_i(R_i, R_{-i}) < \phi_i(R_i', R_{-i}).$$

Assume $\sum_{j \in N} x^*(R_j) \geqslant 1$ (the other case is similar). By efficiency, $\phi_i(R_i, R_{-i}) \leqslant x^*(R_i)$ and $\phi_i(R_i', R_{-i}) \leqslant x^*(R_i') = x^*(R_i)$. So we get from (4):

$$\phi_i(R_i, R_{-i}) < \phi_i(R_i', R_{-i}) \leqslant x^*(R_i') = x^*(R_i).$$

Clearly $\phi_i(R_i', R_{-i}) P_i \phi_i(R_i, R_{-i})$, violating strategy-proofness. We have established property (3).

(b) Assume now that ϕ: $S^N \to [0,1]^N$ is an efficient and strategy-proof allocation rule violating continuity, say:

$\exists R_1 \in S$, $R_{-1} \in S^{N \setminus 1}$ and $\varepsilon > 0$ such that for all $\delta > 0$ there is R_1' such that $0 < d(R_1, R_1') < \delta$ and $|\phi_1^{R_{-1}}(R_1) - \phi_1^{R_{-1}}(R_1')| \geqslant \varepsilon$.

We want to derive a contradiction.

Consider first the case $\sum_{i \in N} x^*(R_i) = 1$. By efficiency, $\phi_1^{R_{-1}}(R_1) = x^*(R_1)$. Suppose that R_1' is such that $x^*(R_1') \geqslant x^*(R_1)$ (the other case is similar). By efficiency again, $\phi_1^{R_{-1}}(R_1') \leqslant x^*(R_1')$. Choosing $\delta < \varepsilon$, we can rule out the eventuality that $x^*(R_1) = \phi_1^{R_{-1}}(R_1) < \phi_1^{R_{-1}}(R_1') \leqslant x^*(R_1')$ (indeed, $d(R_1, R_1') < \delta < \varepsilon$ implies $|x^*(R_1) - x^*(R_1')| < \varepsilon$, while on the other hand, $|\phi_1^{R_{-1}}(R_1)-\phi_1^{R_{-1}}(R_1')|\geqslant\varepsilon$). Therefore we must have:

$$\phi_1^{R_{-1}}(R_1')<\phi_1^{R_{-1}}(R_1)=x^*(R_1)\leqslant x^*(R_1').$$

Obviously $\phi_1^{R_{-1}}(R_1)P_1'\phi_1^{R_{-1}}(R_1')$, contradicting strategy-proofness.

Consider next the case $\sum_{i\in N}x^*(R_i)>1$ (the case $\sum_{i\in N}x^*(R_i)<1$ is treated in the same way). By taking δ small enough, we can make sure that R_1 satisfies $x^*(R_1')+\sum_{i\in N\setminus 1}x^*(R_i)>1$. Efficiency thus requires $\phi_1^{R_{-1}}(R_1)\leqslant x^*(R_1)$ and $\phi_1^{R_{-1}}(R_1')\leqslant x^*(R_1')$. There are 6 cases:

(i) $\phi_1^{R_{-1}}(R_1')<\phi_1^{R_{-1}}(R_1)\leqslant x^*(R_1')\leqslant x^*(R_1)$,

(ii) $\phi_1^{R_{-1}}(R_1)<\phi_1^{R_{-1}}(R_1')\leqslant x^*(R_1')\leqslant x^*(R_1)$,

(iii) $\phi_1^{R_{-1}}(R_1')\leqslant x^*(R_1')\leqslant \phi_1^{R_{-1}}(R_1)\leqslant x^*(R_1)$ with $\phi_1^{R_{-1}}(R_1')<\phi_1^{R_{-1}}(R_1)$,

and three other similar cases obtained by permuting the symbols R_1 and R_1'.

