CS 6002: Selected Areas of Mechanism Design — Jan-Apr 2025

Lecture 13: Bargaining Games and TU Games

Lecturer: Swaprava Nath — *Scribe(s): Aditya Sanapala, Aditya Vema Reddy Kesari*

13.1 Setup of the Game

A two-person bargain game consists of:

- A feasibility set $S \in \mathbb{R}^2$ that is often assumed to be nonempty, compact, and convex, the elements of which are interpreted as agreements.
- A disagreement, or threat, point $\mathbf{d} = (d_1, d_2)$, where d_1 and d_2 are the respective payoffs to player 1 and player 2, which they are guaranteed to receive if they cannot come to a mutual agreement.

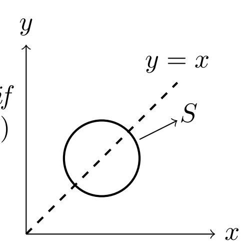
13.2 Desirable Properties

13.2.1 Symmetry

Definition 13.1 *A bargaining game* $(S, \boldsymbol{d}) \in \mathcal{F}$ *is* symmetric *if the following 2 properties are satisfied:*

1. $d_1 = d_2$ *(the disagreement point is symmetric).*
2. *If* $\boldsymbol{x} = (x_1, x_2) \in S$*, then* $(x_2, x_1) \in S$.

Definition 13.2 *A solution concept* ϕ *is* symmetric *(or satisfies the symmetry property) if for every symmetric bargaining game* $(S, \boldsymbol{d}) \in \mathcal{F}$ *the vector* $\varphi(S, \boldsymbol{d}) = (\varphi_1(S, \boldsymbol{d}), \varphi_2(S, \boldsymbol{d}))$ *satisfies* $\varphi_1(S, \boldsymbol{d}) = \varphi_2(S, \boldsymbol{d})$.



13.2.2 (Pareto) Efficiency

Definition 13.3 *An alternative* $\boldsymbol{x} \in S$ *is called an* efficient point *of* S *if there does not exist an alternative* $\boldsymbol{y} \in S, \boldsymbol{y} \neq \boldsymbol{x}$*, such that* $\boldsymbol{y} \geq \boldsymbol{x}$.

Denote by $PO(S)$ the set of efficient points of S (Pareto optimum).

Definition 13.4 *An alternative* $\boldsymbol{x} \in S$ *is called* weakly efficient *in* S *if there is no alternative* $\boldsymbol{y} \in S, \boldsymbol{y} \neq \boldsymbol{x}$, *satisfying* $\boldsymbol{y} \gg \boldsymbol{x}$.

Denote the set of weakly efficient points in S by $PO^W(S)$. It follows by definition that $PO(S) \subseteq PO^W(S)$ for each set $S \subseteq \mathbb{R}^2$; as the following example shows, this set inclusion can be a proper inclusion.

Example: Consider the bargaining game in Figure (13.1). The set of possible outcomes that cannot be improved from the perspective of at least one player, i.e., $PO(S)$, appears in bold in part A. The set of possible outcomes that cannot be improved from the perspective of both players, i.e., $PO^W(S)$, appears in bold in part B. For example, the outcome $(30, 100)$ is inefficient, since the outcome $(40, 100)$ is better from the perspective of Player 1. On the other hand, there is no outcome that is strictly better for both players than $(30, 100)$. In other words, $(30, 100) \in PO^W(S)$, but $(30, 100) \notin PO(S)$.

