

Temporal Cooperative Games

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Abstract

Classical cooperative game theory assumes that the worth of a coalition is determined solely by the *set* of agents involved. In practice, however, the worth may also depend on the *order* in which agents arrive. Motivated by such scenarios, we consider *temporal cooperative games* (TCG), where the worth of a set of agents depends on their order of arrival, i.e., the worth v becomes a function of the sequence of agents π , rather than just the set S of agents. This shift requires a fundamental rethinking of the desired axioms.

A key property in this temporal framework is the *incentive for optimal arrival* (I4OA), which encourages agents to join in the order that maximizes the total worth. Alongside this, we define two additional natural properties: *online individual rationality* (OIR), which incentivizes earlier agents to invite additional agents, and *sequential efficiency* (SE), which requires that the total worth for any sequence is fully distributed among its agents. We identify a class of reward-sharing mechanisms uniquely characterized by these three properties. The celebrated Shapley value does not directly apply here, and we construct natural analogs of the Shapley value in two variants: the *sequential* world, where rewards are defined for each sequence-player pair, and the *extended* world, where rewards are defined for each player alone. We show that properties *efficiency*, *additivity*, and *null player* uniquely determine the Shapley analogs in both worlds. Crucially, the Shapley analogs are disjoint from those satisfying the properties I4OA, OIR, and efficiency. The conflict persists even for the restricted classes of *convex* and *simple* temporal cooperative games.

Our results thus reveal a fundamental tension: when players arrive sequentially, reward-sharing mechanisms that satisfy desirable temporal properties must differ in nature from the Shapley analogs. This opens up a bigger research question of finding *good* solution concepts for TCGs.

1 Introduction

Organizations and institutions benefit from the complementary skill sets of their employees. However, the growth and overall value of an institution critically depend on the *order* in which employees join. Early joiners establish the vision and direction of the institution, which subsequently guides the hiring of core domain experts, followed by process enablers. Support staff and marketing personnel are then integrated in a sequence that aligns with the institutions structure. A different order of joining can result in markedly different growth trajectories and institutional value. Classical cooperative games (Maschler et al., 2020) often overlook this phenomenon, as their worth functions are defined only for coalitions, ignoring the *order of the agents* within the coalition. While Nowak and Radzik (1994); Sánchez and Bergantiños (1997) incorporate agent order through *generalized* characteristic functions, their primary goal is to extend the Shapley value to that context. In contrast, we focus on cooperative games in which the worth is sequence-dependent, which we term *temporal cooperative games* (TCGs),

and our primary objective is to study the properties desirable in this new space as well as their simultaneous satisfaction with Shapley-like properties.

The first two properties we define for TCGs are inspired by similar notions in Ge et al. (2024). They introduce *incentive for early arrivals*, which encourages players to join as soon as they arrive a notion suited for classical cooperative games with dynamic arrivals. In this paper, we aim to incentivize agents to join in the *optimal* order that maximizes institutional worth. We term this property *incentive for optimal arrival* (I4OA). Without it, agents might join in arbitrary order, driven, for example, by short-term salary gains for high-profile roles such as CEO or CTO, rather than by their contribution to institutional value, which is undesirable.

The second property, *online individual rationality* (OIR), is identical to that in Ge et al. (2024). OIR ensures that agents joining earlier receive weakly better payoffs as subsequent agents join, thereby encouraging early agents to facilitate the arrival of additional participants.

We also consider a standard third property, *sequential efficiency* (SE), which guarantees that the worth of any sequence is fully allocated among the agents in that sequence. This property is crucial when rewards must be distributed immediately upon agent arrivals, without waiting for future participants. In this paper, we characterize a class of solution concepts that satisfy these three properties.

Ideally, one would like a solution concept that satisfies these properties while also adhering to Shapley-inspired axioms in TCGs. However, we prove that this is *impossible*: solution concepts that uniquely satisfy Shapley-inspired properties such as efficiency, additivity, and the null-player condition are disjoint from those that satisfy TCG-appropriate properties like I4OA, OIR, and efficiency (Figure 6). Remarkably, this impossibility persists even for restricted TCG classes such as *convex* and *simple* games, highlighting a stark contrast with classical cooperative games.

1.1 Our Contributions

The contributions of this paper are as follows:

- We study the framework of *temporal cooperative games* (TCGs), originally proposed by Nowak and Radzik (1994), where worth is assigned to sequences of agents rather than coalitions. We introduce properties that are natural in this setting, namely I4OA, OIR, and SE (Section 2).
- We design a class of reward-sharing mechanisms (SEQSHARE, Algorithm 1) that uniquely satisfies these properties. Furthermore, we identify a condition, termed *basis*, that is *necessary* for their satisfaction (Lemma 1).
- To compare our results with Shapleys axioms, we formulate suitable adaptations of *additivity*, *null-player*, and *efficiency* for TCGs, where solution concepts are defined for every sequence (Section 4). We show that a Shapley-analogous solution concept, MARGSOL, uniquely satisfies these properties (Section 5.1).
- Since the Shapley value averages over sequences to give a single value to every player, we extend our solution concepts to define *extended* counterparts. We prove that extended additivity, null-player, and efficiency uniquely identify the EXT-SHAP solution concept (Section 5.2).
- We prove that a property analogous to *symmetry* follows directly from the three earlier properties (Section 6), underscoring both the richness of the domain and the strictness of the axioms in this setting.

- Finally, we show that EXT-SHAP is disjoint from the extended version of SEQSHARE, establishing that the desirable properties of TCGs are fundamentally in conflict with Shapley-inspired properties, even in special classes such as *convex* and *simple* games (Section 7).

1.2 Related Work

Classical cooperative game theory originated in the pioneering work of [Von Neumann and Morgenstern \(1947\)](#). Soon after, foundational solution concepts such as the core ([Gillies, 1959](#)), the nucleolus ([Schmeidler, 1969](#)), and the Shapley value ([Shapley, 1953](#)) were introduced. The study of online coalition formation is relatively recent: [Flammini et al. \(2021\)](#) addressed the problem in its general form, [Bullinger and Romen \(2023\)](#) considered random arrivals, and [Bullinger and Romen \(2025\)](#) studied stability in this setting. In parallel, [Lehrer and Scarsini \(2013\)](#) examined dynamic cooperative games in which the worth of a coalition varies over time, while [Habis and Jean-Jacques Herings \(2010\)](#); [Kranich et al. \(2005\)](#) extended the concept of the core to dynamic environments. Some other works consider the characterisation of solutions concepts that satisfy modification of the Shapley properties for classical cooperative games. [Van Den Brink \(2002\)](#) use Efficiency Null Player and an additional property fairness defined appropriately for cooperative games, to characterize the Shapley value. [Casajus and Huettner \(2013\)](#) modify the null player property to obtain a class of solutions. Several other works propose alternative properties that characterize the Shapley Value ([Hamiache, 2001](#); [Casajus, 2014](#)).

Online cooperative games, where agents arrive sequentially and are incentivised to satisfy properties such as immediate participation, have been studied in monotone coalitional games by [Ge et al. \(2024\)](#) and in cost-sharing games by [Zhang et al. \(2025a\)](#). Axiomatic approaches for online cooperative games were explored by [Aziz et al. \(2025\)](#), while [Zhang et al. \(2025b\)](#) developed stable online coalition formation mechanisms aimed at maximizing social welfare. However, in all these models, the worth is still defined for coalitions of agents, not for sequences. By contrast, our work focuses on cooperative games in which the worth depends explicitly on sequences, and one of the objectives is to incentivize an optimal arrival sequence rather than merely incentivizing early arrival.

The idea of generalized characteristic functions defined over ordered subsets was first proposed by [Nowak and Radzik \(1994\)](#) and subsequently extended by [Sánchez and Bergantiños \(1997, 1999\)](#); [Bergantiños and Sánchez \(2001\)](#). While these functions define worth in a sequence-dependent manner, the associated solution concepts remain sequence-independent. In particular, [Nowak and Radzik \(1994\)](#) extended Shapley-like properties, such as *additivity*, *null player*, and *efficiency*, to temporal cooperative games (TCGs) and proposed a solution concept that uniquely satisfies them. These results are closely related to some of ours, which we obtained independently before becoming aware of their work. Our focus, however, is on identifying and characterizing properties that are desirable in TCGs and demonstrating their conflict with Shapley-inspired properties, even within restricted settings such as convex and simple games. Furthermore, we establish connections between sequence-dependent and sequence-independent solution concepts, leading to two unique solutions: MARGSOL and EXT-SHAP. Finally, [Sánchez and Bergantiños \(1997\)](#) proposed a weaker version of the null-player property than ours (for details, see Section 4) and required the introduction of symmetry to obtain a unique solution that satisfies all four properties, which we do not. So, even though this literature is closest to our work, there remain significant differences in both modeling choices and results.