In case (i), $\phi_1^{R_{-1}}(R_1)P_1'\phi_1^{R_{-1}}(R_1')$ while in case (ii), $\phi_1^{R_{-1}}(R_1')P_1\phi_1^{R_{-1}}(R_1)$: ϕ is manipulable in both cases.

In case (iii) we can make the first inequality strict by choosing δ sufficiently small. If $\phi_1^{R_{-1}}(R_1)P_1'\phi_1^{R_{-1}}(R_1')$, then ϕ is not strategy-proof. On the other hand, if $\phi_1^{R_{-1}}(R_1')R_1'\phi_1^{R_{-1}}(R_1)$, we can construct $R_1''\in S$ such that

$$\phi_1^{R_{-1}}(R_1)P_1''\phi_1^{R_{-1}}(R_1') \qquad \text{and} \qquad x^*(R_1'')=x^*(R_1').$$

Now ϕ must satisfy property (3), i.e. $\phi_1^{R_{-1}}(R_1'')=\phi_1^{R_{-1}}(R_1')$. Therefore $\phi_1^{R_{-1}}(R_1)P_1''$ $\phi_1^{R_{-1}}(R_1'')$, contradicting strategy-proofness again and thereby completing the proof of Lemma 1. *Q.E.D.*

We shall prove next a fundamental one-agent result about strategy-proofness. Call a function $f\colon S\to[0,1]$ strategy-proof if and only if $f(R)Rf(R')$ for all $R,R'\in S$. For any three numbers $a,b,c\in[0,1]$, denote by $\text{med}\{a,b,c\}$ the median of these numbers. We have the following lemma.

LEMMA 2: *The function $f\colon S\to[0,1]$ is strategy-proof and continuous if and only if there exist two real numbers a and b, $0\leqslant a\leqslant b\leqslant 1$, such that $f(R)=\text{med}\{a,b,x^*(R)\}$ for all $R\in S$.*

PROOF: The "if" part is just a matter of checking. We prove here the converse statement.

Step 1—We show that if f is strategy-proof, then $f(S)$ is closed. Assume not. Then there exists some sequence $\{x_n\}\subset f(S)$ with $x_n\to x_0\notin f(S)$. Consider then a preference $R\in S$ with $x^*(R)=x^0$. Since $x^0\notin f(S)$, $f(R)\neq x^0$. But we can always find some $x'\in f(S)$ that will be closer to x^0 than $f(R)$ is. Hence by reporting some preference $R'\in S(x')$, our agent ends up better off: $f(R')Pf(R)$. This contradicts strategy-proofness.

Step 2—Since $f(S)$ is closed, one and only one of the following statements must be true:

(i) $f(S)=[a,b]$ for some $a,b\in[0,1]$.

(ii) $\exists(a,b)\subset[0,1]\setminus f(S)$ with $a,b\in f(S)$.

Consider case (i) first. If $x^*(R)\in[a,b]$, it is clear that $f(R)=x^*(R)$, for otherwise the agent could report some preference $R'\in S$ such that $f(R')=x^*(R)$ and end up better off. If $x^*(R)\in[0,a)$, we must have $f(R)=a$, since otherwise the agent could report $R'\in S$ with $f(R')=a$. Similarly $f(R)=b$ if $x^*(R)\in(b,1]$. Therefore $f(R)=\text{med}\{a,b,x^*(R)\}$.

Step 3—Consider now case (ii). Let thus $(a,b)\subset[0,1]\setminus f(S)$ with $a,b\in f(S)$. We will show that this contradicts our assumption that f is continuous.

Choose $R \in S$ such that $x^*(R) \in (a,b)$ and aIb. Since f is strategy-proof, either $f(R)=a$ or $f(R)=b$. Assume $f(R)=a$ (the case $f(R)=b$ is similar). We now construct a preference R_θ in S that is arbitrarily close to R but is such that $bP_\theta a$. Define R_θ to be the unique preference preordering in S satisfying:

(i) For all $x \in [0, x^*(R)]$, $e_{R_\theta}(x) = \min\{(1+\theta|e_R(x) - x^*(R)|)e_R(x), 1\}$.

