CS 6002: Selected Areas of Mechanism Design **Jan-Apr 2025**

Lecture 15: Market Games

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15.1 Market Games

A classical game where the players are producers/manufacturers who can create value by appropriately redistributing their commodities. Example: Chip manufacturer, Silicon supplier, Technology provider for creating VLSI designs, Computer/phone manufacturer.

15.1.1 Producers, Commodities, and Commodity allocation

Denote $N = \{1, 2, \ldots, n\}$ to be the set of producers and $C = \{1, 2, \ldots, L\}$ to be set of commodities e.g., different types of raw material, electricity, formalities, human resources, expertise (scientific).

A Commodity allocation is denoted via a matrix x:

$$x = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1L} \\ x_{21} & \ddots & \ddots & \vdots \\ x_{31} & \ddots & \ddots & \vdots \\ x_{n1} & \ddots & \ddots & \ddots \end{bmatrix},$$

where x_{ij} = amount of commodity j that agent i owns. Note that the rows of this matrix, i.e., agent i's bundle, is denoted as $x_i \in \mathbb{R}^L_{\geq 0}$, the columns are denoted as $x_j \rightarrow j^{th}$ commodity vector, and $x_{ij} \geq 0, \forall i, j$, can be fractional.

15.1.2 Utility Functions and Endowments

Each agent has a utility function from its bundle $u_i(x_i) \in \mathbb{R}$. Example: If there is a price p in the market, then $p^T x_i$ can be its utility. However, it can be nonlinear in x_i too. Each producer comes to the market with an initial endowment $a_i \in \mathbb{R}_{\geq 0}$. The objective is to redistribute the initial endowments efficiently to maximize overall utility and yet be coalitionally stable.

15.1.3 Coalitional Strategy

If a coalition S forms, the members trade commodities among themselves.

Total endowment of S, $a(S) = \sum_{i \in S} a_i$. A feasible reallocation of commodities is:

$$x(S) = \sum_{i \in S} x_i = \sum_{i \in S} a_i$$

Collective utility (social welfare) is

$$\sum_{i \in S} u_i(x_i) : (x_i)_{i \in S} \in X^S,$$

where $X^S = \{(x_i)_{i \in S} : \sum_{i \in S} x_i = \sum_{i \in S} a_i\}$ and $x_i \in \mathbb{R}^L_{\geq 0}, \forall i \in S$.

15.2 Market Definition

Definition 15.1. *A market is given by a vector $(N, C, (a_i, u_i))$ where:*

1. *$N = \{1, \ldots, n\}$ set of producers.*
2. *$C = \{1, \ldots, L\}$ set of commodities.*
3. *$\forall i \in N, a_i \in \mathbb{R}^L_{\geq 0}$ is the initial endowment of producer i.*
4. *$\forall i \in N, u_i : \mathbb{R}^L_{\geq 0} \to \mathbb{R}$ is the utility/production function of i.*
5. *Assumption: production functions are continuous.*

Result: $\forall S \subseteq N, X^S = \{(x_i)_{i \in S} \in \mathbb{R}^{|S|}_{\geq 0} : x(S) = a(S)\}$ is compact, i.e., closed and bounded. Note that X^S is the feasible redistributed commodity set.

15.2.1 Worth/Value of a Coalition

The value of a coalition is defined by

$$v(S) = \max_{(x_i)_{i \in S} \in X^s} \sum_{i \in S} u_i(x_i). \tag{15.1}$$

Note that u_is are continuous functions and X^s is a compact set. Therefore, $v(S)$ exists and $\exists (x_i^*)_{i \in S} \in X^S$ where the maximum is attained. Hence, $v(S) = \sum_{i \in S} u_i(x_i^*)$.

Example:

$$N = \{1, 2, 3\}, C = \{1, 2\}$$

$$a_1 = (1, 0), a_2 = (0, 1), a_3 = (2, 2)$$

$$u_1(x_1) = x_{11} + x_{12}, \quad u_2(x_2) = x_{21} + 2x_{22}, \quad u_3(x_3) = \sqrt{x_{31}} + \sqrt{x_{32}}$$

$$v(1) = 1, \quad v(2) = 2, \quad v(3) = 2\sqrt{2}$$

$$v(123) = ?$$

$$\sum_{i=1}^{3} u_i(x_i) = x_{11} + x_{12} + x_{21} + 2x_{22} + \sqrt{x_{31}} + \sqrt{x_{32}}$$

$$x_{11} + x_{21} + x_{31} = 3$$

$$x_{12} + x_{22} + x_{32} = 3$$

For players 1 and 2, commodity 1 has same as utility to both and commodity 2 has twice as much value for 2 than 1. In the optimal welfare the entire share of player 1 can be transferred to 2. So, the division is only between 2 and 3.

