Two definitions:

Achievable: A man m and a woman w are achievable for each other if there exists some stable matching where they are matched to each other.

Because of strict preferences, among all the achievable men/women of a woman/man (which can appear in different stable matches), there exists exactly one favorite achievable man/woman.

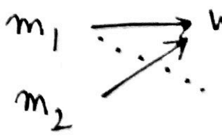
Consider a function $f^m: M \rightarrow W$ which maps every man to its favorite achievable woman — call this men-optimal function.

Similarly $f^W: W \rightarrow M$, maps every woman to her favorite achievable man — women-optimal function.

Since these functions (e.g., men-optimal) work over different ~~sea~~ stable matchings, no reason to believe that they will map to different women/men for each man/woman. But indeed both functions are bijections.

Thm: Men-optimal function is a matching

Suppose not, say $f^m(m_1) = f^m(m_2) = w$



also suppose w prefers $m_1 > m_2$.

But ~~since~~ by definition of achievability, there must be some stable matching μ where m_2 and w are matched.

In that matching μ, m_1 must be matched to some woman he prefers less than w (since w is the favorite achievable woman of m_1). (m_1, w) blocks μ.

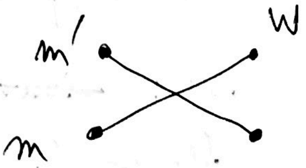
5-1 Algorithms to compute men-optimal matching (Gale-Shapley 62)

Theorem: For every preference profile P, the matching computed by the men-proposing DA is men-optimal.

[~~Similar~~ Analogus for women-proposing]

Proof: Sufficient to show that in the men-proposing DA a man is never rejected by his ~~8~~ favorite achievable woman.

- Suppose not, say m is the first man to be rejected by his favorite achievable woman w.
- w must have recived a proposal from m' which is above m according to w. $m' \, P_w \, m$
- When m' proposes to w, his past rejections (if any) must be all from women that are unachievable for him. (Since m is the first to be rejected by achievable woman)
- $\exists$ some stable matching μ where m and w are matched. Since m and w are achievable for each other.
- Under μ, m' must have been matched to someone below w in his preference. (Since all women above w in his preference list ~~is~~ are unachievable for m')



- Hence (m', w) is a blocking pair.

For any distinct stable matchings μ and μ', if all men prefer μ at least as good as μ', then all women prefer μ' at least as good as μ. (Knuth 75)

Suppose not, $\exists$ some w who finds μ better than μ'.

Let m be w's partner in μ, and m' in μ'

$$\mu: \quad m - w \qquad\qquad \mu': \quad \begin{matrix} m & \times & w \\ m' & & w' \end{matrix}$$

m prefers $\mu > \mu'$ hence $w \, P_m \, w'$

w prefers $\mu > \mu'$ hence $m \, P_w \, m'$

Then (m, w) is a blocking pair of μ'. $\square$

The above result gives the conclusion "if" the conditions hold.
It does not say anything when the matchings were incomparable.
But we can use such incomparable matchings to come up (construct) new matchings (actually stable) where there is a consensus among the men (and women).
Starting from an incomparable pair of matchings, go to a new matching than all men prefer more (and women prefer less) and vice versa.

Generalization of the pointing function idea.
man points to his favorite achievable woman $\Rightarrow$ men optimal
woman points to her favorite achievable man $\Rightarrow$ women-optimal
How about any arbitrary pair of stable matchings?

5-3

Let P and Q be any pair of <u>stable matchings</u>

<u>Define:</u> mapping $\max_{P,Q}$ s.t.

(a) each man maps/points to his <u>more</u> preferred woman between P and Q.

(b) each woman to her <u>less</u> preferred man between P and Q.

Using the same function from both ends $\max_{P,Q}(m)$ and $\max_{P,Q}(w)$ [slight abuse of notation]

Lemma 1: $\max_{P,Q}$ yields a matching.

Proof: suffices to show that for any pair of m and w, $\max_{P,Q}(m) = w \Leftrightarrow \max_{P,Q}(w) = m$.

($\Rightarrow$) Suppose not, i.e., $\max_{P,Q}(m) = w$

but $\max_{P,Q}(w) = m' \neq m$

~~Say~~ Since $\max_{P,Q}(m) = w$, there must be one matching between P and Q where m is matched to w, say it is P.

P: m — w Q: m — w', m' — w $\Big\}$ below w in m's pref.

Since $\max_{P,Q}(w) = m' \neq m$

then it has to be below m in w's preference by definition of $\max_{P,Q}$.

Q is not stable : (m, w) is a blocking pair.

($\Leftarrow$) Using the first part, we can claim that two distinct men can't point to the same woman — since that woman will point back to exactly one man. $\max_{P,Q}$ is a well-defined function.

Also each man points to a woman. Hence it has to be distinct. Since the cardinality $|M| = |W|$ hence, the mapping $\max_{P,Q}$ must be a matching.

(Alternative way of thinking about it)

$\max_{P,Q}$ is a mapping from $M \to W$ (from men side)

first part shows that it is 1-1 (injective)

Since a 1-1 mapping between two finite sets of equal cardinality must be bijective, the function must be bijection (hence a matching)

uses comes after lemma 2

Use: incomparable stable matchings, hence no consensus. But all the men/women can immediately construct a different stable matching that they can agree on

$(1,1,2,2)$ $\max_{P,Q}$

$(1,2,3,2)$ $\quad$ $(2,1,2,3)$

$(2,2,3,3)$ $\min_{P,Q}$

example comes after the next lemma.

Lattice Theorem: The mappings $\max_{P,Q}$ and $\min_{P,Q}$ induce stable matchings.

$\boxed{\text{Lemma 2:}}$ $\max_{P,Q}$ yields a stable matching.

Proof: Suppose not, (m, w) blocks $\max_{P,Q}$

Since m prefers w over $\max_{P,Q}(m) = w_1$ (say)

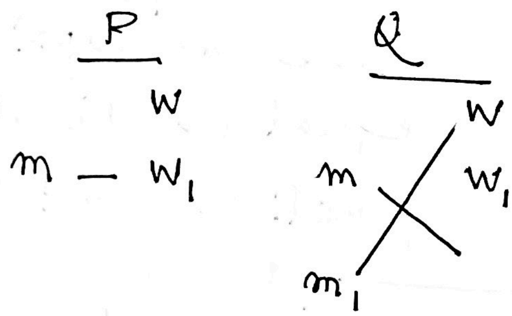
w_1 is the more preferred woman for m in P and Q

$\therefore$ w will be even above that. $\therefore w \, P_m \max_{P,Q}(m)$.

Now consider w. Say she prefers P over Q.

Hence $\max_{P,Q}(w)$ is her Q-matching. Suppose that man is m_1. Claim: m_1 is below m in w's preference, otherwise (m, w) can't be a blocking pair of $\max_{P,Q}$.

If m_1 = worse man between P and Q matchings of w is above m in w's preference then (m, w) can't make a blocking pair of $\max_{P,Q}$



$$m \, P_w \max_{P,Q}(w).$$

Regardless of which matching gives the worse match for w, that matching is blocked by (m, w).

But both P and Q are stable matchings. Hence contradiction. $\square$

Similarly, $\min_{P,Q}$ mapping can be defined. ~~for~~

A mirror opposite from the women side.

Can show that $\min_{P,Q}$ is also a stable matching.