Mechanism Design with Monetary Transfers

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Indian Statistical Institute, New Delhi

Workshop on Static and Dynamic Mechanism Design
Indian Statistical Institute, New Delhi

August 2, 2015
Outline of the Talk

1. The Setup
   - Unrestricted Preferences
   - Restricted Preferences

2. Mechanisms in Quasi-linear Domain
   - Structure of a Mechanism
   - Some Definitions

3. Results
   - Groves Class of Mechanisms
   - What Other Mechanisms are Incentive Compatible
   - Revenue Equivalence
   - Uniqueness of Groves for Efficiency
   - Budget Balance
   - Bayesian Incentive Compatibility

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4 Summary
The Gibbard-Satterthwaite Setting

- Voters can have arbitrary *strict ordinal* preferences over the set of alternatives
- Set of alternatives $X = \{a, b, c, d\}$

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**Theorem (Gibbard (1973), Satterthwaite (1975))**

*If $|X| \geq 3$, an onto social choice function is strategyproof if and only if it is dictatorial.*
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Quasi-linear Preferences

- An alternative $x \in X$ is a tuple $(a, p)$
- Allocation $a$ belongs to the set of allocations $A$
Quasi-linear Preferences

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Quasi-linear Preferences

- An alternative $x \in X$ is a tuple $(a, p)$
- Allocation $a$ belongs to the set of allocations $A$
- Payments $p$ belong to $\mathbb{R}^n$
- Each agent $i$ has a valuation function $v_i : A \to \mathbb{R}$ belonging to the set $V_i$
- Agents’ utilities are given by

$$u_i(x) = u_i(a, p) = v_i(a) - p_i$$
Example: Public Good

Alternatives →

Alice  10  80
Bob    100 30
Carol  40  50

Photo courtesy: wikimedia.org and nimsuniversity.org
Example: Public Good

Alternatives →

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Valuations: $v_A(F) = 10, v_A(L) = 80$

Social planner takes the decision of building F or L

Can tax people differently depending on their preferences

Quasi-linear preferences
### Example: Resource Allocation

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Photo courtesy: individual organizations
### Example: Resource Allocation

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- *Set of allocations* $A = \{ x \in [0, 1]^{n \times m} : \sum_{j=1}^{m} x_{i,j} = 1 \}$
- Items are divisible among the agents
- Agents’ valuations reflect their preferences over different allocations
- They are charged monetary transfers for every allocation

**Quasi-linear preferences**
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**Selfish valuations**
Why Quasi-linearity avoids GS Impossibility

- GS theorem is valid for unrestricted preferences
- In quasi-linear domain, agents’ preferences are restricted
Why Quasi-linearity avoids GS Impossibility

- GS theorem is valid for unrestricted preferences
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Example:
- Set of alternatives $X = A \times \mathbb{R}^n$ consists of $(a, p)$ pairs
- Allocation $a \in A$ and payment vector $p \in \mathbb{R}^n$
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  - Consider two alternatives $x_1 = (a, p_1)$ and $x_2 = (a, p_2)$, where $p_1 < p_2$
  - For all agents, $x_1 \succ x_2$ for any valuation profile
  - There is no preference profile where $x_2 \succ x_1$
An Example of a Truthful Mechanism

Alice  10  80  40  
Bob    100 20  50  
Carol  0   40  30  

Swaprava Nath
Mechanism Design with Monetary Transfers
An Example of a Truthful Mechanism

Consider the mechanism:

- pick the alternative \( a^* \) that maximizes the sum of the valuations (with arbitrary tie-breaking rule)
- pay every agent \( i \) an amount \( \sum_{j \neq i} v_j(a^*) \)

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Consider the mechanism:

- pick the alternative $a^*$ that maximizes the sum of the valuations (with arbitrary tie-breaking rule)
- pay every agent $i$ an amount $\sum_{j \neq i} v_j(a^*)$

The mechanism is truthful, even though not a dictatorship
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4 Summary
Structure of a Mechanism