(ii) For all $x \in [x^*(R), 1]$, $e_{R_\theta}(x) = y \Leftrightarrow y$ is the highest number in $[0, x^*(R)]$ such that $e_{R_\theta}(y) = x$.

Observe that $x^*(R_\theta) = x^*(R)$. Since $bP_\theta a$, Strategy-proofness requires $f(R_\theta) = b$. Next pick ε such that $0 < \varepsilon < b - a$. For any $\delta > 0$, we can make $d(R, R_\theta) < \delta$ by making θ arbitrarily small. However, $|f(R) - f(R_\theta)| = b - a > \varepsilon$. Hence f is not continuous at R, which is the desired contradiction. *Q.E.D.*

Notice that the continuity assumption is necessary. Indeed, consider for instance the function f: $S \to [0,1]$ defined as follows:

$$f(R) = \begin{cases} 0 & \text{if } 0P1, \\ 1 & \text{if } 1R0. \end{cases}$$

This discontinuous function is clearly strategy-proof but cannot be written as a median function as in Lemma 2. There are many other examples.

We now turn to the proof of our characterization theorem.

PROOF OF THE THEOREM: We first show that there is at most one allocation rule ϕ compatible with the axioms stated in the theorem.

Combining Lemmas 1 and 2, we can write for all $i \in N, R \in S^N$: $\phi_i(R) = \operatorname{med}\{a_i(R_{-i}), b_i(R_{-i}), x^*(R_i)\}$, with $0 \leqslant a_i(R_{-i}) \leqslant b_i(R_{-i}) \leqslant 1$ for all R_{-i}.

We claim that anonymity forces $a_i = a$ and $b_i = b$ for all $i \in N$.

To prove this, we first show that for all $i \in N$, the functions a_i and b_i are symmetric in their arguments. Fix $i \in N, R_{-i} \in S^{N \setminus i}$, and let τ be a permutation of $N \setminus i$. Define π to be the permutation of N such that $\pi(i) = i$ and $\pi(j) = \tau(j)$ for all $j \in N \setminus i$. By anonymity, $\phi_i(R^\pi) = \phi_i(R)$, which implies that

$$\begin{aligned} &\operatorname{med}\{a_i(R_{\tau(1)}, \dots, R_{\tau(i-1)}, R_{\pi(i+1)}, \dots, R_{\tau(n)}),\\ &\quad b_i(R_{\tau(1)}, \dots, R_{\tau(i-1)}, R_{\tau(i+1)}, \dots, R_{\tau(n)}), x^*(R_i)\}\\ &\quad = \operatorname{med}\{a_i(R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n),\\ &\qquad b_i(R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n), x^*(R_i)\}. \end{aligned}$$

Choosing R_i such that $x^*(R_i) = 0$ yields the symmetry of a_i and choosing R_i such that $x^*(R_i) = 1$ yields the symmetry of b_i.

To see that the functions a_i and b_i in fact do not depend on i, pick $i_1, i_2 \in N$ and a permutation π of N such that $\pi(i_1) = i_2$, $\pi(i_2) = i_1$ and $\pi(j) = j$ for all $j \neq i_1, i_2$. Then $\phi_{i_1}(R^\pi) = \phi_{i_2}(R)$ for all $R \in S^N$ implies that

