CS 6002: Selected Areas of Mechanism Design **Jan-Apr 2025**

Lecture 16: Limitations of Core

Lecturer: Swaprava Nath *Scribe(s): Aditya Khambete and Sameer Arvind Patil*

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16.1 Limitations of Core

The main problem with core is that it is a set of solutions, which might even be uncountable, which is not a very desirable property. We would like to have a point solution concept for cooperative games.
So for that, first we define some desirable properties we want our solution concept to have.

16.2 Preliminaries

First of all, as usual we denote the game as the ordered pair $< N, v >$ where N is the set of players and v is the characteristic function. For each agent i, let $\phi_i : \mathcal{N} \times \mathcal{V} \to \mathbb{R}$ such that $\phi_i(N, v)$ is the share of worth agent i gets.

Definition 16.1 (Efficiency) *A solution concept ϕ satisfies efficiency (EFF) if for all TU games (N, v)*

$$\sum_{i \in N} \phi_i(N, v) = v(N)$$

i.e. the sum of the worth of all agents is equal to the worth of the grand coalition.

Definition 16.2 (Symmetry) *Agents i, j are said to be symmetric if for every coalition $S \subseteq N \setminus \{i, j\}$*

$$v(S \cup \{i\}) = v(S \cup \{j\}).$$

A solution concept ϕ satisfies symmetry (SYM) if for all TU games (N, v) and for any pair of symmetric agents i, j, we have

$$\phi_i(N, v) = \phi_j(N, v)$$

Definition 16.3 (Null Player Property) *An agent i is said to be a null player if*

$$v(S \cup \{i\}) = v(S) \quad \forall S \subseteq N \setminus \{i\}.$$

A solution concept ϕ satisfies null player property (NULL) if for all TU games (N, v) and for every null player i we have

$$\phi_i(N, v) = 0.$$

Definition 16.4 (Additivity) *A solution concept ϕ satisfies additivity (ADD) if for any two games (N, v) and (N, w), we have*

$$\phi_i(N, v + w) = \phi_i(N, v) + \phi_i(N, w)$$

Note that the $'+'$ in $v + w$ denotes functional addition, i.e. $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$.

16.3 Examples of Solution Concepts

Example 1 Let $\phi_i(N, v) = v(i)$. Let us check which of the above properties are satisfied.

1. *SYMM*
   Let i, j be symmetric agents. Then we have,
   $$v(S \cup \{i\}) = v(S \cup \{j\}) \quad \forall S \subseteq N \setminus \{i, j\}$$
   Put $S = \emptyset$ to get $v(i) = v(j)$, which is just $\phi_i(N, v) = \phi_j(N, v)$, so this solution concept satisfies SYMM.

2. *NULL*
   Let i be a null player. Then we have,
   $$v(S \cup \{i\}) = v(S) \quad \forall S \subseteq N \setminus \{i\}$$
   Just as we did above, put $S = \emptyset$ to get $v(i) = \phi_i(N, v) = v(\emptyset) = 0$, so this solution concept satisfies NULL.

3. *ADD*
   We have
   $$\phi_i(N, v + w) = (v + w)(i) = v(i) + w(i) = \phi_i(N, v) + \phi_i(N, w)$$
   Which is precisely the condition in Definition 16.4, so this solution concept satisfies ADD.

4. *EFF*
   In order for this solution concept to satisfy EFF, we need
   $$\sum_{i \in N} \phi_i(N, v) = \sum_{i \in N} v(i) = v(N)$$
   Which isn't always true since v might not be additive. So this solution concept does not satisfy EFF.

So this solution concept satisfies SYMM, NULL, ADD but not EFF, let us try to do better.

Example 2 Let $\phi_i(N) = max_{S \subset N \setminus \{i\}} [v(S \cup \{i\}) - v(S)]$. Notice this is nothing but the marginal contribution of agent i to the coalition. This satisfies SYMM and NULL which is straightforward to check directly from the definition. However, ϕ does not satisfy ADD and EFF. For EFF, the reasoning is similar to the previous example. It doesn't satisfy ADD because of the simple fact $max(a + b) \neq max(a) + max(b)$.