- Set of agents $N = \{1, \ldots, n\}$
- Set of allocations $A$, finite (for this tutorial)
- Valuation of agent $i$, $v_i : A \rightarrow \mathbb{R}$, the set of valuations is denoted by $V_i$
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A mechanism in quasi-linear (QL) domain is a pair of functions:
- allocation function, $a : \prod_j V_j \to A$
- payment function, $p_i : \prod_j V_j \to \mathbb{R}$, for all $i \in N$

Agent $i$’s payoff is given by:

$$v_i(a(v)) - p_i(v)$$
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- Only direct revelation mechanisms (DRM) (this talk)
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Social Choice Function

Definition (Social Choice Function)

A social choice function (SCF) $f$ is a mapping from the set of valuation profiles to the set of allocations, i.e., $f : V \rightarrow A$, where $V = \prod_j V_j$.

- Note that the outcome is only the allocations
A social choice function (SCF) $f$ is a mapping from the set of valuation profiles to the set of allocations, i.e., $f : V \rightarrow A$, where $V = \prod_j V_j$.

- Note that the outcome is only the allocations.
- In QL domain:
  A mechanism $M = (a, p)$ implements a SCF $f$ if:
  - $a(v) = f(v), \forall v \in V$ and,
  - for every agent $i \in N$, reporting $v_i$ truthfully is an equilibrium.
- Even though the SCF is concerned with only allocations, payments can also be characterized by revenue equivalence (defined later).
Incentive Compatibility

Definition (Dominant Strategy Incentive Compatibility (DSIC))

A mechanism \((f, p_1, \ldots, p_n)\) is dominant strategy incentive compatible if for all \(i \in N\) and for all \(v_{-i} \in V_{-i} := \prod_{j \neq i} V_j\),

\[
v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i}), \ \forall v_i, v'_i \in V_i.
\]

In this case, payments \(p_i, i \in N\) implement \(f\) in dominant strategies.
Incentive Compatibility (Contd.)

- In a Bayesian game, the valuations $v$ are generated through a prior $P$
- Each agent $i$ knows her own realized valuation $v_i$ and $P$
- Her belief on the valuations of other agents $v_{-i}$ is given by $P(v_{-i}|v_i)$ derived by Baye’s rule
In a Bayesian game, the valuations $v$ are generated through a prior $P$.

- Each agent $i$ knows her own realized valuation $v_i$ and $P$.
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**Definition (Bayesian Incentive Compatibility (BIC))**

A mechanism $(f, p_1, \ldots, p_n)$ is *Bayesian incentive compatible* for a prior $P$ if for all $i \in N$,

$$\mathbb{E}_{v_{-i}|v_i}[v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i})] \geq \mathbb{E}_{v_{-i}|v_i}[v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i})]$$

\forall v_i, v'_i \in V_i.

In this case, payments $p_i, i \in N$ implement $f$ in a Bayesian Nash equilibrium.
Observations on IC

- A DSIC mechanism is always BIC
Observations on IC

- A DSIC mechanism is always BIC

For a DSIC mechanism \((f, p_1, \ldots, p_n)\), let valuations of agents other than \(i\) is fixed at \(v_{-i}\)

- If \(v_i, v'_i\) be such that \(f(v_i, v_{-i}) = f(v'_i, v_{-i})\), then \(p_i(v_i, v_{-i}) = p_i(v'_i, v_{-i})\)
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- Consider another payment \(q_i(v_i, v_{-i}) = p_i(v_i, v_{-i}) + h_i(v_{-i})\),

\[
v_i(f(v_i, v_{-i})) - q_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - q_i(v'_i, v_{-i}), \quad \forall v_i, v'_i \in V_i.
\]
Efficiency

Definition (Efficiency)

An SCF $f$ is efficient if for all $v \in V$, 

$$
f(v) \in \arg\max_{a \in A} \sum_{i \in N} v_i(a).$$

An efficient SCF ensures that the ‘social welfare’ is maximized
Revenue Equivalence

