CS 6002: Selected Areas of Mechanism Design **Jan-Apr 2025**

Lecture 14: Core of TU Games

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14.1 TU Games and Solution Concepts

A Transferrable Utility(TU) game is given by the tuple (N, v) where N is the set of agents and $v : 2^N \to R$ is a characteristic function such that $v(S)$ represents the worth of the coalition $S \subseteq N$.

A solution concept for a TU game is a function that assigns to each game a vector $x \in \mathbb{R}^{|N|}$. A vector $x \in \mathbb{R}^{|N|}$ corresponds to monetary transfers made to agents.

14.1.1 Coalitional Rationality

A share of valuations $x \in \mathbb{R}^{|N|}$ (also called an imputation) is said to be coalitionally rational if it satisfies the following two properties:

- For every coalition $S \subseteq N$,
$$\sum_{i \in S} x_i \geq v(S)$$
meaning that the total payments to the members of any coalition is at least as great as the coalition's worth.
- For the coalition N,
$$\sum_{i \in N} x_i = v(N) \quad \text{(definition of imputation)}$$
as the sum of payments to the agents can't be more than the value of grand coalition.

Note : An imputation $x \in R^{|N|}$ is said to be individually rational if

$$x_i \geq v(\{i\}), \forall i \in N$$

Example:
Consider a TU game given by $N = \{1, 2, 3\}$ and function v such that

$$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
$$v(\{1,2\}) = 2, \quad v(\{1,3\}) = 3, \quad v(\{2,3\}) = 4$$
$$v(\{1,2,3\}) = 7$$

If $x = (x_1, x_2, x_3)$ is a coalitionally rational imputation for the above game, the following conditions must be satisfied:

- $x_1, x_2, x_3 \in \mathbb{R}_{\geq 0}$
- $x_1 + x_2 \geq 2$
- $x_1 + x_3 \geq 3$
- $x_2 + x_3 \geq 4$
- $x_1 + x_2 + x_3 = 7$

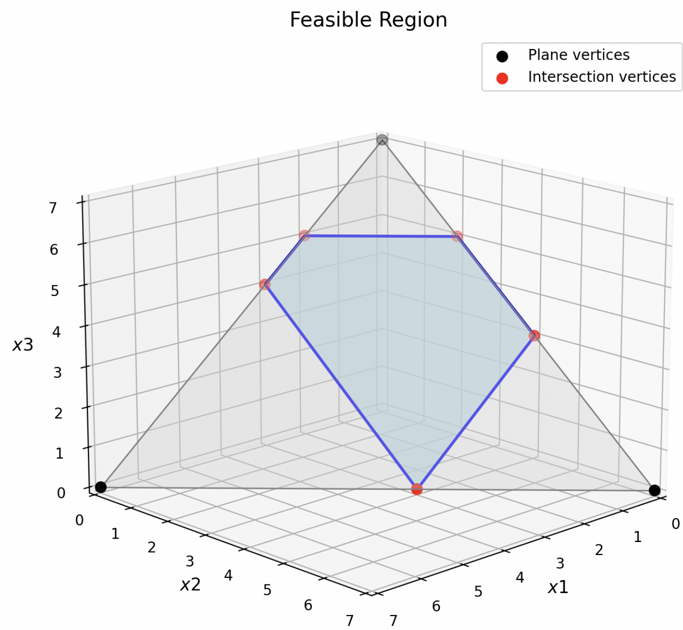


Figure 14.1: Intersection of the constraints $x_1 + x_2 + x_3 = 7$, $x_1 + x_2 \geq 2$, $x_1 + x_3 \geq 3$, and $x_2 + x_3 \geq 4$, $x_1, x_2, x_3 \in \mathbb{R}_{\geq 0}$

In the above figure, the blue region corresponds to the set of all possible coalitionally rational imputations.

While this set is non-empty in this case, there exist games where no such imputations satisfy the coalition rationality conditions, resulting in an empty set.

14.1.2 Core

An imputation $x \in \mathbb{R}^{|N|}$ is in the core of a TU game if it is coalitionally rational.

Let $C(N, v)$ denote the set of all coalitionally rational imputations. We will next try to find the cores of various TU Games discussed in previous lecture 13.

