Algorithmic representation: TTC with endowments

Initialization: Fix an initial endowment a^*

The mechanism maintains the remaining set of objects M^k and the remaining agents N^k in every step k of the mechanism.

Step 1: $M' = M$ and $N' = N$

construct directed graph where every agent points to its most favorite remaining house.

Step 2: Find a cycle in this directed graph (Guaranteed to exist since there are n nodes and n edges). Allocate the houses along this cycle.

Step 3: Remove the allocated agents and houses. Update M^k, N^k accordingly. Repeat Step 1 onwards.

Stop when no more nodes left.

Theorem: TTC with fixed endowment is strategyproof and efficient.

$\boxed{\text{Strategyproofness proof:}}$ Consider an agent i. Suppose if agent i is truthful, she gets her assigned house in round k. The house is her favorite house among the remaining houses in round k.

Two cases can occur if she misreports.

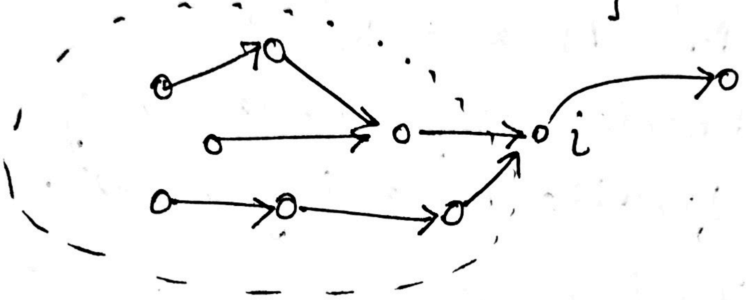
Case 1: Agent i gets a house after round k.

But that is no better than getting the house in round k because she was getting her favorite house in R_k.

Case 2: By misreporting she gets a house in a round $r < k$.

Define $\pi_{ir} = \{$set of nodes that have a directed path ~~to~~ toward i in round $r\}$.

This set only grows with r



The only way i can get assigned a house in round r is if i points to some house owned by an agent in π_{ir}. (Other agents are not changing their actions, therefore there is no cycle if i does not create one). Suppose $i \to i' \in \pi_{ir}$

Point to note: consider the path

$$i' \to i_1 \to i_2 \to \cdots\ i_{\ell} \to i$$

each of these agents are pointing to their most favorite houses. If i does not point to i' in round r then in round k all these options will still be available. These houses won't get allocated any way till k. The fact that agent i's true preference in round k is none of these, ~~the~~ implies that the house i gets in round k is better than all of these. So, agent i gets an inferior house if it points to i'.

Efficiency proof: Proof by contradiction

Suppose ~~house~~ ~~a(i) is given by~~ ~~allocat~~ allocation a is done by TTC and a' ~~by some other~~ is some other allocation s.t. $a' \neq a$ and $a'(i)\, P_i\, a(i)$ or $a'(i) = a(i)$ for all $i \in N$, i.e., every agent gets a better house or the same house than TTC in a'.

Suppose, i is the agent who gets the first house that is different from TTC under a'. Therefore $a'(i) \neq a(i)$, and by assumption $a'(i)\, P_i\, a(i)$.

Since the houses allocated before i got its allocation under TTC are exactly the same, $a'(i)$ was available when $a(i)$ was assigned to i. But that is impossible under TTC: It always gives the most preferred house at that round, never a less preferred one. This is a contradiction.

Observation: TTC is NOT serial dictatorship

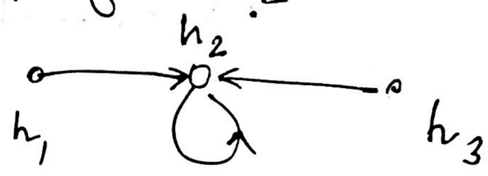
Example (where allocations under TTC and SD are different)

SD order $\sigma = (1, 2, 3)$

Case 1: Suppose each player prefers h_1 the most, Then $1 \to h_1$,

Counterexample

$\boxed{\text{Case 2:}}$ Suppose each player prefers h_2 the most under SD, 1 would have got h_2, but under TTC player 2 gets h_2.



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Stability in House Allocation with initial endowments

Can a subgroup deviate and get a better house allocation than a proposed one?

We saw that efficiency guarantees you can't as the entire group, but what about smaller groups?