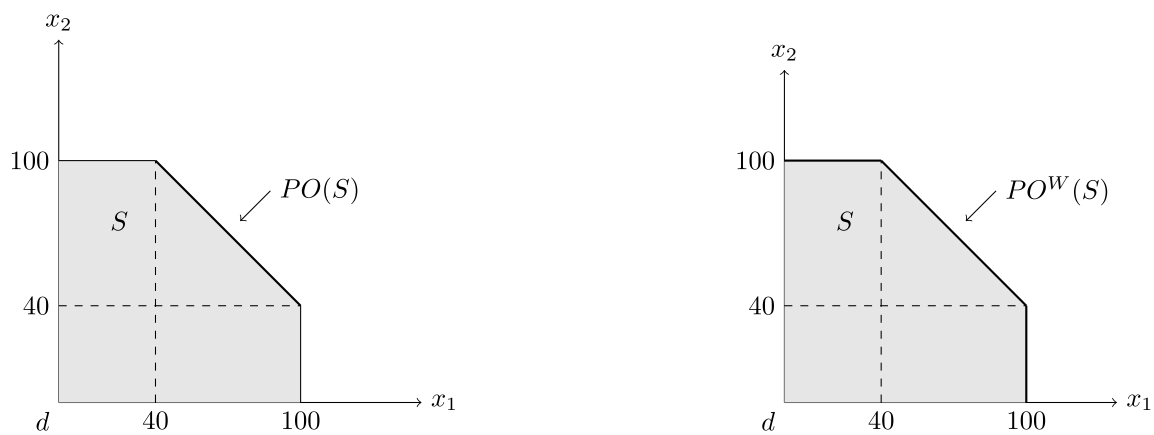


Figure 13.1: The efficient points of S in the previous example

Definition 13.5 *A solution concept* φ *is called* efficient *if* $\varphi(S, \boldsymbol{d}) \in PO(S)$ *for each bargaining game* $(S, \boldsymbol{d}) \in \mathcal{F}$.

Definition 13.6 *A solution concept* φ *is called* weakly efficient *if* $\varphi(S, \boldsymbol{d}) \in PO^W(S)$ *for each bargaining game* $(S, \boldsymbol{d}) \in \mathcal{F}$.

13.2.3 Covariance under Positive Affine Transformation (CPAT)

When the axes of a bargaining game represent monetary payoffs, it is reasonable to require that the solution concept be *independent of the units of measurement.* In other words, if we measure the payoff to one player in cents instead of dollars, we get a different bargaining game (in which the coordinate corresponding to each point is larger by a factor of 100). In this case, we want the coordinate corresponding to the solution to change by the same ratio.

Definition 13.7 *A solution concept* ϕ *is* covariant under positive affine transformations *if for each bargaining game* $(S, \boldsymbol{d}) \in \mathcal{F}$, *and for every vector* $\boldsymbol{a} \in \mathbb{R}^2$ *such that* $\boldsymbol{a} \gg 0$, *and for every vector* $b \in \mathbb{R}^2$,

$$\varphi(\boldsymbol{a}S + \boldsymbol{b}, \boldsymbol{ad} + \boldsymbol{b}) = \boldsymbol{a}\varphi(S, \boldsymbol{d}) + \boldsymbol{b}. \tag{13.1}$$

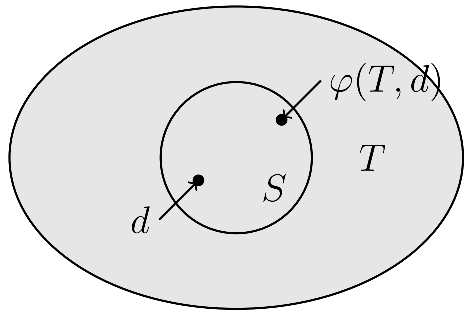


Figure 13.2: Independence of irrelevant alternatives

13.2.4 Independence of Irrelevant Alternatives (IIA)

Definition 13.8 *A solution concept φ satisfies the property of* independence of irrelevant alternatives *(IIA) if for every bargaining game $(T, \boldsymbol{d}) \in \mathcal{F}$, and every subset $S \subseteq T$,*

$$\varphi(T, \boldsymbol{d}) \in S \implies \varphi(S, \boldsymbol{d}) = \varphi(T, \boldsymbol{d}). \tag{13.2}$$

13.3 The Nash Solution

Theorem 13.9 (Nash, 1953) *There exists a unique solution concept $\mathcal{N}$ for the family of bargaining games $S \in \mathcal{F}$ satisfying the four desirable properties.*

