Limitations of the core:

① Many solutions – because it is a set-valued solution concept. The solutions can be uncountable.

Q: Out of these solutions is there a single-valued solution concept?

Q: What are the properties that solution concept satisfies?

Axiomatic approach – similar to Nash Bargaining.

Notation: ϕ be a single valued solution concept.

$\phi_i(N, v)$ is called the allocation of player $i \in N$.

Axioms:

① Efficiency: A solution concept ϕ satisfies efficiency if $\forall$ TU games (N, v)

$$\sum_{i \in N} \phi_i(N, v) = v(N) \qquad \text{(NO WASTAGE)}$$

② Symmetry: Two players i and j are called symmetric players if for every coalition $S \subseteq N \setminus \{i, j\}$.

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

Defn: ϕ is symmetric if $\forall$ TU games (N, v) and for every symmetric players i and j

$$\phi_i(N, v) = \phi_j(N, v)$$

(EQUAL TREATMENT OF EQUALS)

③ Null players property: A player i is a null player if $\forall S \subseteq N$, $v(S) = v(S \cup \{i\})$

→ clearly $v(i) = 0$

Defn: ϕ satisfies null player property if $\forall$ TU games (N, v) and for every null player $i \in N$,

$$\phi_i (N, v) = 0 .$$

④ Additivity: ϕ satisfies additivity if for every pair of coalitional games (N, v) and (N, w)

$$\phi (N, v+w) = \phi(N, v) + \phi(N, w)$$

[TO WHAT EXTENT A SINGLE GAME IS EQUIVALENT TO TWO GAMES INDIVIDUALLY ?]

Examples:

① $\Psi_i (N, v) = v(i)$

- additive ? $\Psi_i (N, v+w) = (v+w) (i)$

$$= v(i) + w(i) = \Psi_i (N, v) + \Psi_i (N, w)$$

- Symmetric ? i and j are symmetric, i.e.,

$$v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall S \subseteq N \setminus \{i, j\}$$

$$\Rightarrow v(i) = v(j) \Rightarrow \Psi_i (N, v) = \Psi_i (N, w)$$

$S = \phi$

- null player ? $\forall$ null player $i \in N$ $\quad v(i) = 0$

$$\Rightarrow \Psi_i (N, v) = 0$$

- efficiency ? $\sum_{i \in N} v(i) = v(N)$? not necessary.

(2) $\Psi_i(N, v) = \max\limits_{\{S : i \notin S\}} \left(v(S \cup \{i\}) - v(S) \right)$

symmetry, null player — yes
efficiency, additivity — no (easy to create counter examples)

(3) $\Psi_i(N, v) = v(1, 2, \ldots, i-1, i) - v(1, 2, \ldots, i-1)$

efficiency — by construction
additivity — The solution concept is linear.
null player — yes
symmetry — consider the game

$v(1) = v(2) = v(3) = v(12) = v(1,3) = 0$

$v(2,3) = v(123) = 1$

which players are symmetric?

1 and 2? $\quad v(\{3\} \cup \{1\}) = 0$

$v(\{3\} \cup \{2\}) = 1$

1 and 3? similar

2 and 3? $\quad v(\{1\} \cup \{2\}) = 0 = v(\{1\} \cup \{3\})$

$v(2) = v(3) = 0$

What is Ψ_2 and Ψ_3?

$\Psi_2 = v(1,2) - v(1) = 0$

$\Psi_3 = v(1,2,3) - v(1,2) = 1$

not symmetric!

The solution concept of 3 can be defined for any order of the players (not just the ~~no~~ natural order)

Say $\Pi(N)$ denote the set of all possible orders over the n players, $|\Pi(N)| = n!$

Call $\pi \in \Pi(N)$ to be one ordering / permutation of the players.

Call the predecessor set of players i in π

as $P_i(\pi) = \{ j \in N : \pi(j) < \pi(i) \}$

→ $P_i(\pi) = \phi \Rightarrow \pi(i) = 1$

→ $P_i(\pi) \cup \{i\} = P_k(\pi) \Rightarrow \pi(k) = \pi(i) + 1$

Now generalize the solution concept of 3 to

$$\psi_i^{\pi}(N, v) = v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))$$

As before, this satisfies efficiency, null player, additivity but not symmetry.

But it is possible to construct based on this solution concept so that it satisfies all.

Shapley Value (Shapley 1953)

It is the solution concept defined as

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \left[v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) \right]$$

$\forall i \in N$

This is a simple average of the previous solution concept

$$Sh_i(N, v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \Psi_i^{\pi}(N, v), \quad \forall i \in N$$

<u>Theorem</u>: The Shapley Value is the <u>only single-valued</u> solution concept satisfying all four properties.

An equivalent representation of Shapley value.

Since the sum is over all permutations, $P_i(\pi)$ will take values of each $S \subseteq N \setminus \{i\}$

$$\frac{1}{n!} \sum_{\pi \in \Pi(N)} \Psi_i^{\pi}(N, v) = \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} \sum_{\pi \in \Pi(N) : P_i(\pi) = S} \left(v(S \cup \{i\}) - v(S) \right)$$

fixed for the inner sum

S i $N \setminus (S \cup \{i\})$

$$= \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} |S|!\,(n - |S| - 1)!\,\left(v(S \cup \{i\}) - v(S) \right)$$

$$= \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n - |S| - 1)!}{n!} \left(v(S \cup \{i\}) - v(S) \right)$$

"Average marginal contribution to all other coalitions".