- This property characterizes the payment functions

**Definition (Revenue Equivalence)**

An SCF $f$ satisfies *revenue equivalence* if for any two payment rules $p$ and $p'$ that implement $f$, there exist functions $\alpha_i : V_{-i} \to \mathbb{R}$, such that,

$$p_i(v_i, v_{-i}) = p'_i(v_i, v_{-i}) + \alpha_i(v_{-i}), \quad \forall v_i \in V_i, \forall v_{-i} \in V_{-i}, \forall i \in N.$$
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\]

- Saw an example of a payment of agent \( i \) being different by a factor not dependent on \( i \)'s valuation

- This property says more: pick *any* two payments that implement \( f \) - they must be different by a similar factor
Definition (Budget Balance)

A set of payments $p_i : V \rightarrow \mathbb{R}, i \in N$ is budget balanced if,

$$\sum_{i \in N} p_i(v) = 0, \forall v \in V.$$

- This property ensures that the mechanism does not produce any monetary surplus
- Hard to satisfy with incentive compatibility
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Single Indivisible Item Auction

Buyer 1
Metropolitan Museum of Arts

Buyer 2
Louvre
Second Price Auction

- Metropolitan wins, but pays second highest bid
- The mechanism is DSIC (why?)
Groves Class of Mechanisms

- Allocation rule is efficient:

\[ a^*(v) \in \arg\max_{a \in A} \sum_{i \in N} v_i(a) \]

- Payment rule is given by:

\[ p^*_i(v_i, v_{-i}) = h_i(v_{-i}) - \sum_{j \in N \setminus \{i\}} v_j(a^*(v)), \]

where \( h_i : V_{-i} \rightarrow \mathbb{R} \) is any arbitrary function that does not depend on \( v_i \)
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Claim

Groves class of mechanisms are DSIC
Incentive Compatibility of Groves

- Utility of agent $i$ according to Groves class of mechanisms:

\[
\begin{align*}
   u_i^{(a^*, p^*)}(v_i, v_{-i}) &= v_i(a^*(v_i, v_{-i})) - p_i^*(v_i, v_{-i})
\end{align*}
\]
Utility of agent $i$ according to Groves class of mechanisms:

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\begin{align*}
& u_i^{(a^*, p^*)}(v_i, v_{-i}) \\
& = v_i(a^*(v_i, v_{-i})) - p_i^*(v_i, v_{-i}) \\
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$$= \sum_{j \in N} v_j(a^*(v_i, v_{-i})) - h_i(v_{-i})$$

$$\geq \sum_{j \in N} v_j(a^*(v'_i, v_{-i})) - h_i(v_{-i}) \quad \text{(by definition of } a^*)$$
Incentive Compatibility of Groves

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\[
\begin{align*}
    u_i^{(a^*, p^*)}(v_i, v_{-i}) &= v_i(a^*(v_i, v_{-i})) - p_i^*(v_i, v_{-i}) \\
    &= v_i(a^*(v_i, v_{-i})) - h_i(v_{-i}) + \sum_{j \in N \setminus \{i\}} v_j(a^*(v_i, v_{-i})) \\
    &= \sum_{j \in N} v_j(a^*(v_i, v_{-i})) - h_i(v_{-i}) \geq \sum_{j \in N} v_j(a^*(v'_i, v_{-i})) - h_i(v_{-i}) \quad \text{(by definition of } a^*) \\
    &= v_i(a^*(v'_i, v_{-i})) - h_i(v_{-i}) + \sum_{j \in N \setminus \{i\}} v_j(a^*(v'_i, v_{-i}))
\end{align*}
\]
Utility of agent $i$ according to Groves class of mechanisms:

