Two sided Matching

Recall: Matching that is "not good": has blocking pairs

$w_1 > w_2 > w_3$ m_1 — w_1 $m_3 > m_2 > m_1$

$w_2 > w_1 > w_3$ m_2 - - - - - w_2 $m_2 > m_3 > m_1$

$w_1 > w_3 > w_2$ m_3 —— w_3 $m_3 > m_2 > m_1$

(m_2, w_2) forms a blocking pair

How should we create a stable match? Is there an algorithm?
need 4 men and women to properly show the steps

men proposing deferred acceptance (Gale-Shapley 62)

$w_1 > w_4 > w_2 > w_3$ m_1 w_1 $m_2 > m_1 > m_4 > m_3$

$w_3 > w_2 > w_4 > w_1$ m_2 w_2 $m_1 > m_2 > m_3 > m_4$

$w_1 > w_2 > w_3 > w_4$ m_3 w_3 $m_3 > m_1 > m_2 > m_4$

$w_2 > w_1 > w_4 > w_3$ m_4 w_4 $m_4 > m_2 > m_1 > m_3$

Round 1: each ~~agent~~ man approaches ~~their~~ his best woman that has not rejected him.

$m_1 \to w_1$, $m_2 \to w_3$, $m_3 \to w_1$, $m_4 \to w_2$

each woman keeps her best man and reject the rest
hence w_1 retains m_1 and rejects m_3. All others tentatively matched

Round 2: Only m_3 is unmatched. He approaches his next best woman, i.e., w_2. w_2 was matched to m_4, but she prefers $m_3 > m_4$. Hence w_2 accepts m_3 and rejects m_4.

Round 3: Only m_4 is unmatched. He approaches next best, w_1. w_1 is currently matched to m_1 which she prefers

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more than m_4. So, she rejects m_4.

Round 4: m_4 approaches W_4. W_4 has not got any offer so far. So, accepts.

Final allocation/matching

$m_1 - W_1$, $m_2 - W_3$, $m_3 - W_2$, $m_4 - W_4$

Claim 1: DA algorithm always terminate in poly-time.

- At least one proposal is made in every round.
- Each man can make at most n proposals (n = # of men = # of women). Hence together $O(n^2)$ operations (proposals + comparisons) are possible
[Round 1: n proposals and at most $(n-1)$ comparisons,
Round 2: ~~se~~ at most $(n-1)$ proposals and $(n-2)$ comparisons,
...
- No man proposes ~~to~~ a woman that rejected him in a previous round. Hence no proposals are repeated.]
- DA algorithm converges in $O(n^2)$ time.

Claim 2: DA algorithm always returns a perfect matching.

- No woman is matched to more than one man.
- Every woman is either tentatively matched, because she got only one proposal
(OR) she can get multiple and keeps one.
- Once a woman is tentatively matched, she is never unmatched.

<u>Claim 3</u>: DA algorithm always finds a pairwise stable matching.

What is pairwise stable?

- A matching is a bijective map $\mu : M \to W$
- ~~A~~ matching is pairwise unstable if at a preference profile P if ~~with the~~ ~~if~~ ~~with~~ $\exists$ a matching ~~oo~~ $m - w$ and $m' - w'$ s.t.

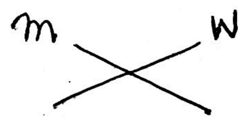
  $$w' \, P_m \, w \quad \text{and} \quad m \, P_{w'} \, m'$$

  - The pair (m, w') is called a <u>blocking pair</u> of this matching at P.

<u>Defn</u>:

- If a matching has no blocking pair at any preference profile P, then it is called pairwise stable.

<u>Proof</u>: Suppose not, $\exists$ some profile P where there is a blocking pair (m, w). So, it must be the case that under DA,



m is matched to a woman w' below w and w is matched to a man below m in their respective preferences. Assuming men proposing DA. Then m has been rejected by w at some round, but then w had a proposal above m. This is impossible since women's ~~prefere~~ matching in DA only improves in DA and hence can't fall below m.

Why pairwise stability? i.e., pairwise blocking and not group blocking?

Defn: Group Blocking

A coalition $S \subseteq M \cup W$ blocks a matching μ at a profile P if $\exists$ another matching μ' s.t.

i) [The coalitional exchange remains in S]

$\forall\, m \in M \cap S$, $\mu'(m) \in W \cap S$ and

$\forall\, w \in W \cap S$, $\mu'^{-1}(w) \in M \cap S$, and

ii) for all $m \in M \cap S$, $\mu'(m)\, P_m\, \mu(m)$ and

for all $w \in W \cap S$, $\mu'^{-1}(w)\, P_w\, \mu^{-1}(w)$.