Proof: $Sh_i(N, v)$ satisfies the four ~~e~~ axioms.

each of ψ_i^{π} satisfies efficiency, additivity, and null player $\Rightarrow$ Sh_i also satisfies them (exercise)

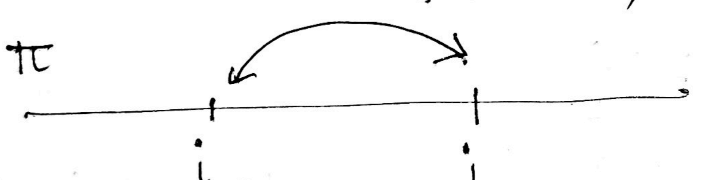
Symmetry: let i and j are symmetric players

i.e. $v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall S \subseteq N \setminus \{i, j\}$

given π, define a small perturbed permutation

$f(\pi)$, $f : \Pi(N) \to \Pi(N)$ s.t.

$$(f(\pi))(k) = \begin{cases} \pi(j) & \text{if } k = i \\ \pi(i) & \text{if } k = j \\ \pi(k) & \text{if } k \neq i, j \end{cases}$$



$f(\pi)$ swaps i and j in the permutation.

clearly $f(\pi)$ is a valid permutation.

for each π, there is a unique $f(\pi)$ and $f(\pi) \neq \pi$

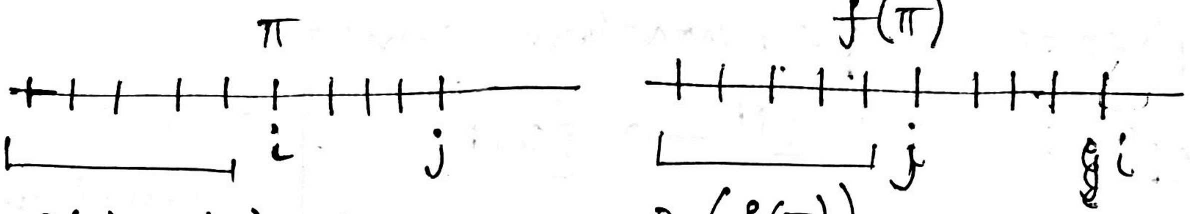
$f(f(\pi)) = \pi$, the sum over $\pi \in \Pi(N)$ sums the same terms while summing over $f(\pi) \in \Pi(N)$ in a different order. ~~Hence~~

We need to show $Sh_i \equiv Sh_j$. Sufficient to show that

$$\psi_i^{\pi}(N, v) = \psi_j^{f(\pi)}(N, v)$$

$$\Leftrightarrow v(P_i(\pi) \cup \{i\}) - v(P_i(\pi))$$
$$= v(P_j(f(\pi)) \cup \{j\}) - v(P_j(f(\pi)))$$

<u>Case 1</u>: Player i comes before j in π, i.e., $j \notin P_i(\pi)$



$P_i(\pi) \xleftrightarrow{\text{equal}} P_j(f(\pi))$

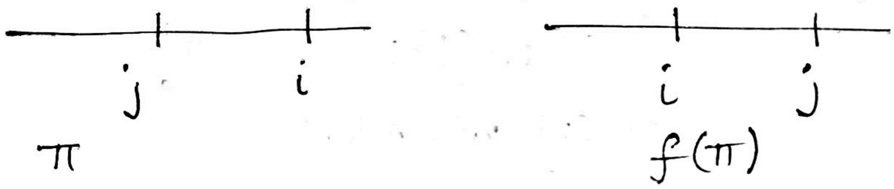
hence $v(P_i(\pi)) = v(P_j(f(\pi)))$

and since i and j are symmetric $\forall S \subseteq N \setminus \{i, j\}$

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

$$\Rightarrow v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\})$$

<u>Case 2</u>:



here $P_i(\pi) \setminus \{j\} = P_j(f(\pi)) \setminus \{i\}$

$$\xRightarrow[\text{symmetry}]{} v(P_i(\pi) \setminus \{j\} \cup \{j\}) = v(P_j(f(\pi)) \setminus \{i\} \cup \{i\})$$

$$\Rightarrow v(P_i(\pi)) = v(P_j(f(\pi)))$$

also $P_i(\pi) \cup \{i\} = P_j(f(\pi)) \cup \{j\}$

$$\Rightarrow v(P_i(\pi) \cup \{i\}) = v(P_j(f(\pi)) \cup \{j\}) \qquad \square$$

The proof of uniqueness is skipped. (MSZ Chap 18)

Shapley value of a convex game always belongs to its core. [Convex games have nonempty core]

Application: Shapley–Shubik power index

Applies to simple, monotone games

Simple: $v : 2^N \to \{0,1\}$ | e.g., a legislation is passed or not

monotone: $v(T) \geq v(S)$, if $S \subseteq T$.

Motivation: political economy, legislation, decisions based on committees.

Defn: The S-S power index is the shapley value of each simple, monotone game

$$Sh_i(N,v) = \sum_{\substack{\{S \subseteq N \setminus \{i\}: \\ S \cup \{i\} \text{ wins but} \\ S \text{ loses}\}}} \frac{|S|!\,(n-|S|-1)!}{n!}$$

Counting all situations where player i is pivotal. This index gives a measure of power of this agent.

Case study: UN security council

UN: International political body, established in 1945, after WW II.

till 1965: Five permanent members, Six non-permanent members

Resolution is accepted if 7 votes in favor but permanent members have to be unanimous. All of them have veto powers.

Debated about unequal distribution of power in the security council.

After 1965: 5 permanent, 10 nonpermanent

Resolution needs 9 votes, veto power with the permanent members

This is a simple, monotone game, what is its S-S index? [Exercise]

~~Power rati~~

	Power ratio nonpermanent to permanent
$\leqslant 1965$	1 : 91.2
> 1965	1 : 105.25

Restructuring actually increased the power of the permanent members.

Not covered:

When core is empty — how to extend the idea — Nucleolus