Example 3 Consider an ordering $1, 2, 3 \ldots n$ of the agents. Let $\phi_i(N, v)$ be defined as follows:

$$\begin{aligned}
\phi_1(N, v) &= v(1) - v(\emptyset) = v(1) \quad \text{(Since } v(\emptyset) = 0\text{)} \\
\phi_2(N, v) &= v(1, 2) - v(1) \\
\phi_3(N, v) &= v(1, 2, 3) - v(1, 2) \\
&\vdots \\
\phi_n(N, v) &= v(1, 2, 3 \ldots n) - v(1, 2, 3 \ldots n - 1)
\end{aligned}$$

Which means, for a general agent i, $\phi_i(N, v) = v(1, 2, 3 \ldots i) - v(1, 2, 3 \ldots i - 1)$. Let's check the properties it satisfies.

1. NULL
   Consider a null player i. Then we have
   $$v(S \cup \{i\}) = v(S) \quad \forall S \subseteq N \setminus \{i\}$$
   put $S = \{1, 2, 3 \ldots i-1\}$ to get $v(1, 2, 3 \ldots i) = v(1, 2, 3 \ldots i-1)$, which means
   $$v(1, 2, 3 \ldots i) - v(1, 2, 3 \ldots i-1) = \phi_i(N, v) = 0$$
   So this solution concept satisfies NULL.

2. EFF
   This is not hard to check. We have
   $$\sum_{i \in N} \phi_i(N, v) = \sum_{i=1}^{n} v(1, 2, 3 \ldots i) - v(1, 2, 3 \ldots i-1) = v(1, 2, 3 \ldots n) - v(\emptyset) = v(1, 2, 3 \ldots n) = v(N)$$
   So, this solution concept satisfies EFF.

3. ADD
   This easily follows from the definition of the characteristic function $v+w$. The solution concept satisfies ADD.

Notice that this solution concept satisfies 3 of the 4 properties we wanted, but the main problem is that it is not symmetric, which can be seen by the following counterexample.

$$v(\emptyset) = v(1) = v(2) = v(3) = v(1, 2) = v(1, 3) = 0 \quad v(2, 3) = v(1, 2, 3) = 1$$

It is easy to see that agent 2,3 are symmetric, but we have

$$\phi_2(N, v) = v(1, 2) - v(1) = 0 \quad \phi_3(N, v) = v(1, 2, 3) - v(1, 2) = 1$$

16.4 Towards Shapley Value

Let $\Pi(N)$ denote the set of all permutations of N.

Definition 16.5 (Predecessor) *For a permutation $\pi \in \Pi(N)$, the set*

$$P_i(\pi) = \{j \in N : \pi(j) < \pi(i)\}$$

is called the predecessor set of agent i in permutation π.

Notice the following properties of the predecessor set.

1. $P_i(\pi) = \emptyset$ iff $\pi(i) = 1$
2. $P_i(\pi) \cup \{i\} = P_k(\pi)$ iff $\pi(k) = \pi(i) + 1$

16.4.1 Shapley Value

The Shapley value is a solution concept that is defined as follows:

$$\phi_i(N,v) = \text{Sh}_i(N,v) = \frac{1}{n!}\sum_{\pi\in\Pi(N)}[v(P_i(\pi)\cup\{i\}) - v(P_i(\pi))] = \frac{1}{n!}\sum_{\pi\in\Pi(N)}\psi_i^\pi(N,v)$$

Theorem 16.6 *If core of a game is non-empty, then the Shapley value is in the core.*

Theorem 16.7 *The Shapley value satisfies EFF, SYMM, NULL, and ADD. Moreover, it is the unique solution concept that satisfies these properties.*

Proof: Clearly, this solution concept satisfies EFF, NULL, ADD (arguments similar to Example 3). The proof of SYMM is as follows. Assume agents i, j are symmetric. Then we have

$$v(S\cup\{i\}) = v(S\cup\{j\}) \quad \forall S\subseteq N\setminus\{i,j\}$$

Define f such that $f(i) = j$ and $f(j) = i$, and $f(k) = k$ for all $k \neq i, j$. We have f is a bijection on N, hence a permutation, that implies $f\circ\pi$ is also a permutation, which swaps the positions of i, j in π.
So to prove SYMM, we need to show that