$$\begin{aligned} &\operatorname{med}\{a_{i_1}(R_i, \dots, R_{i_1-1}, R_{i_1+1}, \dots, R_{i_2-1}, R_{i_1}, R_{i_2+1}, \dots, R_n),\\ &\quad b_{i_1}(R_1, \dots, R_{i_1-1}, R_{i_1+1}, \dots, R_{i_2-1}, R_{i_1}, R_{i_2+1}, \dots, R_n), x^*(R_{i_2})\}\\ &\quad = \operatorname{med}\{a_{i_2}(R_1, \dots, R_{i_1-1}, R_{i_1}, R_{i_1+1}, \dots, R_{i_2-1}, R_{i_2+1}, \dots, R_n),\\ &\qquad b_{i_2}(R_1, \dots, R_{i_1-1}, R_{i_1}, R_{i_1+1}, \dots, R_{i_2-1}, R_{i_2+1}, \dots, R_n), x^*(R_{i_2})\} \end{aligned}$$

for all $R \in S^N$. Choose R such that $x^*(R_{i_2}) = 0$ and invoke the symmetry of a_{i_1}, a_{i_2} to conclude $a_{i_1} = a_{i_2}$. Choose R such that $x^*(R_{i_2}) = 1$ and use the symmetry of b_{i_1}, b_{i_2} to get $b_{i_1} = b_{i_2}$. Since i_1 and i_2 were chosen arbitrarily, we are done.

So we get:

$$\text{(5)} \qquad \phi_i(R) = \operatorname{med}\{a(R_{-i}), b(R_{-i}), x^*(R_i)\} \qquad \text{for all} \quad i \in N, R \in S^N$$

with $0 \leqslant a \leqslant b \leqslant 1$.

By feasibility we have:

$$\sum_{i \in N} \operatorname{med}\{a(R_{-i}), b(R_{-i}), x^*(R_i)\} = 1 \qquad \text{for all} \quad R \in S^N.$$

We first show that the function b is unique. This will be established via a recursive argument. By anonymity, the value of b can be computed when all $n-1$ components of R_{-i} are identical. We may then keep $n-2$ of these components unchanged and use the feasibility condition to compute the value of b as the last component varies. Repeating the procedure determines the function b.

Formally, fix $R_1 = \cdots = R_m = R^1 \in S(1)$, where $1 \leqslant m \leqslant n-1$. Then,

$$\underbrace{b\Big(\underbrace{R^1, \ldots, R^1}_{m-1}, R_{m+1}, \ldots, R_n\Big) + \cdots + b\Big(\underbrace{R^1, \ldots, R^1}_{m-1}, R_{m+1}, \ldots, R_n\Big)}_{m \text{ times}}$$

$$+ \operatorname{med}\Big\{a\Big(\underbrace{R^1, \ldots, R^1}_{m}, R_{m+2}, \ldots, R_n\Big), b\Big(\underbrace{R^1, \ldots, R^1}_{m}, R_{m+2}, \ldots, R_n\Big), x^*(R_{m+1})\Big\}$$

$$+ \operatorname{med}\Big\{a\Big(\underbrace{R^1, \ldots, R^1}_{m}, R_{m+1}, \ldots, R_{n-1}\Big), b\Big(\underbrace{R^1, \ldots, R^1}_{m}, R_{m+1}, \ldots, R_{n-1}\Big), x^*(R_n)\Big\}$$

$$= 1.$$

From this we get, for $1 \leqslant m \leqslant n-1$:

$$\text{(6)} \qquad b\Big(\underbrace{R^1, \ldots, R^1}_{m-1}, R_{m+1}, \ldots, R_n\Big)$$

$$= \frac{1}{m}\left[1 - \sum_{j=m+1}^{n} \operatorname{med}\Big\{a\Big(\underbrace{R^1, \ldots, R^1}_{m}, R_{m+1}, \ldots, R_{j-1}, R_{j+1}, \ldots, R_n\Big), b\Big(\underbrace{R^1, \ldots, R^1}_{m}, R_{m+1}, \ldots, R_{j-1}, R_{j+1}, \ldots, R_n\Big), x^*(R_j)\Big\}\right].$$

