Multi person cooperative games ($n > 2$) :

$(S, (d_1, d_2, \ldots, d_n))$ defines the game in this setting, $S \in \mathbb{R}^n$.

One can extend the bargaining solution to n-player setting as well and "almost" all results extend.

However, there are more "possible choices" to every agent in an n-player game that a bargaining game model is unable to capture.

Ex.1 Divide the ~~dollar~~ money (ver 1)

$N = \{1, 2, 3\}$, want to divide ₹ 300

Each player can propose a division of this money.

Feasible set, $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \geq 0, i = 1, 2, 3$
$\sum_{i=1}^{3} x_i \leq 300\}$

[DTM game]

disagreement point

~~eg~~ $d_1 = d_2 = d_3 = 0$

In ver 1 of the game, the players has to unanimously agree to the division, only then the negotiation succeeds

$$u_i(s_1, s_2, s_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = s_3 = (x_1, x_2, x_3) \\ 0 & \text{ow} \end{cases}$$

every player has equal power in this game.

Nash bargaining solution gives (100, 100, 100) - which seems reasonable. No group can deviate and be better off.

Ex 2 : DTM game (ver 2)

$$u_i(x_1, x_2, x_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ 0 & \text{ow} \end{cases}$$

Nash bargaining solution still remains $(100, 100, 100)$
But in this game, players 1 and 2 has more power than 3. They will deviate from this allocation and may propose $(150, 150, 0)$.

Ex. 3 : DTM (ver 3)

$$u_i(x_1, x_2, x_3) = \begin{cases} x_i & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ & \quad\ s_1 = s_3 = (x_1, x_2, x_3) \\ 0 & \text{ow} \end{cases}$$

Both $\{1,2\}$ and $\{1,3\}$ has profitable deviation from Nash Bargaining. Also player 1 has more power in this game than the other ~~for~~ two.

Ex. 4 DTM (ver 4)

$$u_i(x_1, x_2, x_3) = \begin{cases} x_i & \text{if } s_j = s_k = (x_1, x_2, x_3) \\ & \quad \text{for some } j \neq k \\ 0 & \text{ow} \end{cases}$$

Any two agents agree on a division, that will be final. But if $(100, 100, 100)$ is proposed, agents 1 and 2 can propose differently, say $(150, 150, 0)$, then 3 can approach 1 or 2 and offer $(200, 0, 100)$, ... and the negotiation can continue indefinitely.

Turns out that we need a better axiomatic solution.

Transferrable Utility Games (TU Games)

A fluid commodity that can transfer utility – is money. With the transfer possible, we can define a cooperative game by a characteristic function.

$$v : 2^N \rightarrow \mathbb{R} \quad , \quad N : \text{set of players}$$

$v(S)$: value of the coalition $S \subseteq N$

$v(\phi) = 0$.

Defn: A TU game is given by the tuple (N, v) where N is the set of players and v is the characteristic function.

Examples: DTM.v1 : $v(\{1,2,3\}) = 300$,

$v(\{1\}) = v\{2\} = v(\{3\}) = v(\{1,2\}) \cdots = v(\{2,3\}) = 0$

DTM.v2 : $v(\{1,2\}) = v(\{1,2,3\}) = 300$

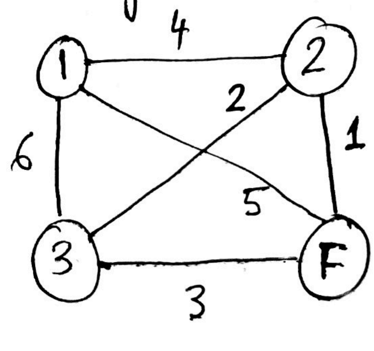
all other coalitions have value $= 0$

DTM.v3 : $v(\{1,2\}) = v(\{1,3\}) = v(\{1,2,3\}) = 300$

DTM.v4 : $v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\})$
$= v(\{1,2,3\}) = 300$

Ex. 2: Minimum cost spanning tree game

Every coalition tries to find the minimum cost spanning tree involving those agents and F.



Value of each coalition is the aggregate benefit − aggregate cost. e.g.,

$v(\{1\}) = 10-5,\ v(\{2\}) = 10-1,\ v(\{1,2\}) = 20-5.$

. . . . .

Ex. 3 Bankruptcy game (E, c)

$E \geq 0$ is the market value of an estate/company that was bankrupt. c denotes the claim vector of different stakeholders of the estate, $c \in \mathbb{R}^n_{\geq 0}$

Value is the difference between the market value and the amount to pay to the rest of the stakeholders to gain the company.

$$v(S) = \Big[E - \sum_{i \in N \setminus S} c_i\Big]^+ \qquad x^+ = \max\{0, x\}$$

Say, $N = \{1,2,3\}$, $c = (10, 50, 70)$, $E = 100$

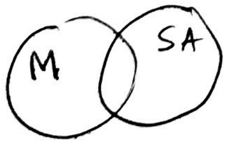
$v(1) = 0,\ v(2) = 20,\ v(3) = 40$

$v(12) = 30,\ v(23) = 90,\ v(13) = 50,\ v(1,2,3) = 100.$

Special classes of TU games

① Monotonic: $v(C) \leq v(D) \quad \forall C \subset D \subseteq N.$

② Superadditive: $v(C \cup D) \cancel{+v(C\cap D)} \geq v(C) + v(D)$
$\forall C, D \subseteq N$ s.t. $C \cap D = \phi.$



③ Convex: $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$
$\forall C, D \subseteq N.$

Proposition: Convex games are always superadditive

Prop 2: (N, v) is convex iff $v(C \cup \{i\}) - v(C) \leq v(D \cup \{i\}) - v(D)$
$C \subseteq D \subseteq N \setminus \{i\},\ \forall i \in N.$