# Market Games

A classical game where the players are producers / manufacturers who can create value by ~~appriately~~ appropriately redistributing their commodities.

Example: Chip manufacturer, Silicon supplier, Technology provider for creating VLSI designs. Computer / mobile phone manufacturer.

Producers : $N = \{1, 2, \dots, n\}$

Commodities: ~~ ~~ $C = \{1, 2, \dots, L\}$

Example: different types of raw material, electricity, foundries, human resources, expertise (scientific)

Commodity allocation is denoted via a matrix $x$ [overloading the notation for transfers]

$$x = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \cdots & & L \end{array} \\ \left[ \begin{array}{ccccc} & & & & \\ & & & & \\ & & x_{ij} & & \\ & & & & \\ & & & & \end{array} \right] \end{array}$$

← commodities

agents

$x_{ij}$ = amount of commodity $j$ that agent $i$ has

$x_{ij} \geq 0$ can be fractional.

- $i^{th}$ row of this matrix, i.e., agent $i$'s "bundle" is denoted as $x_i \in \mathbb{R}^L_{\geq 0}$

- $j^{th}$ column is denoted ~~a~~ as $x_j$ → $j^{th}$ commodity vector.

Each agent has an utility function from its bundle

$u_i(x_i) \in \mathbb{R}$

e.g., if there is a price $p$ in the market then $p^T x_i$

~~&~~ however, it can be non linear too.

Each producer $i$ comes to the market with ~~and~~ an initial endowment $a_i \in \mathbb{R}^L_{\geq 0}$

The objective is to redistribute the initial endowments efficiently → to maximize the overall utility, and yet be ~~st~~ coalitionally stable.

Coalitional Strategy:

~~A~~ If a coalition $S$ forms, the members ~~trade~~ exchange commodities among them.

Total endowment of $S$, $a(S) = \sum_{i \in S} a_i$

A feasible reallocation of the commodities is

$$x(S) = \sum_{i \in S} x_i = \sum_{i \in S} a_i$$

Collective ~~wel~~ utility (social welfare)

$$\sum_{i \in S} u_i(x_i) \qquad (x_i)_{i \in S} \in X^S$$

$$X^S = \left\{ (x_i)_{i \in S} : \sum_{i \in S} x_i = \sum_{i \in S} a_i, \; x_i \in \mathbb{R}^L_{\geq 0} \; \forall i \in S \right\} \quad \text{— (1)}$$

**Defn:** A market is given by a vector $(N, C, (a_i, u_i)_{i \in N})$ where

- $N = \{1, \dots, n\}$ set of producers
- $C = \{1, \dots, L\}$ set of commodities
- $\forall i \in N$, $a_i \in \mathbb{R}^L_{\geq 0}$ is the initial endowment of producer $i$.
- $\forall i \in N$, $u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$ is the utility/production function of $i$.

**Result:** $\forall S \subseteq N$, $X^S = \{(x_i)_{i \in S} \in \mathbb{R}^{|S|}_{\geq 0} : x(S) = a(S)\}$ is compact, i.e., closed and bounded.

$X^S$: feasible redistributed commodity set.

Assumption: production functions are continuous.

Worth/value of a coalition

$$v(S) = \max_{(x_i)_{i \in S} \in X^S} \sum_{i \in S} u_i(x_i) \quad \text{--- (2)}$$

continuous function

compact set

$v(S)$ exists and $\exists (x_i^*)_{i \in S} \in X^S$ where the maxima is attained. Hence, $v(S) = \sum_{i \in S} u_i(x_i^*)$

**Example:** $N = \{1, 2, 3\}$, $C = \{1, 2\}$

$a_1 = (1, 0)$, $a_2 = (0, 1)$, $a_3 = (2, 2)$

$u_1(x_1) = x_{11} + x_{12}, \quad u_2(x_2) = x_{21} + 2x_{22}$

$u_3(x_3) = \sqrt{x_{31}} + \sqrt{x_{32}}$

$\cancel{v_1(a_1)}\ v(1) = 1, \ v(2) = 2, \ v(3) = 2\sqrt{2}$

$v(123) = ?$

$$\sum_{i=1}^{3} u_i(x_i) = x_{11} + x_{12} + x_{21} + 2x_{22} + \sqrt{x_{31}} + \sqrt{\cancel{8}x_{32}}$$

$$x_{11} + x_{21} + x_{31} = 3$$

$$x_{12} + x_{22} + x_{32} = 3$$

For players 1 and 2, commodity 1 has same ~~va~~ utility to both and com 2 has twice as much value for 2 than 1. In the optimal welfare, ~~there should no will~~ The entire share of player 1 can be transferred to 2. So, the division is only between 2 and 3

$$\max\left\{ x_{21} + \sqrt{3 - x_{21}} + x_{22} + \sqrt{3 - x_{22}} \right\}$$

$$0 \leq x_{21} \leq 3, \quad 0 \leq x_{22} \leq 3$$

$$x_2 = \left(\frac{11}{4}, \frac{47}{16}\right) \qquad x_3 = \left(\frac{1}{4}, \frac{1}{4}\right)$$