Example:

		1	2	3	4	5	6	
initial endowment	a^*	h_1	h_3	h_2	h_4	h_5	h_6	
consider allocation		h_1	h_2	h_3	h_4	h_5	h_6	allocation --- a

P_3	P_4
h_4	h_2
$\vdots$	$\vdots$
h_3	h_4

Players $\{3,4\}$ can reject the proposed allocation and exchange their houses to get h_4 and h_2 respectively that they prefer more than h_3 and h_4.

Allocation a is not "stable" since the group $\{3,4\}$ blocks such an allocation.

Formal definitions

a^* : The matching reflecting the initial endowment

a^S : denotes the matching of the agents in $S \subseteq N$ over the houses owned by the agents in S.

Blocking coalition:

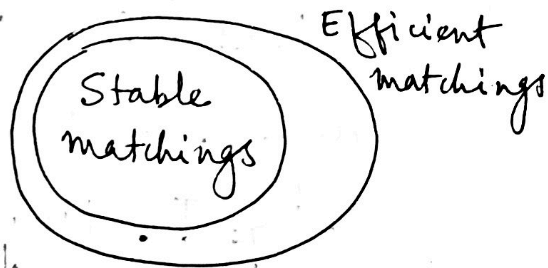
A coalition $S \subseteq N$, $S \neq \phi$, can block a matching a at a preference profile P if $\exists$ a matching a^S s.t. either $a^S(i) \, P_i \, a(i)$ or $a^S(i) = a(i)$ $\forall i \in S$ and $a^S(j) \, P_j \, a(j)$ for some $j \in S$.

Core A matching is in the core at a profile P if no coalition can block it at P.

Stability An SCF f is stable if $f(P)$ is in the core of P, $\forall P$.

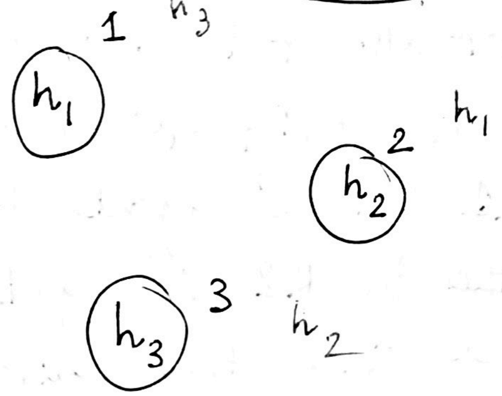
Question: What is the relationship between stability and efficiency?

Stability ensures no blocking coalition, for every size of coalitions, it trivially includes the grand coalition which implies efficiency.



Counterexample: Efficient but not stable

P_1	P_2	P_3
h_1	h_1	h_2
h_2	h_2	h_1
h_3	h_3	h_3



$a(1) = h_3$, $a(2) = h_1$, $a(3) = h_2$

2 and 3 get top choices, no other allocation can improve them, ~~1 can't do much~~ But 1 can just retain his house.

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Theorem: The TTC mechanism is stable. Moreover there is a unique core matching for every preference profile.

$\boxed{\text{Proof:}}$ Suppose TTC is not stable. $\exists$ some ~~allocation~~ coalition $S \subseteq N$ ~~s.t.~~ ~~a^S block~~ that blocks a^{TTC} at some profile P.
That means, $\exists$ some allocation a^S (involving only the agents and houses of agents in S) s.t.

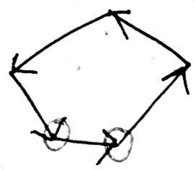
$$a^S(i)\ P_i\ a(i) \text{ or } a^S(i) = a(i) \quad \forall i \in S \text{ and}$$

$\exists j \in S$ s.t. $a^S(j)\ P_j\ a(j)$. Therefore the set

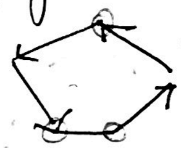
$$T = \{ j \in S : a^S(j)\ P_j\ a(j) \} \neq \phi \quad \text{(can't be empty).}$$

Consider the agents from S that got allocated in round 1 of TTC; call them S', they got their most favorite

allocated in R1 of TTC



S'



S^2

allocated in R2 of TTC.

house. So, they can't be improved, hence ~~$S' \notin T$~~ $S' \notin T$. Now, consider S^2, the agents from S that got allocated in round 2. These agents may have their most preferred from the houses allocated from R1 but S' agents do not prefer their houses. ~~The~~ In R2, S^2 agents get their next best houses and since they can't improve over it (as S' agents will not deviate with them) $S^2 \notin T$. Using induction we find that $S \notin T$ hence $T = \phi$, which is a contradiction.