$$\max\left\{x_{21} + \sqrt{3 - x_{21}} + x_{22} + \sqrt{3 - x_{22}}\right\}$$

$$0 \le x_{21} \le 3, \quad 0 \le x_{22} \le 3$$

$$x_2 = \left(\frac{11}{4}, \frac{47}{16}\right), \quad x_3 = \left(\frac{1}{4}, \frac{1}{4}\right)$$

Definition 15.2. *A coalitional game (N, v) is a market game if $\exists L > 0$, and for every player $i \in N$, there is an initial endowment $a_i \in \mathbb{R}^L_{\ge 0}$, and a continuous and concave utility function $u_i : \mathbb{R}^L_{\ge 0} \to \mathbb{R}$ such that* (15.1) *is satisfied for every $S \subseteq N$.*

15.3 Core of Market Games

Theorem 15.3 (Shapley & Shubik (1969))**.** *The core of a market game is non-empty.*

If we use B-S characterization, this is equivalent to a balanced game.

A balanced game is a TU game (N, v) where for all balanced weights $\lambda(S)$, $S \subseteq N$:

$$v(N) \ge \sum_{S \subseteq N} \lambda(S) v(S).$$

Proof. Let $\lambda = (\lambda(S))_{S \subseteq N}$ be a balanced set of weights. The key idea is to define a weighted distribution of the commodities s.t. above inequalities show up.

$v(S)$ is attained at some reallocation x^S by choice of continuity and compactness:

$$x^S \in \text{argmax}_{(x_i)_{i \in S} \in X^S} \left(\sum_{i \in S} u_i(x_i)\right).$$

Define,

$$z_i = \sum_{S \subseteq N: i \in S} \lambda(S) x_i^S.$$

This is a convex combination, since

$$\sum_{S \subseteq N: i \in S} \lambda(S) = 1, \forall i \in N \quad (\lambda \text{ is balanced}).$$

Claim:z_i is a feasible reallocation over the entire set N i.e., $\sum_{i \in N} z_i = a(N)$.

$$\sum_{i \in N} z_i = \sum_{i \in N} \sum_{S \subseteq N} I\{i \in S\} \lambda(S) x_i^S = \sum_{S \subseteq N} \sum_{i \in S} \lambda(S) x_i^S$$

$$= \sum_{S \subseteq N} \lambda(S) \sum_{i \in S} x_i^S \quad (\sum_{i \in S} x_i^S = a(S) \text{ by definition of } x_i^S)$$

$$= \sum_{S \subseteq N} \lambda(S) \sum_{i \in N} a_i \cdot I\{i \in S\} = \sum_{i \in N} a_i \sum_{S \subseteq N} I\{i \in S\} \lambda(S)$$

$$= \sum_{i \in N} a_i = a(N).$$

Now, $v(N) = \sum_{i \in N} u_i(x_i^*)$, where x^* is optimal reallocation over the entire N. This implies,

$$v(N) \geq \sum_{i \in N} u_i(z_i) = \sum_{i \in N} u_i(\sum_{S \subseteq N: i \in S} \lambda(S) x_i^S)$$

$$\geq \sum_{i \in N} \sum_{S \subseteq N: i \in S} \lambda(S) u_i(x_i^S) \quad (\text{Since } u_i \text{ is Concave})$$

$$= \sum_{i \in N} \sum_{S \subseteq N} I\{i \in S\} \lambda(S) u_i(x_i^S)$$

$$= \sum_{S \subseteq N} \sum_{i \in N} I\{i \in S\} \lambda(S) u_i(x_i^S)$$

$$= \sum_{S \subseteq N} \lambda(S) \sum_{i \in S} u_i(x_i^S)$$

$$= \sum_{S \subseteq N} \lambda(S) v(S). \quad (\text{game is balanced})$$

□

Note that the properties defined here are downward compatible.

$(N, C, (a_i, u_i)_{i\in N})$ reduced to $(S, C, (a_i, u_i)_{i\in S})$ define a restriction of v to S and all properties hold. In particular, the subgame is also balanced. Such games are called totally balanced.

Corollary 15.4 (Shapley-Shubik)**.** *If (N, v) is a market game, every subgame (S, v) of it is a market game, and is balanced.*

Every market game is totally balanced.

Future discussions will cover limitations and alternative solution concepts.