2 Preliminaries

Consider a set of agents $N = \{1, \dots, n\}$. Denote 2^N to be the set of all possible subsets of N . Let $\text{PERM}(S)$ denote all possible permutations of the players in S , where $S \in 2^N \setminus \emptyset$. Define all possible sequences of any length by $\Pi := \{\pi : \pi \in \text{PERM}(S), S \in 2^N \setminus \emptyset\}$. We denote the set of agents in a sequence π by $P(\pi)$ and the player at position i in π as $\pi(i)$. We define a characteristic function $v : \Pi \rightarrow \mathbb{R}$, which assigns a worth to *every sequence of agents*. A *temporal cooperative game (TCG)* is thus described by a tuple $\langle N, v \rangle$. We also define $\Pi_{-N'} := \{\pi : \pi \in \text{PERM}(S), S \in 2^{N \setminus N'}\}$, i.e., the set of all agent sequences except the agents in set N' . Also, denote the last player of a sequence π as $\ell(\pi)$.

We call π' to be a *prefix* of π if the first $|P(\pi')|$ players of π are the same as that of π' , i.e., $\pi(i) = \pi'(i), \forall i = 1, 2, \dots, |P(\pi')|$. A prefix π' of π is denoted as $\pi' \sqsubset \pi$. Note that $|P(\pi')| < |P(\pi)|$, otherwise they become the same sequence. We assume that the characteristic function is *monotone*, i.e., $v(\pi') \leq v(\pi)$ for all $\pi' \sqsubset \pi$. We represent the *predecessor* of i in π with π_i which denotes the longest prefix of π not containing i . A sequence π is a *full sequence* if it contains all the players, $P(\pi) = N$. Denote the set of all monotone characteristic functions by V .

The goal of this paper is to find desirable axioms under this setting and obtain a solution concept $\phi_i, i \in N$, which divides the worth among the players and satisfies these axioms. The term $\phi_i(\pi, v)$ denotes the share of value to agent $i \in P(\pi)$ when arriving in the sequence π in the TCG $\langle N, v \rangle$. We denote by $\phi(\pi, v)$ the vector $(\phi_i(\pi, v), i \in N)$, with the convention that the solution concept assigns zero reward to the agents outside π , i.e., $\phi_j(\pi, v) = 0, \forall j \in N \setminus P(\pi)$. For the rest of the paper, we use $\phi(\pi)$ to denote the reward vector, dropping v when it is clear from context. We denote the optimal sequence $\pi^*(v)$ to be the one that maximizes the worth of this game, i.e., $\pi^*(v) \in \arg\max_{\pi \in \Pi} v(\pi)$. WLOG, we assume that the tie-breaking rule is such that $P(\pi^*(v)) = N$ (with arbitrary tie-breaking among the full length sequences), since the game is monotone. However, our results hold true even when $P(\pi^*(v)) \subset N$, and the proof can be easily adapted.

We now define certain desirable properties specific to temporal cooperative games. The players arrive sequentially in a TCG, thus the rewards must also be given as they arrive. To an existing sequence π , when a new player i arrives, we denote the new sequence by $\pi + i$. The worth generated by the sequence $\pi + i$ must be distributed among the players present in the sequence. Our first property ensures that the earlier players are incentivized when a new player joins, i.e., the solution concept should never decrease the reward of a player as new players join.

Definition 1 (Online Individual Rationality). A solution concept ϕ is *online individually rational* (OIR) if for every TCG $\langle N, v \rangle$

$$\phi_i(\pi, v) \leq \phi_i(\pi', v), \forall i \in P(\pi), \forall \pi \sqsubset \pi', \text{ where } \pi, \pi' \in \Pi.$$

The sequential nature of the game also creates the possibility of the players to stop arriving. Also the next player's arrival is uncertain beforehand. Therefore, the solution concept should ensure the the worth of a given sequence π is given out to all the players in $P(\pi)$ for every sequence π .

Definition 2 (Sequential Efficiency (SE)). A solution concept ϕ is *sequentially efficient* if for every TCG $\langle N, v \rangle$

$$\sum_{i \in P(\pi)} \phi_i(\pi, v) = v(\pi), \forall \pi \in \Pi.$$

Since the number of agents are finite, every TCG will have an *optimal sequence* that generates the most value. We denote the *optimal sequence* of a TCG $\langle N, v \rangle$ by $\pi^*(v)$. Our next

property incentivizes players to arrive in this optimal order. We use π^* to denote the optimal sequence when the game $\langle N, v \rangle$ is clear from the context.

Definition 3 (Incentive for Optimal Arrival (I4OA)). A solution concept ϕ satisfies *incentive for optimal arrival* (I4OA) if for every TCG $\langle N, v \rangle$

$$\phi_i(\pi^*(v), v) \geq \phi_i(\pi, v), \forall \pi \in \Pi, i \in N.$$

In the first part of this paper, we characterize reward sharing mechanisms that satisfy OIR, I4OA, and SE for TCGs. In the second part, we define properties analogous to the celebrated Shapley properties in classical cooperative games and characterize the solution concepts that satisfy them.

3 The SeqShare Class of Mechanisms

In this section, we introduce a class of reward sharing mechanisms that we later show to uniquely satisfy OIR, I4OA, and SE. As a precursor to this class, we first define two conditions called the *basis* conditions for a TCG $\langle N, v \rangle$.

Definition 4 (Basis conditions). A TCG $\langle N, v \rangle$ satisfies *basis conditions* if there exists a vector $x \in \mathbb{R}^n$ such that the following conditions hold

$$\begin{aligned} (i) \quad & \sum_{i \in P(\pi)} x_i \geq v(\pi), \forall \pi \in \Pi, \\ (ii) \quad & \sum_{i \in P(\pi^*(v))} x_i = v(\pi^*(v)). \end{aligned} \tag{1}$$

We call the vector $x \in \mathbb{R}^n$ that satisfies the above conditions a *basis solution*.

The importance of the basis condition can be understood from the following result that shows that this condition is necessary for two desirable properties of TCGs. The basis conditions are analogous to the *core* of classical cooperative games. Certain desirable properties are contingent on these sets to be non-empty, while it has a chance of being empty too.

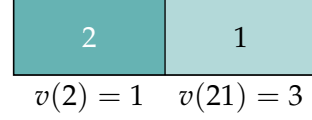
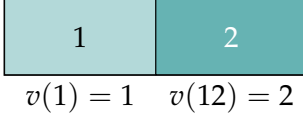
Lemma 1. *For a TCG $\langle N, v \rangle$, if a solution concept ϕ satisfies I4OA and SE then there must be a basis solution.*

Proof. Consider a TCG $\langle N, v \rangle$ and a solution concept ϕ that satisfies I4OA and SE in the game. We claim that $x = \phi(\pi^*)$ satisfies the basis condition.

- Condition (ii) of [Equation \(1\)](#) is satisfied by applying SE of ϕ for π^* .
- Since ϕ also satisfies I4OA, $\phi_i(\pi) \leq \phi_i(\pi^*), \forall \pi \in \Pi, \forall i \in N$. Hence, $\sum_{i \in P(\pi)} \phi_i(\pi) \leq \sum_{i \in P(\pi)} \phi_i(\pi^*)$. By SE of ϕ , $v(\pi) = \sum_{i \in P(\pi)} \phi_i(\pi)$. Therefore, we get $\sum_{i \in P(\pi)} \phi_i(\pi^*) \geq v(\pi)$, which is condition (i) of [Equation \(1\)](#).

Hence $\phi(\pi^*)$ satisfies basis conditions. □

Remark. The result above does not guarantee existence of a solution concept satisfying I4OA and SE if a basis solution exists. In the following example we show examples of solutions that satisfy the basis and use the basis solution as the reward for π^* but still fall short of at least one of the given properties. Existence of a basis solution does not guarantee that trivial solution concepts can satisfy the 3 properties together, as we illustrate in the following example. This shows that method we propose needs to do a non-trivial work to ensure all three.



π	$v(\pi)$	ϕ	ϕ'	ϕ''
1	1	(0,0)	(1,0)	(1,0)
2	1	(0,0)	(0,1)	(0,1)
12	2	(0,0)	(0,2)	(2,0)
21	3	(1,2)	(1,2)	(1,2)

Example 1. Consider the TCG as shown in the figure below. It is easy to check that the game has a basis solution, but the reward share ϕ given below does not satisfy the OIR property.