**Defn:** A coalitional game $(N, v)$ is a market game if $\exists\, L > 0$, ~~such~~ and for every player $i \in N$ an initial endowment $a_i \in \mathbb{R}^L_{\geq 0}$, and a continuous and concave utility function $u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$ s.t. Eq.(2) is satisfied for every $S \subseteq N$.

Theorem (Shapley & Shubik (1969))

The core of a market game is non-empty.

If we use B-S characterization, this is equivalent to a balanced game.

A balanced game is a TU game $(N, v)$ where for every balanced weights $\lambda(S)$, $S \subseteq N$

$$v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S).$$

Proof: Let $\lambda = (\lambda(S))_{S \subseteq N}$ be a balanced set of weights.

Key idea: define a weighted distribution of the commodities s.t. the above inequalities show up.

$v(S)$ is attained at some reallocation $x^S$ by choice of continuity & compactness

$$x^S \in \underset{(x_i)_{i \in S} \in X^S}{\operatorname{argmax}} \left( \sum_{i \in S} u_i(x_i) \right)$$

define, $$z_i = \sum_{S \subseteq N : i \in S} \lambda(S) x_i^S$$

this is a convex combination, since

$$\sum_{S \subseteq N : i \in S} \lambda(S) = 1 \quad (\lambda \text{ is balanced}) \quad \forall i \in N.$$

<u>Claim</u>: $z_i$ is a feasible reallocation over the entire set $N$.

$$\sum_{i \in N} z_i = a(N)$$

$$\sum_{i\in N} z_i = \sum_{i \in N} \sum_{S \subseteq N} I\{i \in S\} \lambda(S)\, x_i^S$$

$$= \sum_{S \subseteq N} \sum_{i \in S} \lambda(S)\, x_i^S$$

$$= \sum_{S \subseteq N} \lambda(S) \underbrace{\sum_{i \in S} x_i^S}_{= a(S) \text{ by definition of } x_i}$$

$$= \sum_{S \subseteq N} \lambda(S) \sum_{i \in N} a_i \cdot I\{i \in S\}$$

$$= \sum_{i\in N} a_i \underbrace{\sum_{S \subseteq N} I\{i \in S\} \lambda(S)}_{= \sum_{S \subseteq N : i \in S} \lambda(S) = 1}$$

$$= \sum_{i \in N} a_i = a(N)$$

Now, $v(N) = \sum_{i\in N} u_i(x_i^*)$ || $x^*$ is the optimal realloc over the entire $N$.

$$\geq \sum_{i \in N} u_i(z_i) = \sum_{i \in N} u_i\Big(\sum_{S \subseteq N : i \in S} \lambda(S)\, x_i^S\Big)$$

$u_i$ is concave

$$\geq \sum_{i \in N} \sum_{S \subseteq N : i \in S} \lambda(S)\, u_i(x_i^S)$$

$$= \sum_{i \in N} \sum_{S \subseteq N} I\{i \in S\} \lambda(S)\, u_i(x_i^S)$$

$$= \sum_{S \subseteq N} \sum_{i \in N} I\{i \in S\} \lambda(S) u_i(x_i^S)$$

$$= \sum_{S \subseteq N} \lambda(S) \underbrace{\sum_{i \in S} u_i(x_i^S)}_{= v(S)}$$

$$= \sum_{S \subseteq N} \lambda(S) v(S). \qquad \square$$

(game is balanced)

---

Note that the properties defined here are downward compatible.

$(N, c, (a_i, u_i)_{i \in N})$ reduced to $(S, c, (a_i, u_i)_{i \in S})$

define a restriction of $v$ to $S$ and all properties hold. In particular, the subgame is also balanced. Such games are called totally balanced.

Corollary of Shapley-Shubik

If $(N, v)$ is a market game, every subgame $(S, v)$ of it is a market game, and is balanced.

Every market game is totally balanced.

Next time: limitations of core and their solution concepts.