The solutions satisfying these properties are exactly the points $\boldsymbol{x} = (x, y) \in S$ which maximize the following expression:

$$\mathcal{N}(S, \boldsymbol{d}) = \arg \max_{\boldsymbol{x} \in S, \boldsymbol{x} \geq \boldsymbol{d}} (x_1 - d_1)(x_2 - d_2) \tag{13.3}$$

Lemma 13.10 *For every bargaining game $(S, \boldsymbol{d}) \in \mathcal{F}$, there exists a unique point in the set, $\mathcal{N}(S, \boldsymbol{d})$ (i.e., the Nash Solution is unique).*

Proof: If we translate all the points in the plane by adding $-\mathbf{d}$ to each point, we get the bargaining game $(S - \mathbf{d}, (0, 0))$. Since the area of a rectangle is unchanged by translation, the points at which the Nash product is maximized for the bargaining game $(S, \mathbf{d})$ are translated to the points at which the Nash product is maximized in the bargaining game $(S - \mathbf{d}, (0, 0))$. We can therefore assume that, without loss of generality, $\mathbf{d} = (0, 0)$, and then

$$f(x) = x_1 x_2.$$

The set of individually rational points in S, which we denote by

$$D := \mathbf{x} \in S, \mathbf{x} \geq \mathbf{d},$$

is the intersection of the compact and convex set S with the closed and convex set

$$\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \geq \mathbf{d},$$

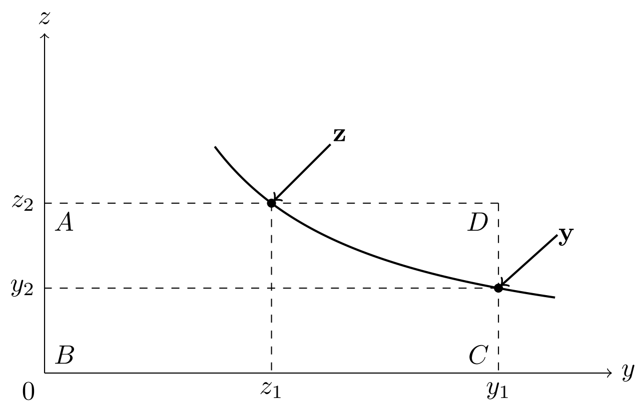


Figure 13.3: The areas of the rectangles defined by $\mathbf{y}$ and $\mathbf{z}$ are equal

so that is compact and convex as well. As we already noted, the set D is nonempty because it contains the disagreement point $\mathbf{d}$.

Since the function f is continuous, and the set D is compact, there exists at least one point y in D at which the maximum is attained. Suppose by contradiction that there exist two distinct points $\mathbf{y}$ and $\mathbf{z}$ in D at which the maximum of f is attained. In particular,

$$y_1 y_2 = z_1 z_2. \tag{13.4}$$

Define

$$\mathbf{w} := \frac{1}{2}\mathbf{y} + \frac{1}{2}\mathbf{z}. \tag{13.5}$$

Since D is convex, and $\mathbf{y}, \mathbf{z} \in D$, it follows that $\mathbf{w} \in D$. We will show that

$$f(\mathbf{w}) > f(\mathbf{z}), \tag{13.6}$$

contradicting the fact that the Nash product is maximized at $\mathbf{y}$ (and at $\mathbf{z}$). The assumption that $\mathbf{y} \neq \mathbf{z}$ therefore leads to a contradiction, hence $\mathbf{y} = \mathbf{z}$, and we will be able to conclude that the Nash product is maximized at a unique point.

One way to prove Equation (13.6) is to note that for every $c > 0$ the function $x_2 = \frac{c}{x_1}$ is strictly convex. For $c = y_1 y_2$, both $(\mathbf{y}, f(\mathbf{y}))$ and $(\mathbf{z}, f(\mathbf{z}))$, are on the graph of the function, and therefore $(\mathbf{w}, f(\mathbf{w}))$ is above the graph. In particular, $w_1 w_2 > c = y_1 y_2$.