In this example, $\pi^* = 21$, $x_1 = 1$, $x_2 = 2$. Now, consider the following solution concepts ϕ . ϕ satisfies OIR and I4OA but not SE since $v(12) \neq \phi_1(12) + \phi_2(12)$. ϕ' satisfies SE and I4OA but not OIR, $\phi(1) > \phi(12)$. ϕ'' satisfies SE and OIR but not I4OA $\phi_1(12) > \phi_1(21)$. $\square$

Our next endeavor is to construct a solution concept that will satisfy both and OIR.

Our desired class of solution concepts start with the *basis* conditions to check if a solution exists. If the basis conditions hold, **Algorithm 1** yields a class of solution concepts that satisfies OIR, I4OA, and SE. It is a class because there are multiple possible solution concept ϕ that can be given by this algorithm. However, we will show that each of them satisfies the three above desirable properties, and any solution concept satisfying the three is a valid solution concept given by this algorithm. The class SEQSHARE performs the following steps:

1. It checks and returns a basis solution if it exists. The returned value x_i is treated as the upper bound for agent i 's reward for all sequences. Hence, $\phi_i(\pi^*)$ is assigned x_i .
2. Given a sequence π where $\pi(1) = i$, the algorithm assigns $\phi_i(i) = v(i)$. This value now serves as a lower bound for $\phi_i(\pi)$ for every sequence having i as a prefix.
3. It computes the marginal contribution made by $\pi(2) = j$, and divides it among i and j such that their rewards $\phi_i(ij)$ and $\phi_j(ij)$ lies within the upper and lower bounds set for them. The lower bounds for the agents i and j are then updated to $\phi_i(ij)$ and $\phi_j(ij)$ respectively for every sequence containing ij as a prefix.
4. The algorithm continues this iterative process of assigning rewards within the lower and upper bounds at the arrival of each agent, followed by updating the lower bound to the newest assigned value.

Figure 2 provides a visual representation of SEQSHARE. The non-uniqueness of the solution concept given by SEQSHARE has two distinct sources. First, the basis solution may not be unique as **Equation (1)** may be satisfied by multiple x vectors. Second, given a basis solution, there may be multiple functions that are suitable candidates for ϕ as the IMPROVIZE function returns a non-unique y vector. Hence, SEQSHARE is a class based on these two freedom of choice. Once a solution concept ϕ is obtained via SEQSHARE, it can be implemented in an online fashion. This is because ϕ gives a reward share for every player in every sequence $\pi \in \Pi$. Hence, whenever an agent appears in any sequence, ϕ has a reward share for that agent at that point of arrival.

Algorithm 1 Class of Mechanisms SEQSHARE

Require: TCG $\langle N, v \rangle$

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1: Find  $\pi^*(v) = \operatorname{argmax}_{\pi} v(\pi)$ 
2: Let  $x = \text{BASIS}(N, v, \pi^*(v))$ 
3: if  $x = \text{NULL}$  then
4:   Output NULL, EXIT
5: else
6:    $\phi(\pi^*(v)) = x$ 
7:    $\phi(\pi) = \mathbf{0}, \forall \pi \neq \pi^*(v)$ 
8: end if
9: for  $i \in N$  do
10:   $\phi_i(\{i\}) = v(\{i\})$ 
11: end for
12: for  $k = 2, \dots, n$  do
13:   for  $\pi \in \Pi, |\pi| = k - 1$  do
14:    for  $j \in N \setminus P(\pi)$  do
15:      $y =$ 
      IMPROVIZE( $j, \pi, \phi(\pi), \phi(\pi^*(v))$ )
16:      $\phi_i(\pi + j) = \phi_i(\pi) + y_i, \forall i \in P(\pi)$ 
17:      $\phi_j(\pi + j) = y_j$ 
18:   end for
19: end for
20: end for
21: Output:  $\phi$ 

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1: procedure BASIS( $N, v, \pi^*(v)$ )
2:   if solution to Equation (1) exists then
3:     return  $x$  of Equation (1) w.r.t.  $\pi^*(v)$ 
4:   else
5:     return NULL
6:   end if
7: end procedure

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1: procedure IMPROVIZE( $j, \pi, \phi(\pi), \phi(\pi^*)$ )
2:    $y_i \in [0, \phi_i(\pi^*) - \phi_i(\pi)], \forall i \in P(\pi + j)$ 
3:    $\sum_{i \in P(\pi + j)} y_i = v(\pi + i) - v(\pi)$ 
4:   return  $y$ 
5: end procedure

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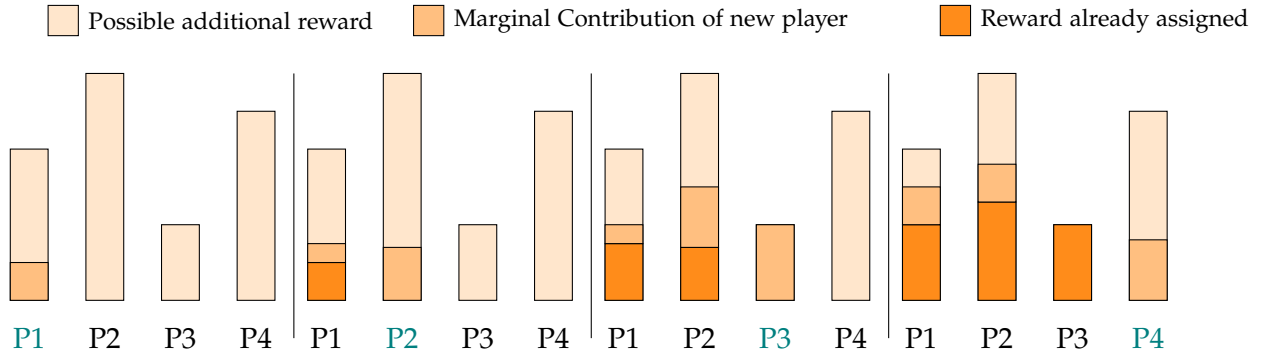


Figure 2: (a) The first player arrives and all the value generated is assigned to player P1. (b) The second player P2 arrives and the marginal value generated by the arrival of this player is distributed among P1 and P2 in a way that ensures the total reward of each agent is less than the upper bound. The total reward for any agent is the sum of the reward in the previous round and the share of marginal increase assigned in the current round. (c) The third player P3 arrives and the marginal value generated by P3 is distributed among P1, P2 and P3 keeping total rewards lower than the upper bound. (d) Finally, P4 arrives and the marginal value generated by this player can only be divided among P1, P2 and P4 as P3 has already reached its maximum possible reward.

Remark. While we use a basis solution as part of the algorithm, it should be noted that the existence of a basis solution is a property of a given TCG $\langle N, v \rangle$ and not that of a solution concept.

3.1 Properties of SeqShare

In this section, we present the results of the properties of SEQSHARE. We first show that the existence of a basis solutions is sufficient to ensure a solution that satisfies OIR, I4OA, and SE using the procedures of SEQSHARE. Further, we show that if a game has a solution that satisfies OIR, I4OA, and SE it is a solution concept in the SEQSHARE class. We have already proved [Lemma 1](#) that states that the existence of a basis solution is necessary to satisfy I4OA and SE. The following theorem completes this lemma with the OIR property.

Theorem 1. *Given a TCG $\langle N, v \rangle$ a reward function ϕ satisfies OIR, I4OA, and SE iff it is from the class SEQSHARE.*

Proof. If the game $\langle N, v \rangle$ does not satisfy the basis conditions, then from [Lemma 1](#) we know that no ϕ can satisfy I4OA and SE. In that case, SEQSHARE returns NULL and hence the theorem holds. For the rest of the proof, we will assume that the game satisfies the basis conditions.

First, we prove the reverse direction, constructively via [Algorithm 1](#). The function BASIS returns a valid *partial* allocation x , which is assigned to the value of the solution concept for the specific sequence $\pi^*(v)$, i.e., $\phi(\pi^*(v))$, and ϕ is initialized to zero for all other sequences (including the empty sequence). The rest of the proof is via induction on the length of the sequence π .

First, observe that for all unit length sequences, the allocation given by [Algorithm 1](#) trivially satisfies SE. Property I4OA is satisfied due to the fact that $\phi_i(\{i\}) = v(\{i\}) \leq \phi_i(\pi^*(v))$, $\forall i \in N$, given by the BASIS conditions. OIR is trivial too, since the game is monotone and $\phi_i(\emptyset) = 0, \forall i \in N$.