Now we claim that for all $j = m+1, \ldots, n$ and all $R_{m+1}, \ldots, R_{j-1}, R_{j+1}, \ldots, R_n$, $a(R^1, \ldots, R^1, R_{m+1}, \ldots, R_{j-1}, R_{j+1}, \ldots, R_n) = 0$. Indeed, let $R = (R^1, \ldots, R^1, R_{m+1}, \ldots, R_{j-1}, R^0, R_{j+1}, \ldots, R_n)$, where $R^0 \in S(0)$. By efficiency, $\phi_j(R) = \operatorname{med}\{a(R_{-j}), b(R_{-j}), 0\} = a(R_{-j}) \leqslant x^*(R_j) = 0$. Since ϕ_j cannot take on negative values, the claim follows. Therefore (6) can be rewritten:

$$(7)\qquad b\Big(\underbrace{R^1,\dots,R^1}_{m-1},R_{m+1},\dots,R_n\Big)$$
$$=\frac{1}{m}\left[1-\sum_{j=m+1}^{n}\min\Big\{b\Big(\underbrace{R^1,\dots,R^1}_{m},R_{m+1},\dots,R_{j-1},R_{j+1},\dots,R_n\Big),x^*(R_j)\Big\}\right].$$

Furthermore, setting $R_1=\cdots=R_n=R^1$ yields $b(R^1,\dots,R^1)=1/n$. So $b(R_{-i})$ is uniquely determined by the recursive formula (7) for any $R_{-i}\in S^{N\setminus i}$.

Next we show that a is unique. Fix $R_1=\cdots=R_m=R^0\in S(0)$, $1\leqslant m\leqslant n-1$, and derive from the feasibility condition the following recursive formula:

$$(8)\qquad a\Big(\underbrace{R^0,\dots,R^0}_{m-1},R_{m+1},\dots,R_n\Big)$$
$$=\frac{1}{m}\left[1-\sum_{j=m+1}^{n}\operatorname{med}\Big\{a\Big(\underbrace{R^0,\dots,R^0}_{m},R_{m+1},\dots,R_{j-1},R_{j+1},\dots,R_n\Big),\right.$$
$$\left.b\Big(\underbrace{R^0,\dots,R^0}_{m},R_{m+1},\dots,R_{j-1},R_{j+1},\dots,R_n\Big),x^*(R_j)\Big\}\right]$$

with initial value $a(R^0,\dots,R^0)=1/n$ obtained by setting $R_1=\cdots=R_n=R^0$.

Since b is unique, formula (8) establishes that a is unique as well. It then follows from (5) that ϕ must be unique. This completes the first part of the proof.

The second part is to show that ϕ^* is an allocation rule and that it satisfies our axioms. Feasibility, efficiency, and anonymity are obvious. We check strategy-proofness. Consider an arbitrary $i\in N$ with true preference R_i^*. The case $x^*(R_i^*)+\sum_{j\in N\setminus i}x^*(R_j)=1$ poses no problem. Assume next $x^*(R_i^*)+\sum_{j\in N\setminus i}x^*(R_j)>1$. Agent i might have an incentive to lie only if $x^*(R_i^*)>\lambda(R_i^*,R_{-i})=\phi_i^*(R_i^*,R_{-i})$. If he reports some R_i with $x^*(R_i)\geqslant x^*(R_i^*)$, he gets $\lambda(R_i,R_{-i})=\lambda(R_i^*,R_{-i})$: there is no improvement. If he reports R_i with $x^*(R_i)<x^*(R_i^*)$, there are two cases:

(i) If $\sum_{j\in N}x^*(R_j)\geqslant 1$, then i gets $\min\{x^*(R_i),\lambda(R)\}$. But in order to have $\lambda(R)>\lambda(R_i^*,R_{-i})$, he should report R_i such that $x^*(R_i)<\lambda(R_i^*,R_{-i})$. He would thus be worse off.