14.1.2.1 DTM Version 1

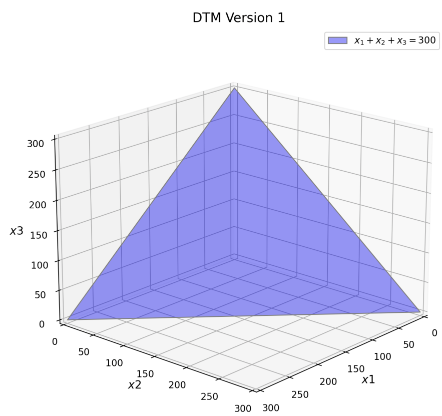
$$C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 + x_2 + x_3 = 300, \quad x_1, x_2, x_3 \in \mathbb{R}_{\geq 0}\}$$

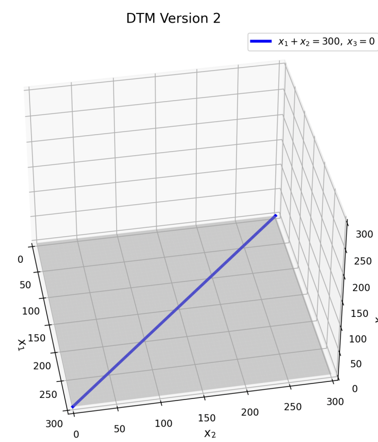
14.1.2.2 DTM Version 2

$$C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 + x_2 + x_3 = 300, \ x_1 + x_2 \geq 300, \ \ x_1, x_2, x_3 \in \mathbb{R}_{\geq 0}\}$$

This is equivalent to

$$C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_3 = 0, \ x_1 + x_2 = 300, \ \ x_1, x_2 \in \mathbb{R}_{\geq 0}\}$$





14.1.2.3 DTM Version 3

$$C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 + x_2 + x_3 = 300, \ x_1 + x_2 \geq 300, \ x_1 + x_3 \geq 300, \ x_1, x_2, x_3 \in \mathbb{R}_{\geq 0}\}$$

$$\implies C(N, v) = \{(300, 0, 0)\}$$

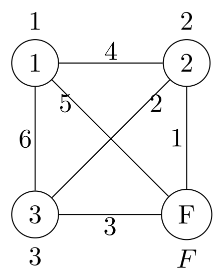
14.1.2.4 DTM Version 4

$$C(N, v) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1+x_2+x_3 = 300, \ x_1+x_2 \geq 300, \ x_1+x_3 \geq 300, \ x_2+x_3 \geq 300, \ x_1, x_2, x_3 \in \mathbb{R}_{\geq 0}\}$$

$$\implies C(N, v) = \phi$$

14.1.2.5 MST Game

In the MST (Minimum Spanning Tree) game, we consider a network where nodes represent players and one special node (labeled F) could represent a facility or a common connection point. The game is defined on a graph where the cost or value associated with connections influences the monetary transfers.



The values are given by:

$$v(1) = 5, \quad v(2) = 9, \quad v(3) = 7$$

$$v(12) = 15, \quad v(13) = 12, \quad v(23) = 17, \quad v(123) = 23$$

Additionally, the individual rationality constraints imply:

$$x_1 \geq 5, \quad x_2 \geq 9, \quad x_3 \geq 7$$

Other coalition constraints yield:

$$x_1 + x_2 \geq 15 \quad \Rightarrow \quad x_3 \leq 8,$$

$$x_2 + x_3 \geq 17 \quad \Rightarrow \quad x_1 \leq 6,$$

$$x_3 + x_1 \geq 12 \quad \Rightarrow \quad x_2 \leq 11.$$

These constraints together show that a core allocation exists—no coalition can obtain a better payoff by deviating.

But under what conditions does a TU game have a non-empty core? We next discuss a characterization result that answers this questions.

14.2 Balanced weights and Bondareva-Shapley Theorem

Balanced weights are nonnegative numbers assigned to every coalition such that each player's total weight over all coalitions they belong to is exactly 1:

$$\sum_{\substack{S \subseteq N \\ i \in S}} \lambda(S) = 1, \quad \forall i \in N.$$

These weights are instrumental in analyzing the core via duality in linear programming. They "balance" the contributions of each player across different coalitions.