Hence, by the induction hypothesis, assume that the forward implication of the theorem is true for all π such that $|\pi| = k - 1, k \geq 2$. We show that the implication is true for every $\pi + i$ of length k as well. Consider the function IMPROVIZE. For every π of length $k - 1$ and the existing partial share $\phi(\pi)$, IMPROVIZE allocates the reward to agents in such a way that it is at least $\phi_i(\pi)$ (required by OIR) and at most $\phi_i(\pi^*)$ (required by I4OA). This allocation is always possible since by the monotonicity of the game ensures $v(\pi + i) \geq v(\pi)$ and $v(\pi + i) \leq v(\pi^*)$ (by definition of π^*). Hence, I4OA and OIR are satisfied for each $i \in N$. Also, the construction of the y vectors (reward shares) is such that the sum of rewards of all agents $j \in \pi + i$ equals $v(\pi + i)$, satisfying SE.

Next, we prove the forward direction. We prove this via contradiction. Suppose there is a solution concept ϕ for a given game $\langle N, v \rangle$ that satisfies I4OA, SE, and OIR but is not from the class SEQSHARE. Then we will construct a solution concept $\tilde{\phi} \in \text{SEQSHARE}$ which matches ϕ for every sequence and for every player, leading to the contradiction that $\phi \in \text{SEQSHARE}$.

For every solution ϕ' from class SEQSHARE, there is a shortest length sequence $\pi \in \Pi$ such that $\phi'_i(\pi) \neq \phi_i(\pi)$ for some $i \in P(\pi)$ while $\phi'_i(\pi') = \phi_i(\pi')$ for all $i \in P(\pi')$ and $\pi' \in \Pi : |\pi'| < |\pi|$, i.e., π has the minimum length where the solution concepts ϕ and ϕ' differ.

Choose the ϕ' where the length of the sequence π is the longest over all ϕ' s (breaking ties arbitrarily). Call this longest (over solution concepts) shortest (over sequences) sequence to be $\pi_{\text{longshort}}$. Note that, by assumption, no $\phi'' \in \text{SEQSHARE}$ can satisfy $\phi''_i(\pi) = \phi_i(\pi)$ for all $i \in P(\pi)$ where $\pi \in \Pi$ and $|\pi| \leq |\pi_{\text{longshort}}|$. We will construct a $\tilde{\phi}$ that achieves this and belongs to SEQSHARE to show a contradiction.

Start with a game where the required properties are satisfiable. Since this game has a solution concept ϕ that satisfies I4OA and SE, BASIS must be non-empty ([Lemma 1](#)). Also, $\phi(\pi^*)$ will be a solution to the basis by [Lemma 1](#). Hence, set $\tilde{\phi}$ such that $\tilde{\phi}(\pi^*) = \phi(\pi^*)$.

Construct $\tilde{\phi} \in \text{SEQSHARE}$ as follows: $\tilde{\phi}_i(\pi') = \phi_i(\pi'), \forall i \in P(\pi'), \forall \pi' : |\pi'| < |\pi_{\text{longshort}}|$. Clearly, this is feasible, since there are $\phi'' \in \text{SEQSHARE}$ that satisfy these conditions and the construction of ϕ'' is always from shorter to longer sequences.

Since ϕ , the solution concept not in SEQSHARE , satisfies OIR, $\phi_i(\pi_{\text{longshort}}) - \phi_i(\pi_{\text{longshort}} - \ell(\pi_{\text{longshort}})) = y_i \geq 0$ for all $i \in P(\pi_{\text{longshort}})$, where $\ell(\pi)$ is the last agent of the sequence π and the subtraction $\pi - \ell(\pi)$ denotes that the last agent of π is removed from the end of that sequence. Since, ϕ satisfies I4OA, $\phi_i(\pi_{\text{longshort}}) \leq \phi_i(\pi^*)$ for all $i \in P(\pi_{\text{longshort}})$.

Now, set, $\forall i \in P(\pi_{\text{longshort}})$

$$\begin{aligned}\tilde{\phi}_i(\pi_{\text{longshort}}) &= \tilde{\phi}_i(\pi_{\text{longshort}} - \ell(\pi_{\text{longshort}})) + y_i \\ &= \phi_i(\pi_{\text{longshort}} - \ell(\pi_{\text{longshort}})) + y_i \\ &= \phi_i(\pi_{\text{longshort}}).\end{aligned}$$

Note that this retains the solution concept $\tilde{\phi}$ in SEQSHARE since it adds y_i on top of the previously allocated reward to the sequence $\pi_{\text{longshort}} - \ell(\pi_{\text{longshort}})$, since the value that y_i can take is according to the function IMPROVIZE. After adding $y_{\ell(\pi_{\text{longshort}})}$ it gives $\phi_{\ell(\pi_{\text{longshort}})}(\pi_{\text{longshort}})$ which is at most $\phi_{\ell(\pi_{\text{longshort}})}(\pi^*) = \tilde{\phi}_{\ell(\pi_{\text{longshort}})}(\pi^*)$. The second equality above comes because by construction $\tilde{\phi}$ matches ϕ for all sequences of length less than that of $\pi_{\text{longshort}}$.

Since $\pi_{\text{longshort}}$ was chosen arbitrarily among all possible longest shortest sequences, and this process of assigning $\tilde{\phi}_i(\pi_{\text{longshort}})$ does not affect other sequences of the same length, we can do the same for all other sequences of the same length simultaneously. Hence, $\tilde{\phi}$ matches ϕ for all sequences of that length and lower. This completes the proof. $\square$

To position this class of solution concepts with that of the classical cooperative games, we consider the analogs of the Shapley properties in the context of temporal cooperative games.

4 Properties Inspired by Shapley in TCGs

Classical cooperative games and the classic solution concept due to Shapley operates on the setting where the characteristic function v depends on a coalition. In TCG, the function v depends on a sequence of players. To compare the solution concepts with that of the Shapley properties for a TCG $\langle N, v \rangle$, we first need to distinguish the solution concepts where it depends on the sequence π (we will call this *solution concepts* as we did so far in this paper) and the ones where it *does not* depend on the sequence (we will call this *extended solution concepts* in the rest of this paper). The space of solution concepts is thus given by $\Phi = \{\phi : \Pi \times V \rightarrow \mathbb{R}^n\}$, and the space of *extended solution concepts* is given by $\Psi = \{\psi : V \rightarrow \mathbb{R}^n\}$, where V is the set of all monotone characteristic functions. A solution concept, however, can be *reduced* to an extended solution concept by averaging it over all possible full sequences, i.e., $\pi : P(\pi) = N$, as follows.

Definition 5. Given a TCG $\langle N, v \rangle$ and a solution concept $\phi \in \Phi$, the reduction of ϕ to an extended solution concept $\bar{\phi} \in \Psi$ is defined as

$$\bar{\phi}_i = \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \phi_i(\pi). \quad (2)$$

In parity with the Shapley properties, we will now define the *additivity*, *efficiency* and *null player* in the context of both types of solution concepts. To distinguish them, the properties for the solution concept that depends on the sequence are called *sequential* properties, while those for the extended solution concept are called *extended* solution concepts. We begin with the sequential properties. The sequential efficiency property is already defined in [Definition 2](#).

Definition 6 (Sequential Additivity (SA)). A solution concept $\phi \in \Phi$ is *sequentially additive* if for every pair of TCGs $\langle N, v \rangle$ and $\langle N, u \rangle$,

$$\phi_i(\pi, u) + \phi_i(\pi, v) = \phi_i(\pi, u + v), \forall \pi \in \Pi, \forall i \in N. \quad (3)$$

Definition 7 (Sequential Null Player (SNP)). A solution concept $\phi \in \Phi$ satisfies *sequential null player* if for every TCG $\langle N, v \rangle$

$$v(\pi + i) = v(\pi), \forall \pi \in \Pi_{-i} \implies \phi_i(\pi) = 0, \forall \pi \in \Pi. \quad (4)$$

We now turn to the extended properties defined as follows.

Definition 8 (Extended Efficiency (EE)). An extended solution concept $\psi \in \Psi$ satisfies *extended efficiency* if for every TCG $\langle N, v \rangle$

$$\sum_{i \in N} \psi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi) = N} v(\pi). \quad (5)$$

The null player property in the extended space ensures that the player's extended reward is zero.