Core matching: A matching μ is in the core of a profile P if no coalition can block μ at P.

Thm: A matching is pairwise stable iff it belongs to the core of that profile.

Proof: $\Leftarrow$ [core $\Rightarrow$ pairwise stable]

Core implies that no coalition of any size can block the matching. Clearly, no coalition of size 2 can do that. This is the trivial direction.

$\Rightarrow$ [pairwise stable $\Rightarrow$ core]

We prove $\neg$ core $\Rightarrow$ $\neg$ pairwise stable.

$\neg$ core: $\exists$ some coalition that group blocks the given matching μ at some profile.

Focus on the "for all" part in condition (ii) of the group blocking definition. It implies that for every such men and women in S, the new matching μ' is better. Consider $\mu'(m) = w_1$ (the woman matched to m under the new matching μ'). For this woman, her current match is better than that given by μ. Hence

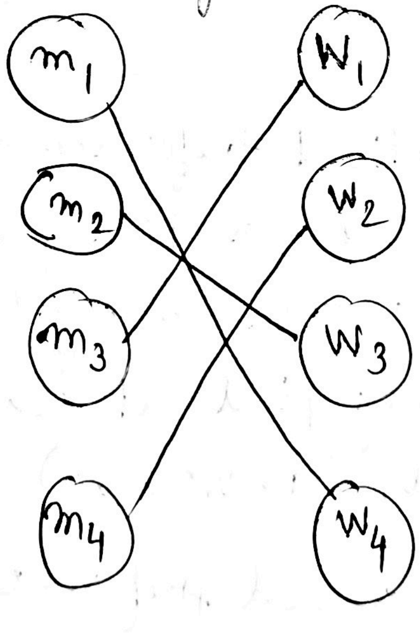
$$w_1 \; P_m \; \mu(m) \quad \text{and} \quad m \; P_{w_1} \; \mu^{-1}(w_1)$$

hence (m, w_1) forms a blocking pair of μ in profile P. Therefore, μ can't be pairwise stable at P either. $\square$

★ We will refer to stability as pairwise stability for two sided matching.

Structure of Stable matchings

$w_4 > w_1 > w_2 > w_3$	m_1 — w_1	$m_2 > m_1 > m_4 > m_3$
$w_3 > w_2 > w_4 > w_1$	m_2 — w_2	$m_1 > m_2 > m_3 > m_4$
$w_1 > w_2 > w_3 > w_4$	m_3 — w_3	$m_3 > m_1 > m_2 > m_4$
$w_2 > w_1 > w_4 > w_3$	m_4 — w_4	$m_4 > m_2 > m_1 > m_3$



A concise notation for a matching: from the men side $(1,1,1,1)$ w.r.t. the preference above means that man i is matched to $P_i(x)$ where x is the i^{th} entry in this tuple.

Consider the following matchings in the previous profile.

Some ~~allocate~~ matchings are unanimously better than other by all the men. Some matchings are incomparable.
These are the set of all stable matchings for this profile.

$$
\begin{tikzcd}[column sep=small]
 & \text{men-optimal stable matching (women pessimal)}\ (1,1,1,1)\ (4,4,3,3) & \\
 & (1,1,2,2)\ (3,3,3,3) \arrow[u] & \\
(1,2,3,2)\ (3,2,1,3) \arrow[ur] & & (2,1,2,3)\ (2,3,3,1) \arrow[ul] \\
 & (2,2,3,3)\ (2,2,1,1) \arrow[ul] \arrow[ur] & \\
 & (3,4,3,3)\ (1,1,1,1) \arrow[u] & \\
 & \text{men-pessimal = women optimal stable matching} &
\end{tikzcd}
$$

What are the corresponding women-side representation of the same matchings? This flips the direction of the preferences over the matchings from the women-side.

Transform these observations into results

A. Theorem: (Identical Preference over the stable matchings)
There is a stable matching that all men find at least as good as any other stable matching, and one they find at least as bad.
(similarly for women)

B. Theorem: (Reversed preference for men and women)
For any distinct stable matchings P and Q if all men find P at least as good as Q, then all women find Q at least as good as P.