14.2.1 Example:

For $N = \{1, 2, 3\}$, consider the following collection of coalitions:

$$S = \varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}.$$

If we set

$$\lambda(\{1,2\}) = \lambda(\{2,3\}) = \lambda(\{1,3\}) = \frac{1}{2},$$

then each player's weight sums to 1.

14.2.2 Bondareva-Shapley Theorem ('63, '67)

The Bondareva-Shapley theorem provides a necessary and sufficient condition for the core of a TU game to be non-empty. It states that a TU game (N, v) has a non-empty core if and only if, for every balanced collection of weights λ,

$$v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S).$$

Explanation: This theorem connects the feasibility of distributing the total worth $v(N)$ to the possibility of any coalition obtaining more than its allocated share. Its proof uses linear programming duality. In the primal formulation, we minimize the sum of payoffs subject to coalitional rationality:

$$\min \sum_{i \in N} x_i, \tag{14.1}$$

subject to

$$\sum_{i \in S} x_i \geq v(S) \quad \forall S \subseteq N.$$

If the optimal value exceeds $v(N)$, the core is empty; if it equals $v(N)$, then the core is non-empty.

The dual of this linear program is:

$$\max \lambda^T b, \quad \text{subject to } \lambda^T A = c^T, \quad \lambda \geq 0,$$

which can be rewritten as:

$$\max \sum_{S \subseteq N} \lambda(S) v(S) \quad \text{s.t.} \sum_{\substack{S \subseteq N \\ i \in S}} \lambda(S) = 1, \forall i \in N \quad \lambda(S) \geq 0.$$

Thus, the condition

$$\sum_{S \subseteq N} \lambda(S) v(S) \leq v(N)$$

for all balanced weights λ is equivalent to the non-emptiness of the core.

14.2.3 Ex. DTM.v4.

Consider the case where

$$v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = 300 = v(\{1,2,3\}).$$

Choosing balanced weights as

$$\lambda(12) = \lambda(23) = \lambda(13) = \frac{1}{2},$$

we have

$$\sum_{S \subseteq N} \lambda(S) v(S) = \frac{1}{2} \times 300 \times 3 > 300 = v(N).$$

Thus, the condition of the Bondareva-Shapley theorem is violated, and the core is empty in this instance.

14.3 Convex Games

Convex games are a class of TU games with the property that the incentive to join a coalition increases as the coalition grows. Formally, a game is convex if:

$$v(C \cup D) + v(C \cap D) \geq v(C) + v(D), \quad \forall C, D \subseteq N.$$

An equivalent condition is:

$$v(A \cup \{i\}) - v(A) \leq v(B \cup \{i\}) - v(B), \quad \forall A \subseteq B \subseteq N, \forall i \in N \setminus B.$$

Explanation: This inequality means that the marginal contribution of a player i to a coalition is non-decreasing as the coalition becomes larger. Consequently, convex games encourage cooperation, and it can be shown that they always have a non-empty core.

Claim: Convex games have a non-empty core.

Proof: (Using the Bondareva-Shapley characterization, Approach 1): One can construct an imputation by setting:

$$x_1 = v(1), \quad x_2 = v(12) - v(1), \quad x_3 = v(123) - v(12), \quad \dots, \quad x_n = v(N) - v(1, \dots, n-1).$$

For any coalition $S \subseteq N$, it can be shown that:

$$\sum_{i \in S} x_i \geq v(S),$$

and by construction,

$$\sum_{i \in N} x_i = v(N).$$

Thus, the imputation lies in the core.

For example, if we take an arbitrary $S = \{i_1, i_2, \dots, i_k\}$ in lexicographic order, we have:

$$x_{i_1} = v(1, \dots, i_1) - v(1, \dots, i_1 - 1) \geq v(i_1) - v(\emptyset),$$

$$x_{i_2} = v(1, \dots, i_1, i_2) - v(1, \dots, i_2 - 1) \geq v(i_1, i_2) - v(i_1),$$

$$\vdots$$

$$x_{i_k} = v(1, \dots, i_1, \dots, i_k) - v(1, \dots, i_k - 1) \geq v(i_1, i_2, \dots, i_k) - v(i_1, \dots, i_k - 1).$$

Summing these inequalities gives:

$$\sum_{l=1}^{k} x_{i_l} = \sum_{i \in S} x_i \geq v(S),$$

thus proving that the constructed imputation is in the core.