A direct proof of the claim is as follows. In Figure (13.3), the points $\mathbf{y}$ and $\mathbf{z}$ are noted, with A, B, C,and D denoting four rectangular areas. From the figure we see that

$$y_1 z_2 + z_1 y_2 = A + 2B + C + D > A + 2B + C = y_1 y_2 + z_1 z_2. \tag{13.7}$$

Thus we have

$$f(\mathbf{w}) = w_1 w_2 = (\frac{y_1}{2} + \frac{z_1}{2})(\frac{y_2}{2} + \frac{z_2}{2}) \tag{13.8}$$

$$= \frac{y_1 y_2}{4} + \frac{y_1 z_2}{4} + \frac{z_1 y_2}{4} + \frac{z_1 z_2}{4} \tag{13.9}$$

$$> \frac{y_1 y_2}{4} + \frac{y_1 y_2}{4} + \frac{z_1 z_2}{4} + \frac{z_1 z_2}{4} \tag{13.10}$$

$$= \frac{y_1 y_2}{2} + \frac{z_1 z_2}{2} = f(\mathbf{y}), \tag{13.11}$$

where Equation (13.10) follows from Equation (13.7) and Equation (13.11) follows from Equation (13.4). In summary, $f(\mathbf{w}) > f(\mathbf{y})$, which is the desired contradiction. ■

Lemma 13.11 *The solution concept $\mathcal{N}$ satisfies the properties of symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives.*

Proof:

1. $\mathcal{N}$ satisfies **symmetry:**

   Let $(S, \mathbf{d})$ be a symmetric bargaining game, and let $\mathbf{y}^* = (y_1^*, y_2^*)$ be the vector that maximizes the Nash product $(y_1^* - d)(y_2^* - d)$.

   Define the point $\mathbf{z} = (y_2^*, y_1^*)$.

   Since S is symmetric, and $\mathbf{y}^* \in S$, we have $\mathbf{z} \in S$.

   Since $d_1 = d_2$, the area of the rectangle defined by $\mathbf{y}^*$ and $\mathbf{d}$ equals the area of the rectangle defined by $\mathbf{z}$ and $\mathbf{d}$:

   By Lemma (13.10), the maximum of f over S is attained at a unique point.

   Therefore, $\mathbf{y}^* = \mathbf{z}$, leading to $y_1^* = y_2^*$.

   Hence, $\mathcal{N}$ satisfies symmetry.

2. $\mathcal{N}$ satisfies **efficiency: Proof by Contradiction**

   If $\mathbf{y}$ is not efficient in S then there exists $\mathbf{z} \in S$ satisfying

   (a) $\mathbf{z} \geq \mathbf{y}$
   (b) $\mathbf{z} \neq \mathbf{y}$.

   Then the area of the rectangle defined by $\mathbf{z}$ and $\mathbf{d}$ is strictly greater than the area of the rectangle defined by $\mathbf{y}$ and $\mathbf{d}$, and therefore,

   $$\mathcal{N}(S, \mathbf{d}) \neq \mathbf{y}$$

   .

   Hence, $\mathcal{N}$ has to be efficient.

3. $\mathcal{N}$ satisfies **covariance under positive affine transformations**:

   The maximum of the function f over $\{\mathbf{x} \in S, \mathbf{x} \geq \mathbf{d}\}$ is attained at the point $\mathcal{N}(S, d)$. Applying the positive affine transformation $\mathbf{x} \mapsto \mathbf{ax} + \mathbf{b}$ to the plane combines a translation with multiplication by a positive constant at every coordinate. A translation does not change the area of a rectangle and multiplication by $\mathbf{a} = (a_1, a_2)$ multiplies the area of the rectangle by $a_1 a_2$. It follows that if prior to the application of the transformation the Nash product maximizes at $\mathbf{y}$, then after the application of the transformation $\mathbf{x} \mapsto \mathbf{ax} + \mathbf{b}$ the Nash product maximizes at $\mathbf{ay} + \mathbf{b}$.