\[
\begin{align*}
    u_i^{(a^*, p^*)}(v_i, v_{-i}) &= v_i(a^*(v_i, v_{-i})) - p_i^*(v_i, v_{-i}) \\
    &= v_i(a^*(v_i, v_{-i})) - h_i(v_{-i}) + \sum_{j \in N \setminus \{i\}} v_j(a^*(v_i, v_{-i})) \\
    &= \sum_{j \in N} v_j(a^*(v_i, v_{-i})) - h_i(v_{-i}) \\
    &\geq \sum_{j \in N} v_j(a^*(v'_i, v_{-i})) - h_i(v_{-i}) \quad \text{(by definition of } a^*) \\
    &= v_i(a^*(v'_i, v_{-i})) - h_i(v_{-i}) + \sum_{j \in N \setminus \{i\}} v_j(a^*(v'_i, v_{-i})) \\
    &= v_i(a^*(v'_i, v_{-i})) - p_i^*(v'_i, v_{-i})
\end{align*}
\]
Incentive Compatibility of Groves

Utility of agent $i$ according to Groves class of mechanisms:

$$u_i^{(a^*, p^*)}(v_i, v_{-i})$$

$$= v_i(a^*(v_i, v_{-i})) - p_i^*(v_i, v_{-i})$$

$$= v_i(a^*(v_i, v_{-i})) - h_i(v_{-i}) + \sum_{j \in N \setminus \{i\}} v_j(a^*(v_i, v_{-i}))$$

$$= \sum_{j \in N} v_j(a^*(v_i, v_{-i})) - h_i(v_{-i})$$

$$\geq \sum_{j \in N} v_j(a^*(v'_i, v_{-i})) - h_i(v_{-i}) \quad \text{(by definition of } a^*)$$

$$= v_i(a^*(v'_i, v_{-i})) - h_i(v_{-i}) + \sum_{j \in N \setminus \{i\}} v_j(a^*(v'_i, v_{-i}))$$

$$= v_i(a^*(v'_i, v_{-i})) - p_i^*(v'_i, v_{-i})$$

$$= u_i^{(a^*, p^*)}(v'_i, v_{-i})$$
Pivot Mechanism

A special case of Groves class when the payment is given by:

\[ h_i(v_{-i}) = \sum_{j \in N \setminus \{i\}} v_j(a_{-i}^*(v_{-i})) , \]

where the allocation \( a_{-i}^*(v_{-i}) \) is given by:

\[ a_{-i}^*(v_{-i}) \in \arg\max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a) \]

The allocation \( a_{-i}^* \) maximizes the sum of valuations in the absence of agent \( i \).

The function \( h_i \) is the maximum value of this sum.

The payment is therefore:

\[ p_i(v_i, v_{-i}) = \max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a) - \sum_{j \in N \setminus \{i\}} v_j(a^*(v)) \]
Interpretations of the Pivot Mechanism

\[ p_i(v_i, v_{-i}) = \max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a) - \sum_{j \in N \setminus \{i\}} v_j(a^*(v)) \]

Two Interpretations:

1. Externality:
   - \( \max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a) \) is what the agents \( N \setminus \{i\} \) can achieve
   - \( \sum_{j \in N \setminus \{i\}} v_j(a^*(v)) \) is what they achieve under the efficient rule
   - The mechanism asks agent \( i \) to pay the difference
Interpretations of the Pivot Mechanism

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   - \( \sum_{j \in N \setminus \{i\}} v_j(a^*(v)) \) is what they achieve under the efficient rule
   - The mechanism asks agent \( i \) to pay the difference

2. Marginal contribution:
   - Net utility of agent \( i \) in pivot mechanism:
     \[ u_i(v_i, v_{-i}) = \sum_{j \in N} v_j(a^*(v_i, v_{-i})) - \max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a) \]
     i.e., the difference in sum valuation in presence of agent \( i \) and in her absence
   - Net utility is agent \( i \)'s marginal contribution
What is Pivotal about it?

Alternatives →

<table>
<thead>
<tr>
<th>Alice</th>
<th>10</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bob</td>
<td>100</td>
<td>10</td>
</tr>
<tr>
<td>Carol</td>
<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>
What is Pivotal about it?

