8-1

Envy-freeness question

~~Ans~~ Is DS algo EF? No, the earlier agents can envy the later ones

Is EP algo EF? No, easy to create counterexample

These algorithms just ensure $\frac{1}{n}$ allocation, but not much attention to EF.

Consider a 3-agent EF cake cutting protocol
Selfridge - Conway Algorithm

Phase 1:

- Agent A divides the cake into 3 equal pieces (as per V_A)
- Agent B trims his favorite piece to create a tie with his second favorite. (as per V_B)
  - Trimming = S, Main cake = M,
    Original cake = $M \cup S$
- Agent C → B → A pick a piece from main cake M.
  - Agent B must pick the trimmed piece if C doesn't pick it.

T = owner of the trimmed piece (B or C)

$T' = \{B, C\} \setminus T$

Phase 2:

- Agent T' divides the trimmings S into 3 equal pieces (as per $V_{T'}$)
- $T \to A \to T'$ pick a piece each from S.

Allocation is EF?

From C: In M it chooses first

in S, it is either T or T'

if T, gets to choose first, if T'; it is indifferent to the pieces of S.

From B: In M, it gets either the trimmed piece or the other one having same value, the third one is less preferred than both.

In S, similar logic to C.

From A: In M, A never gets the trimmed piece hence the other two pieces are $\leq$ value of his piece. ~~The entire S is only from the trimmed piece. His share in S is only making.~~

~~A~~ In S, A picks after T, but T gets the trimmed piece (the entire share of T in M + the trimming is $\leq$ value than A's ~~B~~ share in M) and A picks before T'.

query complexity

$O(n^{n^{n^{n^{n}}}})$ — Aziz & Mckenzie (FOCS 2016)

Open

$\Omega(n^2)$ — Procaccia (2009)

$O(1)$ queries for $n=3$ — SC

2 queries for $n=2$ — Cut and Choose

Fair allocation of Indivisible goods

Examples : Inheritance, Allocation of hostel rooms among students, Essential supplies, e.g., COVID vaccines, Sharing computing resources among a group, Allocating buildings/real-estate between departments/ Centers etc.

Model :

Objects

Agents	A	B	C	D	E
1	1	2	5	1	1
2	1	0	5	1	1
3	1	1	5	1	1

~~Example~~ Fig. 1

Allocation is an assignment to each object to exactly one agents [not considering wastages]

valuations are defined on the bundles of objects. But to begin with, we assume that these values are additive

$$v_i(\{A,B,D\}) = v_i(\{A\}) + v_i(\{B\}) + v_i(\{D\})$$
$$= 1+2+1 = 4 \text{ if } i=1.$$
$$= 1+0+1 = 2 \text{ if } i=2$$
$$= 1+1+1 = 3 \text{ if } i=3.$$

Agents, $N = \{1, \ldots, n\}$, $M = \{1, \ldots, m\}$ objects

$$v_i : 2^M \to \mathbb{R}$$

Allocation/Division: (feasible)

$A = (A_1, A_2, \ldots, A_n)$ where $A_i \cap A_j = \phi$ if $i \neq j$

and $\bigcup_{i \in N} A_i = M$.

Example allocation from the above example

$A = (\{A, B\}, \{C\}, \{D, E\})$

Envy-freeness:

Each agent prefers her own bundle over any other agent's bundle. [note: envy is on a bundle, not agent]

$$v_i(A_i) \geqslant v_i(A_j) \quad \forall j \in N, \ \forall i \in N$$

	A	B	C
1	2	1	2
2	1	1	3

fig. 2

$A = (\{A, B\}, \{C\})$

$v_1(\{A, B\}) = 3 > v_1(\{C\}) = 2$

$v_2(\{C\}) = 3 > v_2(\{A, B\}) = 2$

What will be an EF allocation in figure 1?

- EF allocation is not guaranteed to exist (unlike cake cutting) in indivisible object allocation setting.
- Aside: checking whether an EF allocation exists is NP-complete. [Aziz, Gaspers, Mackenzie, Walsh '15]

Envy-free upto one good

Envy can be eliminated by removing ~~a~~ a good from the envied bundle

$v_i(A_i) < v_i(A_j)$ ← i envies j's bundle

but $v_i(A_i) \geq v_i(A_j \setminus \{x\})$ for some x.

In fig. 1. $(\{A, B\}, \{D, E\} \{C\})$ is not EF but EF1.

Defn: An allocation $A = (A_1, A_2, \ldots, A_n)$ is EF1 if for every pair of agents i, j $\exists$ an item x_j s.t.

$$v_i(A_i) \geq v_i(A_j \setminus \{x_j\}).$$

Note: The definition does not assume anything about the valuation, i.e., valuation need not be additive.

Good news: guaranteed to exist and efficiently computable for a large class of valuations.

The classes of valuations discussed:

1. Additive valuations — Round robin [dissimilar objects]
2. Monotone valuations — Envy cycle elimination

[example: complementary goods]

Round-Robin Algorithm (for additive valuations)

- Fix a sequence for round robin over the agents. WLOG assume the sequence is $(1, 2, 3, \ldots, n)$
- Ask agents to take turns in that sequence and pick their favorite item from the set of remaining items. $(m \geq n)$

Apply on Fig. 1 with $(1, 2, 3)$ sequence

$1 \to C$ $\;2 \to A$, $3 \to B$, $1 \to D$, $2 \to E$

$(\{C, D\}, \{A, E\}, \{B\})$

<u>Thm</u>: For additive valuations, the allocation computed by RR satisfies EF1.

Proof: Consider 2 agents i and j, considering i's envy towards j.

<u>Case 1</u>: i comes before j in the sequence

R1 - - - - - $i \xrightarrow{\text{better}} j$ - - - - - -

R2 - - - - $i \xrightarrow{\text{better}} j$ - - - -

last round - - - - i - - - - j - - - -

may stop here ①   or here ②   <u>or here</u> this is equivalent to ① for analysis of i and j

<u>Case 2</u>: i comes after j in the sequence

R1 - - - (j) $\xleftarrow{\text{better}}$ - - - i ———— $\quad v_i(A_i) \geq v_i(A_j \setminus \{x_j\})$

x_j

R2 - - - $j \leftarrow$ - - - i - - -

last - - - $j \xleftarrow{\text{better}}$ - - -