

## 8-1 Envy-freeness question

~~Are~~ Is DS algo EF? No, the earlier agents can envy the later ones

Is EP algo EF? No, easy to create counterexample

These algorithms just ensure  $\frac{1}{n}$  allocation, but not much attention to EF.

Consider a 3-agent EF cake cutting protocol  
Selfridge - Conway Algorithm

Phase 1:

- Agent A divides the cake into 3 equal pieces (as per  $v_A$ )
- Agent B trims his favorite piece to create a tie with his second favorite. (as per  $v_B$ )
  - Trimming = S, Main cake = M, Original cake = MUS
- Agent C  $\rightarrow$  B  $\rightarrow$  A pick a piece from main cake M.
  - Agent B must pick the trimmed piece if C doesn't pick it.

T = owner of the trimmed piece (B or C)

$$T' = \{B, C\} \setminus T$$

Phase 2:

- Agent T' divides the trimmings S into 3 equal pieces (as per  $v_{T'}$ )
- T  $\rightarrow$  A  $\rightarrow$  T' pick a piece each from S.

Allocation is EF?

From C: In  $M$  it chooses first

in  $S$ , it is either  $T$  or  $T'$

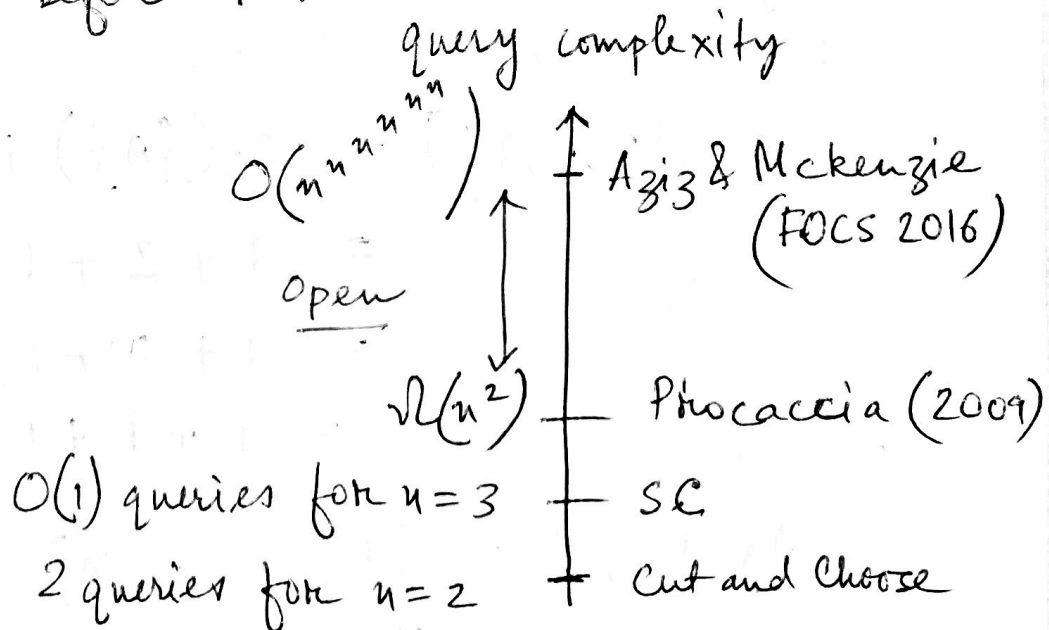
if  $T$ , gets to choose first, if  $T'$ ; it is indifferent to the pieces of  $S$ .

From B: In  $M$ , it gets either the trimmed piece or the other one having same value, the third one is less preferred than both.

In  $S$ , similar logic to C.

From A: In  $M$ , A never gets the trimmed piece hence the other two pieces are  $\leq$  value of his piece. ~~The entire  $S$  is only from the trimmed piece. His share in  $S$  is only making~~

In  $S$ , A picks after  $T$ , but  $T$  gets the trimmed piece (the entire share of  $T$  in  $M$  + the trimming is  $\leq$  value than A's share in  $M$ ) and A picks before  $T'$ .



## 8-3 Fair allocation of Indivisible goods

Examples: Inheritance, Allocation of hostel rooms among students, Essential supplies, e.g., COVID vaccines, Sharing computing resources among a group, Allocating buildings/real-estate between departments/Centers etc.