Alternatives →

Alice 10 70
Bob 100 10
Carol 10 50

Outcome: L
What is Pivotal about it?

Alternatives →

Alice  10  70
Bob  100  10
Carol  10  50

 Outcome: L
Alice pays $(100 + 10) - (10 + 50) = 50$
What is Pivotal about it?

Alternatives →

<table>
<thead>
<tr>
<th></th>
<th>Alice</th>
<th>Bob</th>
<th>Carol</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>10</td>
<td>100</td>
<td>10</td>
</tr>
<tr>
<td>Pay</td>
<td>70</td>
<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>

- **Outcome:** L
- **Alice pays** \((100 + 10) - (10 + 50) = 50\)
- **Carol pays** \((10 + 100) - (70 + 10) = 30\)
What is Pivotal about it?

Alternatives →

<table>
<thead>
<tr>
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</tr>
<tr>
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<td>10</td>
<td>50</td>
</tr>
</tbody>
</table>

- **Outcome: L**
- Alice pays \((100 + 10) - (10 + 50) = 50\)
- Bob pays \((70 + 50) - (70 + 50) = 0\)
- Carol pays \((10 + 100) - (70 + 10) = 30\)
What is Pivotal about it?

Alternatives →

<table>
<thead>
<tr>
<th>Name</th>
<th>Value 1</th>
<th>Value 2</th>
</tr>
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<tbody>
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<td>10</td>
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- **Outcome:** L
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4. Summary
An important class of SCFs is that of affine maximizers

**Definition (Affine Maximizer)**

An SCF \( f : V \rightarrow A \) is an affine maximizer if there exists \( w_i \geq 0, i \in N \), not all zero, and a function \( \kappa : A \rightarrow \mathbb{R} \) such that,

\[
f(v) \in \arg\max_{a \in A} \left( \sum_{i \in N} w_i v_i(a) + \kappa(a) \right).
\]
Affine Maximizers

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**Special cases:**

- $w_i = 1, \forall i$ and $\kappa \equiv 0$: **efficient** SCF
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**Special cases:**

- $w_i = 1, \forall i$ and $\kappa \equiv 0$: **efficient** SCF
- $w_d = 1$, for some $d$, $w_i = 0, \forall i \neq d$ and $\kappa \equiv 0$: **dictatorial** SCF
An affine maximizer $f$ satisfies *independence of irrelevant agents* (IIA) if for every $i$ with $w_i = 0$ and for every $v_{-i} \in V_{-i}$,

$$f(v_i, v_{-i}) = f(v_i', v_{-i}), \forall v_i, v_i' \in V_i$$

This is a consistency condition for tie-breaking.
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Every affine maximizer satisfying IIA is implementable
Affine Maximizers (Contd.)

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- This is a consistency condition for tie-breaking
- Every affine maximizer satisfying IIA is implementable
- In particular, payments are of the following form: for all $i \in N$

$$p_i(v_i, v_{-i}) = \begin{cases} \frac{1}{w_i} \left( \sum_{j \neq i} w_j v_j(f(v)) + \kappa(f(v)) + h_i(v_{-i}) \right), & w_i > 0 \\ 0, & w_i = 0 \end{cases}$$

$f$ is an affine maximizer
Roberts’ Theorem

**Theorem (Roberts 1979)**

Let the allocation space \( A \) be finite with \( |A| \geq 3 \). If the space of valuations \( V \) is unrestricted, then an onto and dominant strategy implementable SCF \( f : V \rightarrow A \) is an affine maximizer.
Theorem (Roberts 1979)

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Understanding Roberts’ Theorem:

- Groves’ or pivotal mechanisms are implementable, but this result is giving a necessary condition for implementability.
Roberts’ Theorem

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- Groves’ or pivotal mechanisms are implementable, but this result is giving a necessary condition for implementability.
- Moreover, it provides a functional form characterization of the DSIC mechanisms (as opposed to Myerson’s monotonicity characterization).
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**Understanding Roberts’ Theorem:**