(ii) In the second case, $\sum_{j\in N}x^*(R_j)<1$. Agent i receives $\max\{x^*(R_i),\mu(R)\}$. This is not more than $\min\{x^*(R_i^*),\lambda(R_i^*,R_{-i})\}=\phi_i^*(R_i^*,R_{-i})$ for otherwise the two equations

$$\begin{cases}\min\{x^*(R_i^*),\lambda(R_i^*,R_{-i})\}+\sum_{j\in N\setminus i}\min\{x^*(R_j),\lambda(R_i^*,R_{-i})\}=1,\\ \max\{x^*(R_i),\mu(R)\}+\sum_{j\in N\setminus i}\max\{x^*(R_j),\mu(R)\}=1,\end{cases}$$

would imply that for some $j\in N\setminus i$, $\max\{x^*(R_j),\mu(R)\}<\min\{x^*(R_j),\lambda(R_i^*,R_{-i})\}$, hence $x^*(R_j)<x^*(R_j)$. So once again i cannot benefit from misreporting.

A similar argument holds when $x^*(R_i^*)+\sum_{j\in N\setminus i}x^*(R_j)<1$, thus completing the proof of the theorem. *Q.E.D.*

Observe that if we drop any of the three properties required by the theorem, new allocation rules emerge.

The egalitarian rule $\phi_i^e(R) = 1/n$ for all $i \in N$ is strategy-proof and anonymous but clearly inefficient.

The proportional rule $\phi_i^p(R) = x^*(R_i)/\sum_{j \in N} x^*(R_j)$ is anonymous and efficient but not strategy-proof.

Finally, the queuing rule ϕ^q defined by $\phi_1^q(R) = x^*(R_1)$ and $\phi_i^q(R) = \min\{x^*(R_i), 1 - \sum_{j<i} \phi_j^q(R)\}$ for $1 < i \leqslant n$ in the case $\sum_{i \in N} x^*(R_i) \geqslant 1$, and by $\phi_i^q(R) = x^*(R_i)$ for $i \leqslant n-1$ and $\phi_n^q(R) = 1 - \sum_{j \leqslant n-1} x^*(R_j)$ in the case $\sum_{i \in N} x^*(R_i) < 1$, is strategy-proof and efficient but not anonymous.

The proof of the theorem may convey the impression that the bounds $\lambda(R)$ and $\mu(R)$ in the definition of the uniform allocation rule are common to all agents because of the anonymity axiom. It turns out, however, that anonymity is not necessary to force $\phi = \phi^*$: it can be replaced by envy-freeness:

ENVY-FREENESS: *For all* $R \in S^N$ *and* $i, j \in N, \phi_i(R) R_i \phi_j(R)$.

This leads to the following variant of our theorem, whose proof is found in the Appendix:

THEOREM BIS: *The allocation rule* ϕ: $S^N \to [0,1]^N$ *is efficient, envy-free, and strategy-proof if and only if* $\phi = \phi^*$.

This new characterization is again tight. The egalitarian rule is an example of an envy-free and strategy-proof rule which violates efficiency, while the queuing method is efficient and strategy-proof but generates envy.

One can also construct methods which are efficient and envy-free, though manipulable. Here is an example. Given the reported preferences R, select those allocations that are feasible, efficient, and don't generate envy. Among those, choose one that maximizes player one's share. Let $\phi_1(R)$ be that maximal value (if it does not exist, then let $\phi_1 = \phi_1^*(R)$). Next, among those allocations which are feasible, efficient, envy-free, and give $\phi_1(R)$ to player one, choose one that maximizes player two's share. And so on. It is easy to check that this rule is indeed manipulable.

4. CONCLUSIONS

This paper has explored the problem of dividing some fixed amount of a good for which individuals have single-peaked preferences. We showed that the requirements of strategy-proofness, efficiency, and anonymity point to a unique rule, namely the uniform allocation rule: everyone gets what he wants within the limits of a lower bound and an upper bound that are common to all agents. This remains true if the anonymity axiom is replaced by envy-freeness.