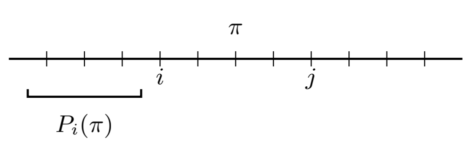
$$Sh_i(N,v) = Sh_j(N,v)$$

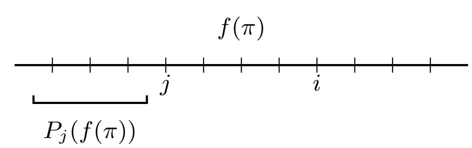
For which it is enough to show that

$$\psi_i^\pi(N,v) = \psi_j^{f\circ\pi}(N,v) \quad \forall\pi\in\Pi(N)$$
$$\iff v(P_i(\pi)\cup\{i\}) - v(P_i(\pi)) = v(P_j(f\circ\pi)\cup\{j\}) - v(P_j(f\circ\pi)) \quad \forall\pi\in\Pi(N)$$

Let us deal in 2 cases

CASE 1: $\pi(i) < \pi(j)$





Notice from the diagram that $P_i(\pi) = P_j(f\circ\pi)$, which implies

$$v(P_i(\pi)) = v(P_j(f\circ\pi)) \tag{1}$$

And since i, j are symmetric, we have

$$v(P_i(\pi)\cup\{i\}) = v(P_j(f\circ\pi)\cup\{j\}) \tag{2}$$

From (1) from (2) we get

$$v(P_i(\pi)\cup\{i\}) - v(P_i(\pi)) = v(P_j(f\circ\pi)\cup\{j\}) - v(P_j(f\circ\pi)),$$

Which is the desired result.

CASE 2: $\pi(i) > \pi(j)$

From the picture notice that $P_i(\pi) \setminus \{j\} = P_j(f \circ \pi) \setminus \{i\}$, which implies by symmetry

$$v(P_i(\pi) \setminus \{j\} \cup \{j\}) = v(P_j(f \circ \pi) \setminus \{i\} \cup \{i\}) \implies v(P_i(\pi)) = v(P_j(f \circ \pi)) \tag{3}$$

Also, observe that $P_i(\pi) \cup \{i\} = P_j(f \circ \pi) \cup \{j\}$, which implies

$$v(P_i(\pi) \cup \{i\}) = v(P_j(f \circ \pi) \cup \{j\}) \tag{4}$$

From (3) from (4) we get

$$v(P_i(\pi) \cup \{i\}) - v(P_i(\pi)) = v(P_j(f \circ \pi) \cup \{j\}) - v(P_j(f \circ \pi)),$$

Which is the desired result.

This proves that Shapley value is a solution concept that satisfies all the properties we wanted. We next prove a stronger result. ■

Lemma 16.8 *Any solution concept that satisfies EFF, SYMM, NULL, ADD must be the Shapley value.*

The proof of this is a bit involved and we will not go into all the details, however let's see some main tools used in the proof.

Definition 16.9 (Carrier (Basis) Games) *Let $T \subseteq N$ be a non-empty coalition of N. A Carrier game on T is a game $< N, u_T >$, where u_T is defined as follows. For each $S \subseteq N$,*

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

The idea of the carrier games is similar to the idea of basis vectors in linear algebra. In fact we have the following result.

Theorem 16.10 *Any game $< N, v >$ can be written as a linear combination of carrier games i.e.*

$$v(S) = \sum_{T \subseteq N, T \neq \emptyset} \alpha_T u_T(S)$$

The idea of proof of Lemma 16.8 is very similar to linear algebra, we show for the basis (carrier games) in order to satisfy EFF, SYMM, NULL, the solution concept must have the form of the Shapley value. And ADD follows from the linear combination property of the carrier games. For a detailed proof, refer [MSZ].

References

[MSZ] Michael Maschler, Eilon Solan and Shmuel Zamir "Game Theory", Cambridge University Press.