Definition 9 (Extended Null Player (ENP)). An extended solution concept $\psi \in \Psi$ satisfies *extended null player* if for every TCG $\langle N, v \rangle$

$$v(\pi + i) = v(\pi), \forall \pi \in \Pi_{-i} \implies \psi_i(v) = 0. \quad (6)$$

Note that this definition of extended null player is stronger than that in (Sánchez and Bergantiños, 1997), where a player is null only if it does not improve the worth of a sequence if it is inserted in any position of the sequence, i.e., starting from the first, second, till the last position, while our null player only requires the agent to be added at the end. Since the 'if' condition of our definition is weaker, the property ENP is stronger than their null player property. Their characterization of the solution concepts require four (and not three) properties, including symmetry, while we need only three, as we will soon see in Section 5. We show in the following example that even if we use their definitions of the Shapley properties in this setting, the TCG-appropriate properties of solution concepts are in conflict with that of the Shapley-inspired properties, which is the fundamental conclusion of our paper.

Example 2. Consider the TCG given in Figure 3. The only solution that lies in SEQSHARE is $\phi(1) = (1, 0), \phi(2) = (0, 1), \phi(12) = (1, 0), \phi(21) = (1, 1)$. The reduction of this solution is $\bar{\phi} = (1, 0.5)$. The solution characterized by [sanchez1997values] for this game is $\psi = (0.75, 0.75)$. This tells us that there exist games for which no $\phi \in \text{SEQSHARE}$ can be reduced to the solution given by [sanchez1997values]. $\square$



Figure 3: SEQSHARE is not compatible with [sanchez1997values]

Additivity for an extended solution concept implies that the rewards must stay the same whether the agents play two games separately and consolidate their rewards or they play a game with the sum of the rewards of the two games.

Definition 10 (Extended Additivity (EA)). An extended solution concept $\psi \in \Psi$ satisfies *extended additivity* if for every pair of TCGs $\langle N, v \rangle$ and $\langle N, u \rangle$

$$\psi_i(u) + \psi_i(v) = \psi_i(u + v), \forall i \in N. \quad (7)$$

We will drop the argument v from $\psi(v)$ whenever it is clear from context.

One important property for Shapley value is *symmetry*. To define this property in the context of TCGs, we need the concept of *swap*. A swap of agents i and j in a sequence π is: (a) i replaced with j if only $i \in P(\pi)$, or (b) j replaced with i if only $j \in P(\pi)$, or (c) both i and j 's positions swapped if both $i, j \in P(\pi)$. This is denoted by $\pi_{\text{swap}(i,j)}$.

Definition 11 (Sequential Symmetry (SS)). A solution concept $\phi \in \Phi$ satisfies *sequential symmetry* if for every TCG $\langle N, v \rangle$, the following holds:

$$v(\pi) = v(\pi_{\text{swap}(i,j)}), \forall \pi \in \Pi \implies \phi_i(\pi) = \phi_j(\pi_{\text{swap}(i,j)}), \forall \pi \in \Pi. \quad (8)$$

Definition 12 (Extended Symmetry (ES)). An extended solution concept $\psi \in \Psi$ satisfies *extended symmetry* if for every TCG $\langle N, v \rangle$, the following holds:

$$v(\pi) = v(\pi_{\text{swap}(i,j)}), \forall \pi \in \Pi \implies \psi_i = \psi_j. \quad (9)$$

However, we will see in [Section 6](#) that each of these properties is a consequence of the other three properties in the two solution spaces.

Equipped with these definitions, we can now adapt the Shapley value to TCGs and draw connections to the solutions already discussed and that inspired by the idea of marginal contribution.

5 Solutions Satisfying Shapley-inspired Properties

In this section, we introduce two solution concepts, one each in Φ and Ψ respectively. We show that both these solution concepts uniquely satisfy certain set of desired properties defined in the previous section.

We define the solution concept $\text{MARGSOL} \in \Phi$ inspired by the notion of marginal contribution in classical cooperative games.

$$\text{MARGSOL}_i(\pi) = v(\pi_i + i) - v(\pi_i), \forall \pi \in \Pi, \forall i \in N. \quad (10)$$

Define the extended solution concept *Extended Shapley Value* by using the notion of marginal contribution in sequential cooperative games as follows.

$$\text{EXT-SHAP}_i = \frac{1}{n!} \sum_{\pi \in \Pi: |\pi|=n} \text{MARGSOL}_i(\pi) = \frac{1}{n!} \sum_{\pi \in \Pi: |\pi|=n} (v(\pi_i + i) - v(\pi_i)). \quad (11)$$

Observe that $\text{EXT-SHAP} = \overline{\text{MARGSOL}}$, i.e., EXT-SHAP is the reduced version of MARGSOL . The following lemma shows that the sequential properties are stronger than the extended properties, i.e., if a solution $\phi \in \Phi$ satisfies SE, SA, and SNP, its reduced solution $\bar{\phi} \in \Psi$ satisfies EE, EA, and ENP respectively.

Lemma 2. *For every TCG $\langle N, v \rangle$ and every solution concept $\phi \in \Phi$, the following implications hold.*

- (a) ϕ satisfies SE $\implies \bar{\phi}$ satisfies EE,
- (b) ϕ satisfies SA $\implies \bar{\phi}$ satisfies EA, and
- (c) ϕ satisfies SNP $\implies \bar{\phi}$ satisfies ENP.

Proof. Part (a): since ϕ satisfies SE, we have $\sum_{i \in P(\pi)} \phi_i(\pi, v) = v(\pi), \forall \pi \in \Pi$. Averaging this over all full length sequences, we get $\forall \pi \in \Pi$

$$\begin{aligned} \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} v(\pi) &= \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \sum_{i \in P(\pi)} \phi_i(\pi, v) \\ &= \sum_{i \in N} \left(\frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \phi_i(\pi, v) \right) \\ &= \sum_{i \in N} \bar{\phi}_i(\pi, v) \end{aligned}$$

The first equality comes via changing the order of the summation and the second equality is from the definition of the reduced solution concept. Hence, $\bar{\phi}$ satisfies EE.

Part (b): since ϕ satisfies SA, we have $\phi_i(\pi, u) + \phi_i(\pi, w) = \phi_i(\pi, u + w), \forall i \in N$. Averaging this over all full length sequences, we get

$$\begin{aligned} \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} (\phi_i(\pi, u) + \phi_i(\pi, w)) &= \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \phi_i(\pi, u + w), \forall \pi \in \Pi, i \in N \\ \implies \bar{\phi}_i(u) + \bar{\phi}_i(w) &= \bar{\phi}_i(u + w), \forall i \in N \end{aligned}$$

The implication holds from the definition of the reduced solution concept. Hence, $\bar{\phi}$ satisfies EA.

Part (c): since ϕ satisfies SNP, we have $v(\pi + i) = v(\pi), \forall \pi \in \Pi : i \notin P(\pi) \implies \phi_i(\pi, v) = 0, \forall \pi \in \Pi$. For every such player i , averaging over all possible full length sequences, we get $\frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \phi_i(\pi, v) = 0$, which is $\bar{\phi}_i(v) = 0$ by definition. Hence,

$$v(\pi + i) = v(\pi), \forall \pi \in \Pi : i \notin P(\pi) \implies \bar{\phi}_i(v) = 0.$$

Hence, $\bar{\phi}$ satisfies ENP. □

Our next goal is to show that the three properties are uniquely satisfied by MARGSOL. In order to show that, we use a construct similar to Shapley uniqueness proof (Maschler et al., 2020, Chapter 19). We define the *carrier games* in the context of TCG as follows.

Definition 13 (Carrier Game). For any sequence $\pi \in \Pi$, the carrier game over π is the simple game¹ $\langle N, u_\pi \rangle$ where:

$$u_\pi(\pi') = \begin{cases} 1, & \text{if } \pi \sqsubset \pi', \\ 0, & \text{otherwise.} \end{cases}$$

We show that the *carrier games* span the space of all TCGs.

Lemma 3. Every TCG $\langle N, v \rangle$ is a linear combination of carrier games.

Proof. Consider an arbitrary TCG $\langle N, v \rangle$. We show that every v can be written as a linear combination of u_π (Definition 13).

Part 1: we show that $u_\pi, \pi \in \Pi$ are linearly independent. Suppose not, then $\exists \{\alpha_\pi : \pi \in \Pi\}$, not all 0 such that $\sum_{\pi \in \Pi} \alpha_\pi u_\pi(\pi') = 0, \forall \pi' \in \Pi$.