- Groves’ or pivotal mechanisms are implementable, but this result is giving a necessary condition for implementability.
- Moreover, it provides a functional form characterization of the DSIC mechanisms (as opposed to Myerson’s monotonicity characterization).
- If payments are enforced to be zero for every valuation profile $v$, then the only implementable mechanism is dictatorial - GS theorem is a corollary of this result.
If an SCF $f$ is implementable in a valuation space $V$, it is implementable in every valuation space $V' \subseteq V$ - same payments implement them and the number of incentive compatibility constraints reduce.
Some Observations and Implications

- If an SCF $f$ is implementable in a valuation space $V$, it is implementable in every valuation space $V' \subseteq V$ - same payments implement them and the number of incentive compatibility constraints reduce

- Efficient SCF is implementable in any valuation space
Some Observations and Implications

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- Unrestricted valuation space is crucial for Roberts’ theorem - some recent results show that the affine maximizer characterization is true even for certain restricted valuation spaces.
Some Observations and Implications

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- Unrestricted valuation space is crucial for Roberts’ theorem - some recent results show that the affine maximizer characterization is true even for certain restricted valuation spaces.

- Characterization of implementability in restricted domains is an active research area.

[A proof by pictures]
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4. Summary
Revenue Equivalence

If $p$ and $p'$ implement $f$ in dominant strategies, then

$$p_i(v) = p'_i(v) + \alpha_i(v_{-i}), \forall v \in V$$
Revenue Equivalence

If $p$ and $p'$ implement $f$ in dominant strategies, then

$$p_i(v) = p'_i(v) + \alpha_i(v_{-i}), \forall v \in V$$

**Theorem (Rockafeller 1997; Krishna and Maenner (2001))**

If the type space is convex and the valuations are linear in type, then an SCF, implementable in dominant strategies, satisfies revenue equivalence.
Revenue Equivalence

- If \( p \) and \( p' \) implement \( f \) in dominant strategies, then

\[
p_i(v) = p'_i(v) + \alpha_i(v_{-i}), \forall v \in V
\]

**Theorem (Rockafeller 1997; Krishna and Maenner (2001))**

*If the type space is convex and the valuations are linear in type, then an SCF, implementable in dominant strategies, satisfies revenue equivalence.*

**Theorem (Chung and Olszewski (2007))**

*Suppose the type space \( T \subseteq \mathbb{R}^n \) is a connected set, \( A \) is finite and the valuations are continuous in type. If an SCF is implementable in dominant strategies, then it satisfies revenue equivalence.*
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4. Summary
Green-Laffont-Holmström Characterization

- An efficient SCF $f$ chooses an alternative in $\arg\max_{a \in A} \sum_{j \in N} v_j(a)$
Green-Laffont-Holmström Characterization

- An efficient SCF $f$ chooses an alternative in $\arg\max_{a \in A} \sum_{j \in N} v_j(a)$

**Theorem (Green and Laffont (1979), Holmström (1979))**

*If the valuation space is convex and smoothly connected, every efficient and DSIC mechanism is a Groves mechanism.*

- Shows uniqueness of Groves class in the space of efficient, DSIC mechanisms

[A proof outline]
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4. Summary
Green-Laffont Impossibility

Theorem (Green and Laffont (1979))

No Groves mechanism is budget balanced (BB), i.e.,
\[ \exists p_i^{\text{Groves}} \text{ s.t. } \sum_{i \in N} p_i^{\text{Groves}}(v) = 0, \forall v \in V. \]

This leads to the following corollary

Corollary

If the valuation space is convex and smoothly connected, no efficient mechanism can be both DSIC and BB.
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4. Summary
AGV Mechanism