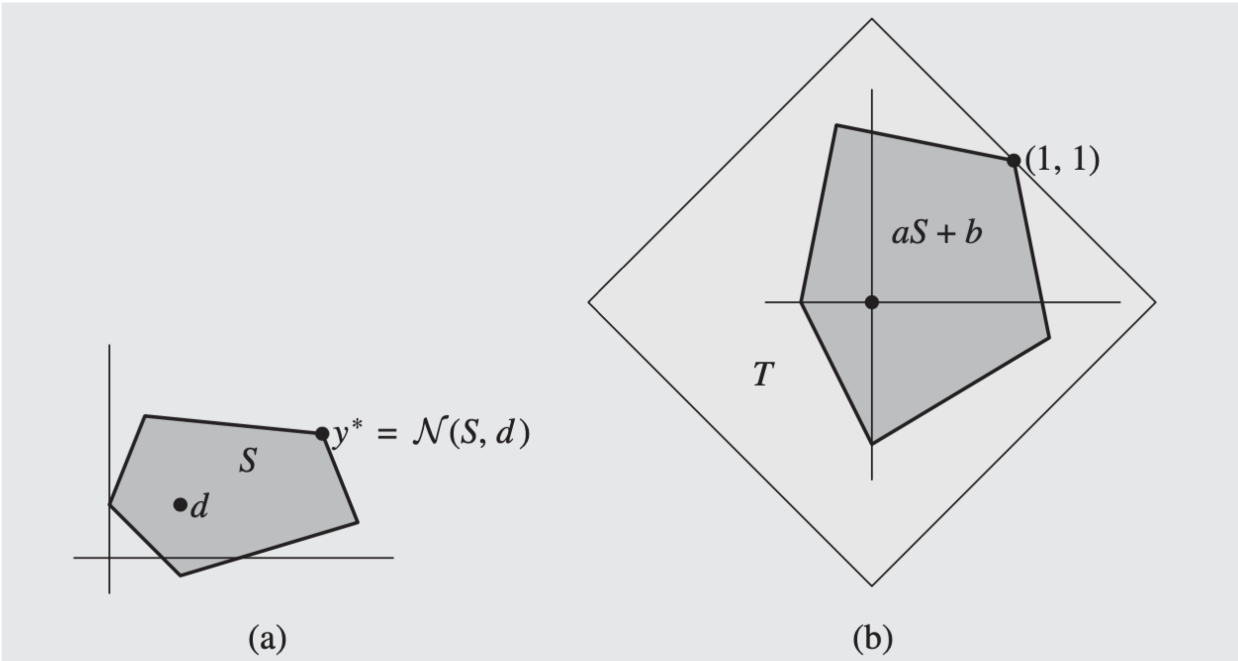


Figure 13.4: The bargaining game $(S, \mathbf{d})$ (a) and the game obtained by implementation of the positive affine transformation L, along with the symmetric square T (b)

4. $\mathcal{N}$ satisfies **independence of irrelevant alternatives:**

   This follows from a general fact:

   Let $S \subseteq T$, let $g : T \to \mathbb{R}$ be a function, and let $\mathbf{w} \in \arg\max_{\mathbf{x}\in T,\mathbf{x}\geq\mathbf{d}} g(\mathbf{x})$. If $\mathbf{w} \in S$, then $\mathbf{w} \in \arg\max_{\mathbf{x}\in T,\mathbf{x}\geq\mathbf{d}} g(\mathbf{x})$ (explain why the claim that $\mathcal{N}$ satisfies independence of irrelevant alternatives follows from this general fact). To see why this claim holds, note that since $\mathbf{w} \in S$ and $S \subseteq T$,

$$\max_{\{\mathbf{x}\in S,\mathbf{x}\geq\mathbf{d}\}} g(\mathbf{x}) \geq g(\mathbf{w}) = \max_{\{\mathbf{x}\in T,\mathbf{x}\geq\mathbf{d}\}} g(\mathbf{x}) \geq \max_{\{\mathbf{x}\in S,\mathbf{x}\geq\mathbf{d}\}} g(\mathbf{x}). \tag{13.12}$$