Let $T = \{\alpha_\pi : \pi \in \Pi, \alpha_\pi \neq 0\}$ be the set of all non-zero α_π 's. Let $\tau = \{\pi : \pi \in \Pi, \alpha_\pi \in T\}$ be the indices of α in T . Observe that, any $\pi \notin \tau$ must have $\alpha_\pi = 0$. Consider the permutation $\pi_0 \in \tau$, such that no prefix of π_0 is in τ . We call this a *minimal sequence* of τ . We can write

$$\sum_{\pi \in \Pi} \alpha_\pi u_\pi(\pi_0) = \sum_{\pi \in \Pi, \pi \sqsubset \pi_0} \alpha_\pi u_\pi(\pi_0) + \alpha_{\pi_0} u_{\pi_0}(\pi_0) + \sum_{\pi \in \Pi, \pi \not\sqsubset \pi_0} \alpha_\pi u_\pi(\pi_0) = 0.$$

¹We use the same terminology from the classical cooperative game: a simple game is a TCG where the worth of any sequence can either be zero or unity.

Note that the first sum in the expanded form does not have any $\pi \in \tau$; so all $\alpha_\pi = 0$ for that sum. In third sum, π is not a prefix of π_0 . Hence by definition of u_π , $u_\pi(\pi_0) = 0$ for them. So we conclude that $\alpha_{\pi_0} u_{\pi_0}(\pi_0) = 0$. But $u_{\pi_0}(\pi_0) = 1$ by definition, and $\alpha_{\pi_0} \neq 0$ since $\pi_0 \in \tau$ which is a contradiction. Hence, u_π 's are linearly independent.

Part 2: we next prove that u_π span the entire space of $\mathbb{R}^{|\Pi|}$. Every sequence $\pi \in \Pi$ denotes a unique carrier game $\langle N, u_\pi \rangle$. Hence, the number of unique u_π are $|\Pi|$. $u_\pi : \Pi \rightarrow \mathbb{R}$, so if they are linearly independent, they must span the entire vector space of $|\Pi|$ dimensions, i.e., $\mathbb{R}^{|\Pi|}$.

Note that from parts 1 and 2, the lemma is immediate, since for any arbitrary TCG $\langle N, v \rangle$, the worth function v lives in $\mathbb{R}^{|\Pi|}$, since $v : \Pi \rightarrow \mathbb{R}$. For a fixed N , Π is finite. Parts 1 and 2 showed that $u_\pi, \pi \in \Pi$ forms a basis of $\mathbb{R}^{|\Pi|}$, and hence any v can be written as a linear combination of u_π . Therefore, every TCG $\langle N, v \rangle$ is a linear combination of the carrier games. $\square$

5.1 Uniqueness of MargSol

To show that MARGSOL is the only solution in Φ that satisfies SE, SA, and SNP simultaneously, we need the following result.

Lemma 4. Consider a TCG $\langle N, u_{\pi, \alpha} \rangle$ for a given $\pi \in \Pi$ and α , where $u_{\pi, \alpha}$ is defined as follows.

$$u_{\pi, \alpha}(\pi') = \begin{cases} \alpha, & \text{if } \pi \sqsubset \pi', \\ 0, & \text{otherwise.} \end{cases}$$

If a solution concept $\phi \in \Phi$ satisfies SE and SNP, it must be true that

$$\phi_i(\pi', u_{\pi, \alpha}) = \begin{cases} \alpha, & \text{if } i = \ell(\pi) \text{ and } \pi \sqsubset \pi', \\ 0, & \text{otherwise.} \end{cases}$$

Intuitively, this result says that all the players except the last player of π , $\ell(\pi)$, are null players in TCG $\langle N, u_{\pi, \alpha} \rangle$. Due to efficiency, it is necessary that the last player must get the whole reward of α . The formal proof is as follows.

Proof. Suppose $i \in N \setminus \{\ell(\pi)\}$ is not the last player of π . In $\langle N, u_{\pi, \alpha} \rangle$, every such player satisfies $u_{\pi, \alpha}(\pi' + i) = u_{\pi, \alpha}(\pi')$, $\forall \pi' \in \Pi_{-i}$, i.e., each such i is a sequential null player. Until player $\ell(\pi)$ arrives in the sequence π , no marginal contribution is generated in this TCG by any other player arriving after. Hence, by SNP, all other players must get $\phi_i(\pi', u_{\pi, \alpha}) = 0, \forall \pi' \in \Pi$. The entire worth of α then must go to $\ell(\pi)$ due to SE. $\square$

Theorem 2. For every TCG $\langle N, v \rangle$, a solution concept $\phi \in \Phi$ satisfies SE, SA, and SNP iff $\phi \equiv \text{MARGSOL}$.

Proof. ($\Leftarrow$) It is trivial to see that if i is a sequential null player, $\text{MARGSOL}_i(\pi) = 0, \forall \pi \in \Pi$. Also, $\text{MARGSOL}_i(\pi, u) + \text{MARGSOL}_i(\pi, v) = u(\pi_i + i) + v(\pi_i + i) - u(\pi_i) - v(\pi) = \text{MARGSOL}_i(\pi, u + v)$. Hence MARGSOL satisfies SA and SNP. To see that it also satisfies SE, note that

$$\begin{aligned} \sum_{i \in P(\pi)} \text{MARGSOL}_i(\pi) &= \sum_{i \in P(\pi)} (v(\pi_i + i) - v(\pi_i)) \\ &= (v(\pi(1)) - v(\emptyset)) + (v(\pi(1)\pi(2)) - v(\pi(1))) + (v(\pi(1)\pi(2)\pi(3)) - v(\pi(1)\pi(2))) + \\ &\quad \dots + (v(\pi(1)\pi(2)\pi(3) \cdots \pi(|P(\pi)|)) - v(\pi(1)\pi(2) \cdots \pi(|P(\pi)| - 1))) = v(\pi). \end{aligned}$$

Where $\pi(k)$ is the agent at the k^{th} position of π .

($\Rightarrow$) Consider an arbitrary TCG $\langle N, v \rangle$. Write it as a sum of its carrier games which is possible due to [Lemma 3](#):

$$v(\pi') = \sum_{\pi \in \Pi} \alpha_{\pi} u_{\pi}(\pi') = \sum_{\pi \in \Pi} u_{\pi, \alpha_{\pi}}(\pi'). \quad (12)$$

For each carrier game, we know the structure of the unique solution concept that satisfies SE and SNP from [Lemma 4](#). Note that MARGSOL satisfies all the three properties for every game $\langle N, v \rangle$, hence it must satisfy them for the carrier games as well. Suppose there exists a different solution concept ϕ that also satisfies the same three properties. We will prove that ϕ must be MARGSOL.

From [Lemma 4](#) and the discussion above, we conclude that ϕ must be same as MARGSOL for the carrier game $\langle N, u_{\pi, \alpha} \rangle$. Hence we have

$$\begin{aligned} & \text{MARGSOL}_i(\pi', u_{\pi, \alpha_{\pi}}) = \phi_i(\pi', u_{\pi, \alpha_{\pi}}), \forall i \in N, \forall \pi, \pi' \in \Pi, \\ \Rightarrow & \sum_{\pi \in \Pi} \text{MARGSOL}_i(\pi', u_{\pi, \alpha_{\pi}}) = \sum_{\pi \in \Pi} \phi_i(\pi', u_{\pi, \alpha_{\pi}}), \forall i \in N, \forall \pi' \in \Pi, \\ \Rightarrow & \text{MARGSOL}_i(\pi', \sum_{\pi \in \Pi} u_{\pi, \alpha_{\pi}}) = \phi_i(\pi', \sum_{\pi \in \Pi} u_{\pi, \alpha_{\pi}}), \forall i \in N, \forall \pi' \in \Pi, \\ \Rightarrow & \text{MARGSOL}_i(\pi', v) = \phi_i(\pi', v), \forall i \in N, \forall \pi' \in \Pi. \end{aligned}$$

The first implication holds by summing over all $\pi \in \Pi$. The second implication is by SA for both solution concepts. The third implication is by [Equation \(12\)](#). Since v was arbitrary, we conclude that ϕ is the same as MARGSOL. $\square$

5.2 Uniqueness of Ext-Shap

EXT-SHAP is an averaged version of MARGSOL ([Equation \(11\)](#)) and hence the mapping to EXT-SHAP may not be unique. Hence, even though MARGSOL uniquely satisfies SE, SA, and SNP ([Theorem 2](#)), that does not automatically imply that EXT-SHAP will uniquely satisfy EE, EA, and ENP. In this section, we show that indeed this implication is true via a proof similar yet independent of that of [Theorem 2](#).