Model:

Agents	Objects				
	A	B	C	D	E
1	1	2	5	1	1
2	1	0	5	1	1
3	1	1	5	1	1

~~Example~~

Fig. 1

Allocation is an assignment to each object to exactly one agent [not considering wastages]

valuations are defined on the bundles of objects. But to begin with, we assume that these values are additive

$$\begin{aligned}v_i(\{A, B, D\}) &= v_i(\{A\}) + v_i(\{B\}) + v_i(\{D\}) \\ &= 1 + 2 + 1 = 4 \text{ if } i = 1. \\ &= 1 + 0 + 1 = 2 \text{ if } i = 2 \\ &= 1 + 1 + 1 = 3 \text{ if } i = 3.\end{aligned}$$

Agents,  $N = \{1, \dots, n\}$ ,  $M = \{1, \dots, m\}$  objects

$$v_i : 2^M \rightarrow \mathbb{R}$$

Allocation/Division: (feasible)

$$A = (A_1, A_2, \dots, A_n) \text{ where } A_i \cap A_j = \emptyset \text{ if } i \neq j$$

$$\text{and } \bigcup_{i \in N} A_i = M.$$

Example allocation from the above example

$$A = (\{A, B\}, \{C\}, \{D, E\})$$

Envy-freeness:

Each agent prefers her own bundle over any other agent's bundle. [note: envy is on a bundle, not agent]

$$v_i(A_i) \geq v_i(A_j) \quad \forall j \in N, \forall i \in N$$

	A	B	C
1	2	1	2
2	1	1	3

fig. 2

$$A = (\{A, B\}, \{C\})$$

$$v_1(\{A, B\}) = 3 > v_1(\{C\}) = 2$$

$$v_2(\{C\}) = 3 > v_2(\{A, B\}) = 2$$

What will be an EF allocation in figure 1?

- EF allocation is not guaranteed to exist (unlike cake cutting) in indivisible object allocation setting.
- Aside: checking whether an EF allocation exists is NP-complete. [Aziz, Gaspers, Mackenzie, Walsh '15]

## Envy-free upto one good

Envy can be eliminated by removing a good from the envied bundle

$$v_i(A_i) < v_i(A_j) \quad \leftarrow i \text{ envies } j\text{'s bundle}$$

but  $v_i(A_i) \geq v_i(A_j \setminus \{x\})$  for some  $x$ .

In fig. 1.  $(\{A, B\}, \{D, E\}, \{C\})$  is not EF but EF1.

Defn: An allocation  $A = (A_1, A_2, \dots, A_n)$  is EF1 if for every pair of agents  $i, j$   $\exists$  an item  $x_j$  s.t.

$$v_i(A_i) \geq v_i(A_j \setminus \{x_j\})$$

Note: The definition does not assume anything about the valuation, i.e., valuation need not be additive.

Good news: guaranteed to exist and efficiently computable for a large class of valuations.

The classes of valuations discussed:

- ① Additive valuations — Round robin  
[dissimilar objects]
- ② Monotone valuations — Envy cycle elimination  
[example: complementary goods]

# Round-Robin Algorithm (for additive valuations)

- Fix a sequence for round robin over the agents. WLOG assume the sequence is  $(1, 2, 3, \dots, n)$
- Ask agents to take turns in that sequence and pick their favorite item from the set of remaining items.  $(m, n)$

Apply on Fig. 1 with  $(1, 2, 3)$  sequence

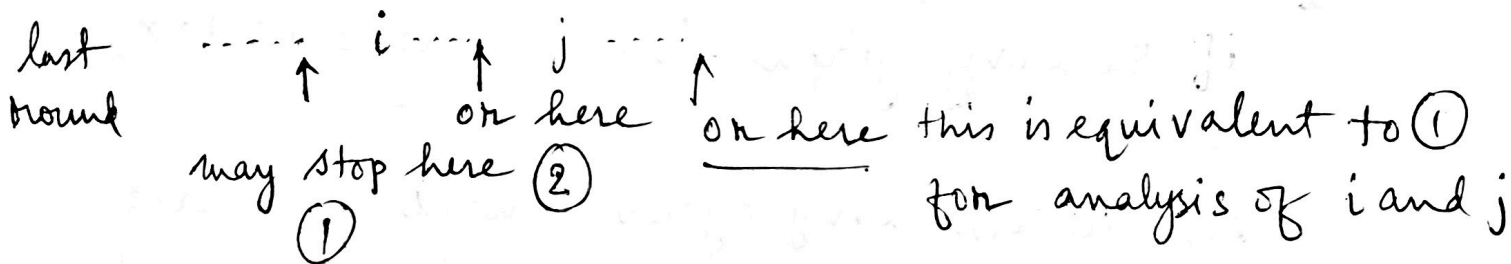
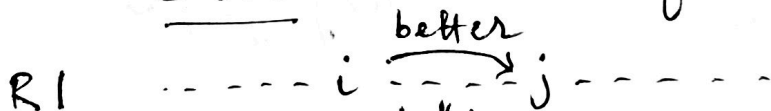
$1 \rightarrow C, 2 \rightarrow A, 3 \rightarrow B, 1 \rightarrow D, 2 \rightarrow E$

$(\{C, D\}, \{A, E\}, \{B\})$

Thm: For additive valuations, the allocation computed by RR satisfies EF1.

Proof: Consider 2 agents  $i$  and  $j$ , considering  $i$ 's envy towards  $j$ .

Case 1:  $i$  comes before  $j$  in the sequence



Case 2:  $i$  comes after  $j$  in the sequence



$$v_i(A_i) \geq v_i(A_j \setminus \{x_j\})$$