   It follows that

$$\max_{\{\mathbf{x}\in T,\mathbf{x}\geq\mathbf{d}\}} g(\mathbf{x}) = \max_{\{\mathbf{x}\in S,\mathbf{x}\geq\mathbf{d}\}} g(\mathbf{x}), \tag{13.13}$$

   and therefore $\mathbf{w} \in \arg\max_{\mathbf{x}\in S,\mathbf{x}\geq\mathbf{d}} g(\mathbf{x})$.

■

Lemma 13.12 *Every solution concept φ satisfying symmetry, efficiency, covariance under positive affine transformations, and independence of irrelevant alternatives is identical to the solution concept $\mathcal{N}$ defined by Equation (13.3).*

Proof: Let φ be a solution concept satisfying the four properties of the statement of the theorem. Let $(S, \mathbf{d})$ be a bargaining game in $\mathcal{F}$, and denote $\mathbf{y}^* := \mathcal{N}(S, \mathbf{d})$. We will show that $\varphi(S, \mathbf{d}) = \mathbf{y}^*$.

Step 1: Applying a positive affine transformation L.

Since there is an alternative $\mathbf{x}$ in S such that $\mathbf{x} \gg \mathbf{d}$, the point $\mathcal{N}(S, \mathbf{d}) = \mathbf{y}^* \in \{\mathbf{z} \in S : \mathbf{z} \geq \mathbf{d}\}$ at which the Nash product is maximized satisfies $\mathbf{y}^* \gg \mathbf{d}$. We can therefore define a positive affine transformation L over the plane shifting $\mathbf{d}$ to the origin, and $\mathbf{y}^*$ to $(1,1)$ (see Figure (13.4)). This function is given by

$$L(x_1, x_2) = \left(\frac{x_1 - d_1}{y_1^* - d_1}, \frac{x_2 - d_2}{y_2^* - d_2} \right) \tag{13.14}$$

Since $y_1^* > d_1$ and $y_2^* > d_2$, the denominators in the definition of L are positive. The function L is of the form $L = \mathbf{a}\mathbf{x} + \mathbf{b}$, where $a_1 = \frac{1}{y_1^* - d_1} > 0$, $a_2 = \frac{1}{y_2^* - d_2} > 0$, $b_1 = \frac{-d_1}{y_1^* - d_1} > 0$, and $b_2 = \frac{-d_2}{y_2^* - d_2} > 0$. Since the solution concept $\mathcal{N}$ satisfies CPAT,

$$\mathcal{N}(\mathbf{a}S + \mathbf{b}, (0,0)) = \mathcal{N}(\mathbf{a}S + \mathbf{b}, \mathbf{a}\mathbf{d} + \mathbf{b}) = \mathbf{a}\mathbf{y}^* + \mathbf{b} = (1,1). \tag{13.15}$$

Step 2: $x_1 + x_2 \leq 2$ for every $\mathbf{x} \in \mathbf{a}S + \mathbf{b}$.

Let $\mathbf{x} \in \mathbf{a}S + \mathbf{b}$. Since S is convex, the set $\mathbf{a}S + \mathbf{b}$ is also convex. Therefore, since both $\mathbf{x}$ and $(1,1)$ are in $\mathbf{a}S + \mathbf{b}$, the interval connecting $\mathbf{x}$ and $(1,1)$ is also in $\mathbf{a}S + \mathbf{b}$. In other words, for every $\varepsilon \in [0,1]$, the point $\mathbf{z}^\varepsilon$ defined by

$$\mathbf{z}^\varepsilon := (1 - \varepsilon)(1,1) + \varepsilon \mathbf{x} = (1 + \varepsilon(x_1 - 1), 1 + \varepsilon(x_2 - 1)) \tag{13.16}$$