As strategy-proofness is the key axiom, the reader may wonder what can be done without the efficiency and fairness conditions. Preliminary results indicate that the class of all strategy-proof allocation rules is extremely rich. Although partial results may easily be derived for the two-player case, we have not been able to reach any general characterization.

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APPENDIX

Proof of Theorem BIS: The basic structure of the proof is the same as the one of our main theorem. There are, however, a few complications.

We show first that there is at most one allocation rule compatible with the axioms of efficiency, envy-freeness, and strategy-proofness. Let $N' \subseteq N$ and $R \in S^{N'}$. For any $T \subset N'$, let R_T denote the restriction of R to T, i.e. the subvector obtained from R by deleting the components corresponding to those i not in T. Let $R^\alpha \in S, \alpha \in [0,1]$, be some preference with peak at α and let M, T form a partition of N'. The notation (R^α_M, R_T) will be used to denote any vector $R \in S^{N'}$ such that $R_i = R^\alpha$ for $i \in M$.

By Lemmas 1 and 2, if ϕ is strategy-proof and efficient, then for all $i \in N, R \in S^N$,

$$\phi_i(R) = \operatorname{med}\{a_i(R_{-i}), b_i(R_{-i}), x^*(R_i)\} \quad \text{with} \quad 0 \leqslant a_i \leqslant b_i \leqslant 1. \tag{15}$$

By feasibility,

$$\sum_{i \in N} \operatorname{med}\{a_i(R_{-i}), b_i(R_{-i}), x^*(R_i)\} = 1. \tag{16}$$

Let now $M \subset N$ be a subset of m agents, with $1 \leqslant m \leqslant n-1$. Pick some $R^1 \in S(1)$ and fix $R_i = R^1$ for all $i \in M$. From (16),

$$\sum_{i \in M} b_i\left(R^1_{M \setminus i}, R_{N \setminus M}\right) + \sum_{j \in N \setminus M} \operatorname{med}\left\{a_j\left(R^1_M, R_{N \setminus M \cup j}\right), b_j\left(R^1_M, R_{N \setminus M \cup j}\right), x^*\left(R_j\right)\right\} = 1.$$

By efficiency, $a_j(R^1_M, R_{N \setminus M \cup j}) = 0$ for all $j \in N \setminus M$, thus leading to:

$$\sum_{i \in M} b_i\left(R^1_{M \setminus i}, R_{N \setminus M}\right) = 1 - \sum_{j \in N \setminus M} \min\left\{b_j\left(R^1_M, R_{N \setminus M \cup j}\right), x^*\left(R_j\right)\right\}.$$

Next, since all $i \in N$ have the same preferences, envy-freeness imposes equal treatment: $\phi_i(R^1_M, R_{N \setminus M})$ must be identical for all $i \in N$. Therefore,

$$b_i\left(R^1_{M \setminus i}, R_{N \setminus M}\right) = \frac{1}{m}\left[1 - \sum_{j \in N \setminus M} \min\left\{b_j\left(R^1_M, R_{N \setminus M \cup j}\right), x^*\left(R_j\right)\right\}\right]$$

for all $i \in M$, all M such that $1 \leqslant m \leqslant n-1$.

Moreover, setting $R = (R^1, \ldots, R^1)$ and using envy-freeness again, we get the initial value $b_i(R^1, \ldots, R^1) = 1/n$ for all $i \in N$. We conclude that for all $i \in N$, b_i is unique.

A similar reasoning shows that a_i is also unique for each i. Taking (15) into account, this completes the first part of the proof.

It remains to show that ϕ^* is envy-free. Suppose $\sum_{i \in N} x^*(R_i) \geqslant 1$ (the other case is handled in the same way). Any arbitrary $i \in N$ gets either $x^*(R_i)$—in which case he does not envy anybody—or $\lambda(R) < x^*(R_i)$. Any other agent j gets either $x^*(R_j) < \lambda(R)$, or $\lambda(R)$, which cannot generate i's envy. *Q.E.D.*

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