Lemma 5. Consider a TCG $\langle N, u_{\pi, \alpha} \rangle$ for a given $\pi \in \Pi$ and α , where $u_{\pi, \alpha}$ is defined as follows.

$$u_{\pi, \alpha}(\pi') = \begin{cases} \alpha, & \text{if } \pi \sqsubset \pi' \\ 0, & \text{otherwise} \end{cases}$$

If an extended solution concept $\psi \in \Psi$ satisfies EE and ENP, it must be true that

$$\psi_i(u_{\pi, \alpha}) = \begin{cases} \frac{(n - |P(\pi)|)!}{n!} \alpha, & \text{if } i = \ell(\pi) \\ 0, & \text{otherwise} \end{cases}$$

Proof. We know from [Lemma 4](#) that every $i \in N \setminus \{\ell(\pi)\}$ is a null player and should get zero reward share by any extended solution concept ψ by ENP. Hence, the only positive reward should go to $\ell(\pi)$. By EE, we get

$$\sum_{i \in P(\pi')} \psi_i(u_{\pi, \alpha}) = \frac{1}{n!} \sum_{\pi' \in \Pi: P(\pi') = N} u_{\pi, \alpha}(\pi').$$

Given the earlier arguments, this reward will go entirely to $\ell(\pi)$, i.e., For i , the last player of p , we get that $\psi_{\ell(\pi)}(u_{\pi, \alpha}) = \frac{1}{n!} \sum_{\pi' \in \Pi: P(\pi') = N} u_{\pi, \alpha}(\pi')$.

By definition of $u_{\pi, \alpha}$, as long as π is a prefix of a given sequence of players π' , the remaining $N \setminus P(\pi)$ players can appear in any sequence. So, there are $(n - |P(\pi)|)!$ different $\pi' \in \Pi$ such that π is a prefix of π' . Therefore, $u_{\pi, \alpha}(\pi') = \alpha$ for all those π' . Hence, $\psi_{\ell(\pi)}(u_{\pi, \alpha}) = \frac{(n - |P(\pi)|)!}{n!} \alpha$. $\square$

Theorem 3. For every TCG $\langle N, v \rangle$, an extended solution concept $\psi \in \Psi$ satisfies EE, EA, and ENP iff $\psi = \text{EXT-SHAP}$.

Proof. ($\Leftarrow$) We have already proved in [Theorem 2](#) that MARGSOL satisfies SE, SA, and SNP, and by [Lemma 2](#), its reduced solution concept EXT-SHAP satisfies EE, EA, and ENP.

($\Rightarrow$) This direction of the proof follows in similar lines of that of [Theorem 2](#). Assume that there exists an extended solution concept $\psi \neq \text{EXT-SHAP}$ that also satisfies the three given properties.

For each carrier game, we know the structure of the unique solution concept that satisfies SE and SNP from [Lemma 5](#). Note that EXT-SHAP satisfies all the three properties for every game $\langle N, v \rangle$, hence it must satisfy them for the carrier games as well. Hence ψ must be same as EXT-SHAP for the carrier game $\langle N, u_{\pi, \alpha} \rangle$. Hence we have

$$\begin{aligned} & \text{EXT-SHAP}_i(u_{\pi, \alpha_\pi}) = \psi_i(u_{\pi, \alpha_\pi}), \forall i \in N, \forall \pi \in \Pi, \\ \Rightarrow & \sum_{\pi \in \Pi} \text{EXT-SHAP}_i(u_{\pi, \alpha_\pi}) = \sum_{\pi \in \Pi} \psi_i(u_{\pi, \alpha_\pi}), \forall i \in N, \\ \Rightarrow & \text{EXT-SHAP}_i(\sum_{\pi \in \Pi} u_{\pi, \alpha_\pi}) = \psi_i(\sum_{\pi \in \Pi} u_{\pi, \alpha_\pi}), \forall i \in N, \\ \Rightarrow & \text{EXT-SHAP}_i(v) = \psi_i(v), \forall i \in N. \end{aligned}$$

The first implication holds by summing over all $\pi \in \Pi$. The second implication is by EA for both solution concepts. The third implication is by [Equation \(12\)](#). Since v was arbitrary, we conclude that ψ is the same as EXT-SHAP. $\square$

Remark. The Shapley value equivalents in the TCG world, MARGSOL and EXT-SHAP, satisfy ‘almost’ similar uniqueness properties like the Shapley value. However, a careful reader can spot the difference since Shapley value was unique for *four* (and not three as we did here) properties which also included *symmetry*. The TCG-equivalent definitions of symmetry are given in [Definitions 11](#) and [12](#). In the following section, we show why these two properties were not necessary to obtain the uniqueness theorems [Theorems 2](#) and [3](#).

6 Why Symmetry Is Not Necessary?

Properties in the TCG framework are more restrictive as they make claims for all *sequences*, making them stronger than claims for all *sets*, as in the classical cooperative games. Hence, three properties are sufficient to ensure unique solution concepts like MARGSOL and EXT-SHAP. In the following, we show that MARGSOL and EXT-SHAP satisfy SS ([Definition 11](#)) and ES ([Definition 12](#)) respectively defined in the spirit of TCGs. These result show that the three other properties imply the symmetry property.

Theorem 4. MARGSOL and EXT-SHAP satisfy SS and ES respectively.

Proof. Part 1 (MARGSOL): Consider a TCG $\langle N, v \rangle$, s.t. $v(\pi) = v(\pi_{\text{swap}(i,j)}), \forall \pi \in \Pi$. We show the following $\forall \pi \in \Pi, \forall i \in N$

$$\begin{aligned} & \text{MARGSOL}_i(\pi) \\ &= v(\pi_i + i) - v(\pi_i) \\ &= v((\pi_i)_{\text{swap}(i,j)} + j) - v((\pi_i)_{\text{swap}(i,j)}) \\ &= v((\pi_{\text{swap}(i,j)})_j + j) - v((\pi_{\text{swap}(i,j)})_j) \\ &= \text{MARGSOL}_j(\pi_{\text{swap}(i,j)}). \end{aligned}$$

The first equality follows from definition. The second equality comes from the ‘if’ condition of the SS property. The third equality follows from the fact that $(\pi_i)_{\text{swap}(i,j)} = (\pi_{\text{swap}(i,j)})_j$. The last equality is again by definition. Hence, we show that MARGSOL satisfies SS.

Part 2 (EXT-SHAP): Since $\text{EXT-SHAP}_i = \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \text{MARGSOL}_i(\pi)$, and we have already shown in Part 1 that $\text{MARGSOL}_i(\pi) = \text{MARGSOL}_j(\pi_{\text{swap}(i,j)})$, it is straightforward to show that

$$\begin{aligned} \text{EXT-SHAP}_i &= \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \text{MARGSOL}_i(\pi) \\ &= \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \text{MARGSOL}_j(\pi_{\text{swap}(i,j)}) \\ &= \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \text{MARGSOL}_j(\pi) \\ &= \text{EXT-SHAP}_j. \end{aligned}$$

The third equality holds since $\pi_{\text{swap}(i,j)}$ just changes the order of summation in the set $\{\pi \in \Pi : P(\pi) = N\}$. Hence, EXT-SHAP satisfies ES. $\square$

Remarks. In classical cooperative games, the Shapley value is the only solution concept satisfying efficiency, additivity, null player property and symmetry. Symmetry is required to make the solution concept unique. The difference in our setting lies in the definition of the null player property for TCGs.

In the classical setting, a carrier game is defined with respect to a critical *set* of players. When the set is present, only then a coalition generates unit positive worth. Dividing the worth among the non-null players requires a property like symmetry.

In TCG framework, a carrier game corresponds to a critical *sequence* of players. When this critical sequence happens to be a prefix of a given sequence, only then it can generate unit positive worth. As a result, only the last player of the critical sequence is non-null, so symmetry is not required. Only one player gets all the worth.

7 SeqShare and Shapley-compliant: Best of Both Worlds

SEQSHARE uniquely satisfies three most desirable properties for TCGs: OIR, SE, and I4OA. On the other hand, EXT-SHAP uniquely characterizes the three most desirable Shapley-equivalent properties: EA, EE, and ENP. A solution concept that satisfies the temporal properties of TCGs and simultaneously has its reduced version satisfying the Shapley properties would therefore be the ideal candidate in a TCG. In this section we show that such a “*best of both worlds*” solution concept does not exist, even in special games like *convex* or *simple* games (Maschler et al., 2020, Chapter 19).

In order to show the result for general TCGs, we first make the following observation about solution concepts in SEQSHARE that reduce to EXT-SHAP.