is in $\mathbf{a}S + \mathbf{b}$. If ε is sufficiently close to 0 then $\mathbf{z}^\varepsilon \geq (0,0)$, and therefore $\mathbf{z}^\varepsilon$ is one of the points in the set $\{\mathbf{w} \in \mathbf{a}S + \mathbf{b}, \mathbf{w} \geq (0,0)\}$. It follows that for each such ε,

$$f(\mathbf{z}^\varepsilon) \leq \max_{\{\mathbf{w} \in \mathbf{a}S + \mathbf{b}, \mathbf{w} \geq (0,0)\}} f(\mathbf{w}) = f(\mathcal{N}(\mathbf{a}S + \mathbf{b}, (0,0))) = f((1,1)) = 1. \tag{13.17}$$

Hence

$$1 \geq f(\mathbf{z}^\varepsilon) = z_1^\varepsilon z_2^\varepsilon = 1 + \varepsilon(x_1 + x_2 - 2) + \varepsilon^2 (x_1 - 1)(x_2 - 1) \tag{13.18}$$

$$= 1 + \varepsilon(x_1 + x_2 - 2 + \varepsilon(x_1 - 1)(x_2 - 1)). \tag{13.19}$$

Therefore, for every $\varepsilon > 0$ sufficiently small,

$$0 \geq \varepsilon(x_1 + x_2 - 2 + \varepsilon(x_1 - 1)(x_2 - 1)), \tag{13.20}$$

leading to the conclusion that

$$2 - \varepsilon(x_1 - 1)(x_2 - 1)) \geq x_1 + x_2. \tag{13.21}$$

Taking the limit as ε approaches 0 yields $2 \geq x_1 + x_2$, which is what we wanted to show.

Finally, let T be a symmetric square relative to the diagonal $x_1 = x_2$ that contains $\mathbf{a}S + \mathbf{b}$, with one side along the line $x_1 + x_2 = 2$ (see Figure (13.4)(b)). Since $\mathbf{a}S + \mathbf{b}$ is compact (and thus bounded), such a square exists. By the symmetry and efficiency of φ, one has $\varphi(T, (0,0)) = (1,1)$. Since the solution concept φ satisfies IIA, and since $\mathbf{a}S + \mathbf{b}$ is a subset of T containing $(1,1)$, it follows that $\varphi(\mathbf{a}S + \mathbf{b}, (0,0)) = (1,1)$.

Since the solution concept φ satisfies CPAT, one can implement the inverse transformation L^{-1} to deduce that $\varphi(S, \mathbf{d}) = \mathbf{y}^*$. Since $\mathbf{y}^* = \mathcal{N}(S, \mathbf{d})$, we conclude that $\varphi(S, \mathbf{d}) = \mathcal{N}(S, \mathbf{d})$, as required. ■

13.4 Multi-Person Cooperative Games ($n > 2$)

A game in this setting is defined as:

$$(S, (d_1, d_2, \dots, d_n))$$

where $S \subseteq \mathbb{R}^M$.

The bargaining solution can be extended to a n-player setting, and almost all results extend. However, there are more possible choices in an n-player game than a bargaining model can capture. We show what the bargaining model cannot capture by an example below

13.4.1 Example 1: Divide the Money (Version 1)

Let $N = \{1, 2, 3\}$ want to divide ₹300. Each player can propose a division of this money. The feasible set S is:

$$S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i \geq 0, \sum_{i=1}^{3} x_i \leq 300 \right\}$$

The disagreement point is given by:

$$d_1 = d_2 = d_3 = 0$$

In this version, all players must unanimously agree to the division for the negotiation to succeed. The utility function is given as:

$$u_i(x_1, x_2, x_3) = \begin{cases} x_i, & \text{if } s_1 = s_2 = s_3 = (x_1, x_2, x_3) \\ 0, & \text{otherwise} \end{cases}$$

Since every player has equal power in this game, the Nash bargaining solution gives $(100, 100, 100)$, which is reasonable as no group can deviate and be better off.