- If the equilibrium condition is relaxed to BIC, we have a positive result
- Payment is defined via a function $\delta_i, i \in N$:

$$\delta_i(v_i) = \mathbb{E}_{v_{-i} | v_i} \left( \sum_{j \in N \setminus \{i\}} v_j(a^*(v)) \right),$$

where $a^*$ is an efficient allocation
- Payment is:

$$p_{AGV}^i(v) = \sum_{j \in N \setminus \{i\}} \delta_j(v_j) - \delta_i(v_i)$$

**Theorem (d’Aspremont and Gerard-Varet (1979), Arrow (1979))**

*The AGV mechanism is BIC, efficient, and budget-balanced*
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4 Summary
Summary

DSIC mechanisms

Valuation / Type Space

Mechanism Space
Summary

DSIC mechanisms

Valuation / Type Space

Mechanism Space
Summary

DSIC mechanisms

Valuation / Type Space

Mechanism Space

Dict
**Summary**

DSIC mechanisms

Valuation / Type Space

Mechanism Space

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Summary

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Summary

DSIC mechanisms

Valuation / Type Space

Mechanism Space

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AM
Thank you!

✉️ swaprava@gmail.com

http://www.isid.ac.in/~swaprava
Value Difference Set

Q: What does affine maximizer mean?
Q: What does affine maximizer mean?
A: If \( f(v) = y \) then
\[
w^\top v(y) + \kappa(y) \geq w^\top v(z) + \kappa(z), \forall z \in A \setminus \{y\}
\]
\[
\Rightarrow w^\top (v(y) - v(z)) \geq \kappa(z) - \kappa(y), \forall z \in A \setminus \{y\}
\]
Q: What does affine maximizer mean?

A: If $f(v) = y$ then

$$w^T v(y) + \kappa(y) \geq w^T v(z) + \kappa(z), \forall z \in A \setminus \{y\}$$

$$\Rightarrow w^T (v(y) - v(z)) \geq \kappa(z) - \kappa(y), \forall z \in A \setminus \{y\}$$

$$w^T \alpha \geq \beta \quad \text{half-space}$$
Q: What does affine maximizer mean?
A: If \( f(v) = y \) then

\[
\begin{align*}
    w^T v(y) + \kappa(y) & \geq w^T v(z) + \kappa(z), \forall z \in A \setminus \{y\} \\
\Rightarrow w^T (v(y) - v(z)) & \geq \kappa(z) - \kappa(y), \forall z \in A \setminus \{y\} \\
    w^T \alpha & \geq \beta
\end{align*}
\]

Define the value difference set for any pair of distinct alternatives \( y, z \in A \).

\[
P(y, z) = \{ \alpha \in \mathbb{R}^n : \exists v \in V \text{ s.t. } v(y) - v(z) = \alpha \text{ and } f(v) = y \}.
\]
Value Difference Set

**Q:** What does affine maximizer mean?

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P(y, z) = \{\alpha \in \mathbb{R}^n : \exists v \in V \text{ s.t. } v(y) - v(z) = \alpha \text{ and } f(v) = y\}.
\]

**Claim**

If \( \alpha \in P(y, z) \), and \( \delta > 0 \in \mathbb{R}^n \), then \( \alpha + \delta \in P(y, z) \), for all distinct \( y, z \in A \).
Graphical Illustration for Two Players

\[ v_2(y) - v_2(z) \]

\[ P(y, z), y, z \in A \]
Complementary Structures of $P(y, z)$ and $P(z, y)$

**Claim**

For every $\alpha, \epsilon \in \mathbb{R}^n$, $\epsilon > 0$, and for all $y, z \in A$,

(a) $\alpha - \epsilon \in P(y, z) \implies -\alpha \notin P(z, y)$.

