Attempt 3: Start with EF1 allocation and make Pareto improvement.

	A	B	C	D	E
1	$\boxed{10}$	8	$\boxed{6}$	4	$\boxed{2}$
2	8	$\boxed{4}$	∅	$\boxed{3}$	∅
			3.9		2

RR gives □ which is EF1

[ ] is a Pareto improvement

but fails EF1.

Nash Social Welfare

The Nash Social Welfare (NSW) of an allocation A is defined as

$$NSW(A) = \left(v_1(A_1)\, v_2(A_2) \cdots v_n(A_n)\right)^{1/n}$$

fig. 3

	A	B	C
1	4	$\boxed{3}$	$\boxed{2}$
2	$\boxed{5}$	2	2

$NSW(O) = 4$

$NSW(\square) = 5$

(Say additive valuation for this discussion)

A Nash Optimal allocation is one that maximizes Nash social welfare.

[If optimal is zero, then find any largest set of agents who can simultaneously be given positive utility and maximize the geometric mean ~~with~~ w.r.t. only those agents]

Any Nash optimal allocation satisfies EF1 + PO.
(Caragiannis et al. 2016)

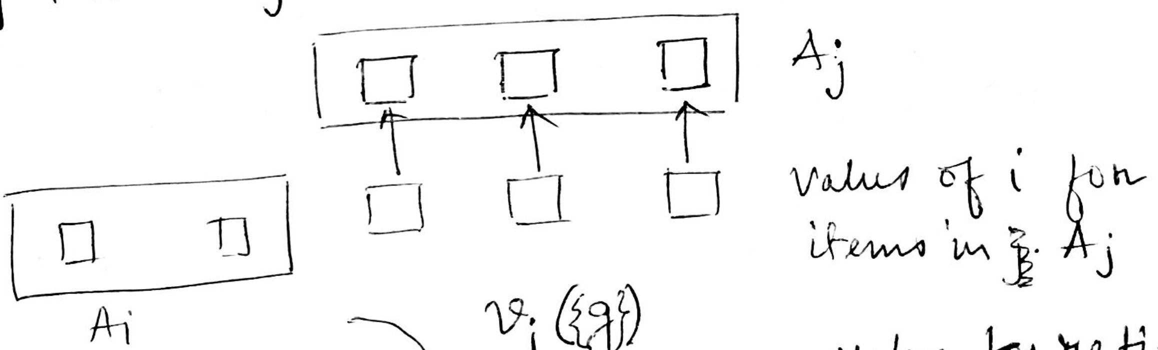
Proof: PO: Pareto improvement improves NSW.
If an allocation is not PO $\Rightarrow$ not NSW maximizing.

EF1: Suppose A is a Nash optimal allocation.
But A is not EF1. Will show it can't be Nash optimal either. Will construct another allocation that beats it in NSW.

$!\,\text{EF1} \Rightarrow \exists\ i, j$ s.t. $\forall\ x_j \in A_j$

$$v_i(A_i) < v_i(A_j \setminus \{x_j\}).$$

dropping any item from j's bundle doesn't make i's envy towards j vanish.



$$\frac{v_j(\{g\})}{v_i(\{g\})}$$ value ratio for $\{g\}$ [bang-per-buck]

define
$$g^* \in \underset{g \in A_j :\ v_i(\{g\}) > 0}{\arg\min} \frac{v_j(\{g\})}{v_i(\{g\})}$$

Existence of such a g^* is guaranteed.
If all $g \in A_j$ were $v_i(\{g\}) = 0$, then i wouldn't envy A_j.

In fig 3, for $i = 2$, under $\square$, $g^* = c$.

Claim: Transfer of g^* from A_j to A_i improves NSW.

[hence contradicts that A was Nash optimal]

A		B	
A_i	A_j	B_i	B_j
□ □ $\cancel{g^*}$ □	□ g^* □	□ □ □ g^*	□ □

Because of additivity:

$$v_i(B_i) = v_i(A_i) + v_i(g^*)$$

$$v_j(B_j) = v_j(A_j) - v_j(g^*)$$

TST: $\quad NSW(B) > NSW(A)$

$$\Leftrightarrow \left(v_i(A_i) + v_i(g^*)\right)\left(v_j(A_j) - v_j(g^*)\right) > v_i(A_i)\, v_j(A_j).$$

$$\Leftrightarrow v_i(g^*)\, v_j(A_j) - v_j(g^*)\left[v_i(A_i) + v_i(g^*)\right] > 0$$

$$\Leftrightarrow \quad v_j(A_j) > \frac{v_j(g^*)}{v_i(g^*)}\left[v_i(A_i) + v_i(g^*)\right] \quad \cdots (1)$$

since $v_i(g^*) > 0$.

Now, by the choice of g^*

$$\frac{v_j(g^*)}{v_i(g^*)} \leqslant \frac{v_j(g)}{v_i(g)} \quad \forall g \in A_j$$

$$\Rightarrow \frac{v_j(g^*)}{v_i(g^*)} \leqslant \frac{\overbrace{\sum_{g \in A_j} v_j(g)}^{= v_j(A_j)}}{\underbrace{\sum_{g \in A_j} v_i(g)}_{= v_i(A_j)}} \quad \cdots (2)$$

and by EF1 violation

$$v_i(A_i) < v_i(A_j) - v_i(g^*) \quad \text{--- (3)}$$

$$\Rightarrow \quad v_i(A_j) > v_i(A_i) + v_i(g^*)$$

From (2) : $\quad v_j(A_j) \geq v_i(A_j) \cdot \dfrac{v_j(g^*)}{v_i(g^*)}$

$$> \frac{v_j(g^*)}{v_i(g^*)} \left(v_i(A_i) + v_i(g^*) \right)$$

[from (3)]

hence (1) is proved $\square$

Existence of EF1 + PO is guaranteed. But computation is not easy.