Example 2: DTM Game (Version 2)

Now, we want to capture x_1 and x_2 have more power than x_3 and therefore as long as both x_1 and x_2 agree on a division, the negotiation succeeds:

$$u_i(x_1, x_2, x_3) = \begin{cases} x_i, & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ 0, & \text{otherwise} \end{cases}$$

The Nash bargaining solution still remains $(100, 100, 100)$. However, in this game, players 1 and 2 have more power than player 3 causing them to deviate from this allocation and propose $(150, 150, 0)$.

Example 3: DTM Game (Version 3)

Now, we want to capture the case where the negotiation is successful if either x_1 and x_2 or x_1 and x_3 agree on a decision. This can be seen as x_1 having maximum power with some power distributed equally between x_2 and x_3.

$$u_i(x_1, x_2, x_3) = \begin{cases} x_i, & \text{if } s_1 = s_2 = (x_1, x_2, x_3) \\ x_i, & \text{if } s_1 = s_3 = (x_1, x_2, x_3) \\ 0, & \text{otherwise} \end{cases}$$

If x_1 and x_2 agree on a distribution, then trivially x_3 receives nothing. However, x_1 and x_3 can renegotiate, allowing x_1 to secure a higher share while granting x_3 a nonzero amount. In response, x_2 is left with nothing and can initiate negotiations with x_1 to obtain a nonzero value. This cycle continues, leading to a final outcome that converges to $(300, 0, 0)$.

Example 4: DTM Game (Version 4)

$$u_i(x_1, x_2, x_3) = \begin{cases} x_i, & \text{if } s_j = s_k = (x_1, x_2, x_3) \text{ for some } j \neq k \\ 0, & \text{otherwise} \end{cases}$$

Any two agents agreeing on a division finalize the decision. However, if $(100, 100, 100)$ is proposed, agents 1 and 2 can propose differently, e.g., $(150, 150, 0)$. Then agent 3 can approach agent 1 or 2 and offer $(200, 0, 100)$, leading to an indefinite negotiation process.

Thus, we conclude that a better axiomatic solution is needed.

13.5 Transferable Utility Games (TU Games)

A fluid commodity that can transfer utility, such as money, allows us to define a cooperative game using a characteristic function.

- $v : 2^N \to \mathbb{R}$, where N is the set of players.
- $v(S)$ represents the value of the coalition $S \subseteq N$.
- $v(\emptyset) = 0$.

Definition: A TU game is given by the tuple (N, v), where N is the set of players and v is the characteristic function.

13.6 DTM Game Variants

13.6.1 DTM Version 1

We calculate the worth of each of the coalitions in the DTM versions we had defined above.

$$v(1,2,3) = 300,\ v(1) = v(2) = v(3) = v(1,2) = v(2,3) = v(1,3) = 0$$

13.6.2 DTM Version 2

$$v(1,2) = v(1,2,3) = 300, v(1) = v(2) = v(3) = v(2,3) = v(1,3) = 0$$

13.6.3 DTM Version 3

$$v(1,2) = v(1,3) = v(1,2,3) = 300, v(1) = v(2) = v(3) = v(2,3) = 0$$

13.6.4 DTM Version 4

$$v(1,2) = v(2,3) = v(1,3) = v(1,2,3) = 300, v(1) = v(2) = v(3) = 0$$

13.6.5 Example - Minimum Cost Spanning Tree Game

This is a game in which each coalition seeks to find the minimum cost spanning tree that connects those agents and the fixed node F.

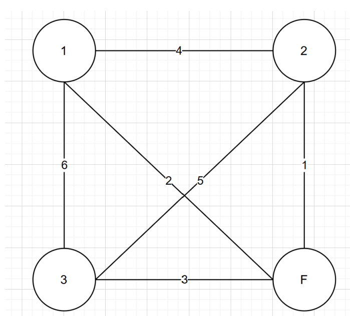


Figure 13.5: Cost graph

The value of each coalition is the aggregate benefit minus the aggregate cost. For example,

$$v(\{1\}) = 10 - 5, v(\{2\}) = 10 - 1, v(\{1,2\}) = 20 - 5$$