(b) $\alpha \notin P(y, z) \implies -\alpha \in P(z, y)$. 
\[ P(y, z), y, z \in A \]

\[ v_1(y) - v_1(z) \]

\[ v_2(y) - v_2(z) \]

\[ P(z, y), y, z \in A \]

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$v_2(y) - v_2(z)$

$P(y, z), y, z \in A$

$\gamma(y, z)$

$\gamma(z, y) = -\gamma(y, z)$

$P(z, y), y, z \in A$

$v_1(y) - v_1(z)$

$P(y, z), y, z \in A$
Independence of $\hat{C}$ from the Alternatives in $A$

- Define the translated set $C(y, z) = P(y, z) - \gamma(y, z)1$
- Denote the ‘interior’ of $C(y, z)$ by $\hat{C}(y, z)$
Independence of $\hat{\mathcal{C}}$ from the Alternatives in $A$

- Define the translated set $\mathcal{C}(y, z) = P(y, z) - \gamma(y, z)1$
- Denote the ‘interior’ of $\mathcal{C}(y, z)$ by $\hat{\mathcal{C}}(y, z)$

Claim

$\hat{\mathcal{C}}(y, z) = \hat{\mathcal{C}}(w, l)$, for any $y, z, w, l \in A$, $y \neq z$ and $w \neq l$. 
Independence of $\hat{C}$ from the Alternatives in $A$

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**Claim**

$\hat{C}(y, z) = \hat{C}(w, l)$, for any $y, z, w, l \in A$, $y \neq z$ and $w \neq l$.

**Remark:** Note that this result, in particular, includes the cases, $\hat{C}(y, z) = \hat{C}(l, z) = \hat{C}(l, y) = \hat{C}(z, y)$. Therefore, the claim holds even without $y, z, w, l$ being all distinct.
\[ \hat{C}(y, z) = \hat{C}'(z, y) = C \]

\[ v_2(y) - v_2(z) \]

\[ v_1(y) - v_1(z) \]
Convexity of $C$

Claim

*The set $C$ is convex.*
Holmström Characterization

- Set of allocations $A = \{a, b\}$
- Social welfares at these two allocations are $\sum_{j \in N} v_j(a)$ and $\sum_{j \in N} v_j(b)$
Holmström Characterization

- Set of allocations $A = \{a, b\}$
- Social welfares at these two allocations are $\sum_{j \in N} v_j(a)$ and $\sum_{j \in N} v_j(b)$
- Efficiency requires that if $a$ is chosen, then $\sum_{j \in N} v_j(a) \geq \sum_{j \in N} v_j(b)$
Holmström Characterization

- Set of allocations $A = \{a, b\}$
- Social welfares at these two allocations are $\sum_{j \in N} v_j(a)$ and $\sum_{j \in N} v_j(b)$
- Efficiency requires that if $a$ is chosen, then $\sum_{j \in N} v_j(a) \geq \sum_{j \in N} v_j(b)$
- Fix valuations of agents other than $i$ at $v_{-i}$
- Fix valuations of agent $i$ except allocation $a$, i.e., at $b$ at $v_i(b)$
Holmström Characterization

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- Consider $v_i(a) = v_i^*(a) + \epsilon$, $\epsilon > 0$, and write the DSIC constraint:
  \[
  v_i^*(a) + \epsilon - p_{i,a} \geq v_i(b) - p_{i,b}
  \] (1)
  - outcome does not change $\Rightarrow$ payment does not change
Consider $v_i(a) = v_i^*(a) - \delta$, $\delta > 0$, and similarly:

$$v_i(b) - p_{i,b} \geq v_i^*(a) - \delta - p_{i,a} \quad (2)$$

Combining Equations (1) and (2) and taking limits $\epsilon, \delta \rightarrow 0$, we get,

$$v_i(b) - p_{i,b} = v_i^*(a) - p_{i,a}$$
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v_i(b) - p_{i,b} = v_i^*(a) - p_{i,a}
\]

- Since \( v_i^*(a) \) is a threshold of change of efficient outcome,

\[
v_i^*(a) + \sum_{j \in N\setminus\{i\}} v_j(a) = v_i(b) + \sum_{j \in N\setminus\{i\}} v_j(b)
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Substituting:

$$p_{i,a} - p_{i,b} = - \left( \sum_{j \in N \setminus \{i\}} v_j(b) - \sum_{j \in N \setminus \{i\}} v_j(a) \right)$$