Thm (Lee, 2017): Maximizing Nash social welfare is APX-hard.

[holds for bounded valuations as well]

Thm (Barman, Krishnamurthy, Vaish, 2018)

An EF1 + PO allocation can be computed in pseudopolynomial time.

- Running time depends on v_{ij} rather than $\log v_{ij}$.
- Poly for bounded valuations.
- 0.69 - approximation to NSW objective.

Fair allocation of Bads/Chores

In more general terms, the definition of goods/chores are in terms of marginal values.

- $v_i(S)$ = value of agent i for the bundle $S \subseteq M$

An item A is a good ~~for~~ for i if

$$v_i(S \cup A) - v_i(S) \geq 0$$

any useful resource cloud, ~~cat~~ spectrum electricity, ...

An item B is a bad for i if

$$v_i(S \cup A) - v_i(S) \leq 0$$

any corrosive item a factory, household chores, jobs in an organisation, a programming assignment,

Another setup is ~~the~~ where items are mixed — goods for some and bads for others.

A specific problem set may ~~be~~ have problems that are goods for some and bads for others.

Equivalent definition: goods: that have monotone non-decreasing valuations for each agent

$$v_i(S) \geq v_i(T) \quad \text{if} \quad S \supseteq T$$

bads: that have monotone non-increasing valuations for each agent.

$$v_i(S) \leq v_i(T) \quad \text{if} \quad S \supseteq T$$

These definitions are equivalent to the marginal value definition [show this]

Envy-freeness was impossible for indivisible objects.
Also difficult to ascertain if an instance has EF solution.

Envy-free upto one good was feasible for any monotone valuation.

- Round Robin for additive
- Envy-cycle elimination for monotone valuation

In the "bads" world

EF upto one chore: (still written EF1)

An allocation $A = (A_1, A_2, \ldots, A_n)$ is EF1 if for each pair of agents i, j, $\exists x_i$ s.t.

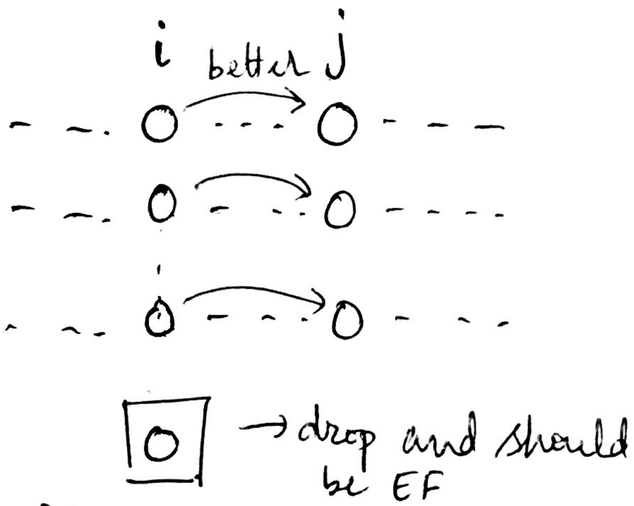
$$v_i(A_i \setminus \{x_i\}) \geq v_i(A_j).$$

Q: What algorithms satisfy EF1 for the chores?

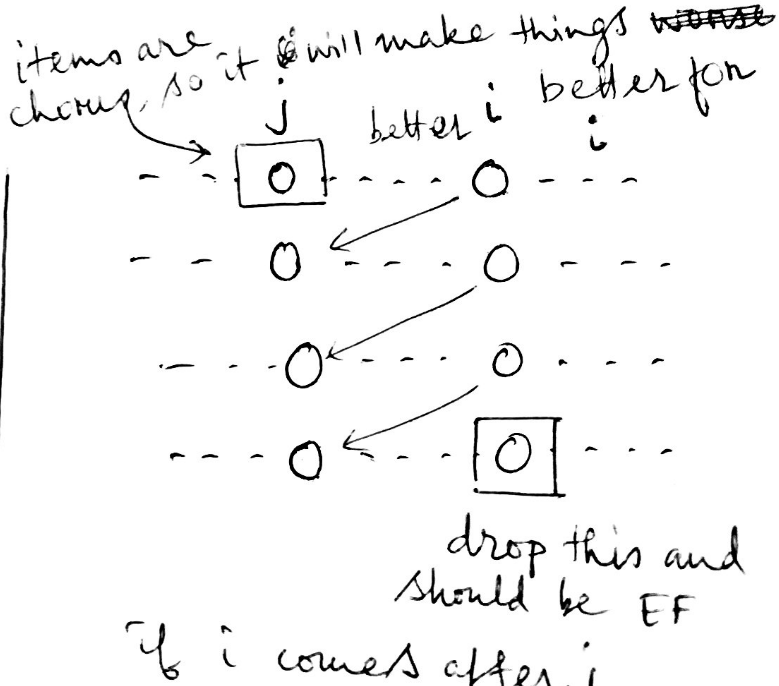
Case 1: Additive chores. Does RR work?

Thm: For additive chores, RR algorithm ensures EF1.

Proof: Consider an arbitrary pair of agents i and j



if i comes before j



If i comes after j