Lemma 6. For a TCG $\langle N, v \rangle$ and a solution concept $\phi \in \text{SEQSHARE}$, if $\bar{\phi} = \text{EXT-SHAP}$ then the following holds $\forall i \in N$

$$\phi_i(\pi^*(v)) \geq \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} v(\pi_i + i) - v(\pi_i) = \text{EXT-SHAP}_i, \quad (13)$$

where $\pi^*(v)$ is the optimal sequence of the TCG.

Proof. Consider a solution concept $\phi \in \text{SEQSHARE}$ such that $\bar{\phi} = \text{EXT-SHAP}$. Suppose $\exists i \in N$ such that $\phi_i(\pi^*(v)) < \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} v(\pi_i + i) - v(\pi_i)$. Since $\phi \in \text{SEQSHARE}$, it satisfies I4OA, i.e.,

$$\begin{aligned} \phi_i(\pi) &\leq \phi_i(\pi^*(v)), \quad \forall \pi \in \Pi \\ \implies \phi_i(\pi) &< \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} v(\pi_i + i) - v(\pi_i), \quad \forall \pi \in \Pi. \end{aligned}$$

Averaging over all possible $\pi \in \Pi$ such that $P(\pi) = N$, we get

$$\bar{\phi}_i = \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} \phi_i(\pi) < \frac{1}{n!} \sum_{\pi \in \Pi: P(\pi)=N} v(\pi_i + i) - v(\pi_i) = \text{EXT-SHAP}_i. \quad (14)$$

This contradicts the hypothesis that $\bar{\phi} = \text{EXT-SHAP}$. $\square$

Hence, we conclude that Equation (13) is necessary to ensure that a solution from SEQSHARE reduces to EXT-SHAP. In the example given below we show a game that does not satisfy Equation (13) and therefore, there does not exist any solution concept $\phi \in \text{SEQSHARE}$ that has a reduction to EXT-SHAP.

1	2	3
$v(1) = 1$	$v(12) = 3$	$v(123) = a$
2	1	3
$v(2) = b$	$v(21) = 4$	$v(213) = 8$
3	1	2
$v(3) = 4$	$v(31) = 5$	$v(312) = a$
1	3	2
$v(1) = 1$	$v(13) = 4$	$v(132) = 6$
2	3	1
$v(2) = b$	$v(23) = 5$	$v(231) = 7$
3	2	1
$v(3) = 4$	$v(32) = 5$	$v(321) = 7$

Figure 4: A generic TCG serving as counterexample for multiple settings for different values of a and b , e.g., no solution in SEQSHARE can be reduced to EXT-SHAP.

Example 3. Consider the TCG shown in Figure 4 with $a = 7, b = 3$. Clearly, $v(\pi^*) = 8$. The basis solution (Equation (1)) dictates that $x_1 \geq v(1) = 1, x_2 \geq v(2) = 3, x_3 \geq v(3) = 4$ and $x_1 + x_2 + x_3 = 8$, which implies that the basis solution must be $x_1 = 1, x_2 = 3, x_3 = 4$. This is the only possible basis solution and therefore any $\phi \in \text{SEQSHARE}$ assigns $\phi_i(\pi^*) = x_i$. Routine calculations for Player 1 gives $\text{EXT-SHAP}_1 = \frac{8}{6}$. However, $\phi_1(\pi^*) = 1 < \frac{8}{6} = \text{EXT-SHAP}_1$, which violates Equation (13). $\square$

Remark. Since no solution in SEQSHARE can be reduced to EXT-SHAP, no solution in SEQSHARE can be MARGSOL. In particular, MARGSOL satisfies SE by construction. Since we are considering only monotone games, the marginal contribution, $v(\pi_i + i) - v(\pi_i)$, is always non-negative for every player $i \in N$, and remains so even after the arrival of future agents, which ensures OIR. Hence, the only property that MARGSOL violates is I4OA. The following example illustrates this. This rules out the possibility of obtaining a solution concept that satisfies both set of properties in both the original and extended spaces.

Example 4. Consider the TCG given in Figure 5. In this example $\pi^* = 12$. Consider the

1	2
$v(1) = 3$	$v(12) = 8$
2	1
$v(2) = 1$	$v(21) = 5$

Figure 5: MARGSOL violates I4OA

MARGSOL solution:

$$\text{MARGSOL}_1(1) = 1, \text{MARGSOL}_2(1) = 0, \text{MARGSOL}_2(2) = 3, \text{MARGSOL}_1(2) = 0,$$

$$\text{MARGSOL}_1(21) = 4, \text{MARGSOL}_2(21) = 1, \text{MARGSOL}_1(12) = 3, \text{MARGSOL}_2(12) = 5.$$

MARGSOL violates I4OA for agent 1 whose reward at $\pi^* = 12$ is less than the reward at 21. ($\text{MARGSOL}_1(12) = 3 < 4 = \text{MARGSOL}_1(21)$). $\square$

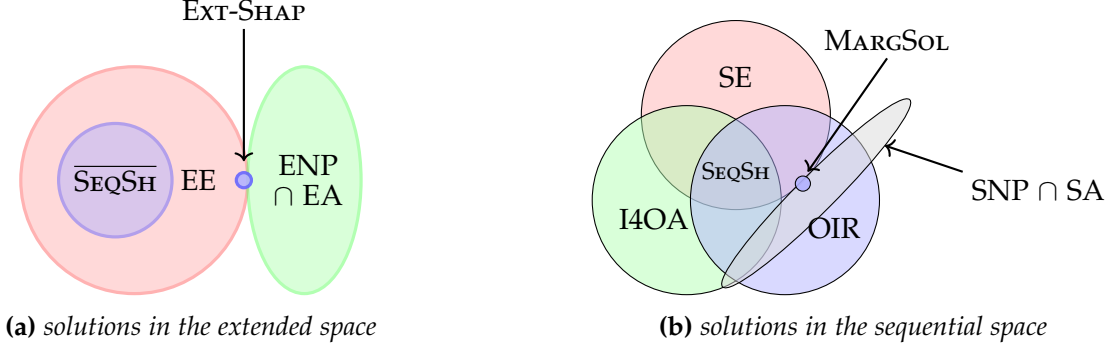


Figure 6: Impossibilities of Shapley inspired properties and solutions in SEQSH.

Since the Shapley-inspired properties and I4OA are incompatible for general games, we aim to find an intersection in the following sections in special classes of TCGs.

7.1 Convex Games

In classical cooperative games, Shapley value exhibit some nice properties in the class of *convex* games. One such property is that for convex games, the Shapley Value always lie in the *core*. Since the basis conditions (Equation (1)) are analogous to the core conditions in classical games, we check for a similar result in the case of TCGs. We also check if there is a solution in SEQSHARE that reduces to EXT-SHAP in this class of games. For that, first we need to define convex TCGs.

Definition 14 (Convex Temporal Cooperative Games). A TCG $\langle N, v \rangle$ is *convex* if the following holds

$$v(\pi + i) - v(\pi) \leq v(\pi' + i) - v(\pi'), \forall \pi', \pi \text{ s.t. } i \notin P(\pi') \text{ and } P(\pi) \subseteq P(\pi'). \quad (15)$$

The answer to the question of having a solution in SEQSHARE that reduces to EXT-SHAP is unfortunately negative. First, the following example illustrates a convex TCG that has no basis solution, unlike the convex games in classical cooperative game where it is always in the core.

Example 5. Consider the convex TCG as shown in Figure 4 with $a = 7, b = 4$. The basis solution (Equation (1)) must satisfy $x_1 \geq 1, x_2 \geq 4, x_3 \geq 4$ and $x_1 + x_2 + x_3 = 8$, that are clearly impossible. Hence, the given convex TCG has no basis solution. $\square$

Our next example shows that even for a convex TCG having a basis solution, there may be no solution in SEQSH that reduces to EXT-SHAP.

Example 6. Consider the TCG as used in Example 3, the game is convex and violates Equation (13). $\square$

In the next section we define simple TCGs and explore the connections between EXT-SHAP and SEQSHARE.

7.2 Simple Games

In classical cooperative games simple games or ‘0-1’ games are defined as the games where the worth of a coalition can either be 0 or 1. In TCGs, we follow an analogous definition where the worth of any sequence can either take 0 or 1 values. We define simple TCGs as follows.

Definition 15 (Simple Temporal Cooperative Games). A TCG $\langle N, v \rangle$ with $v : \Pi \rightarrow \{0, 1\}$ is called a *simple* temporal cooperative game.

Since we consider monotone games in this paper, monotonicity for simple TCGs implies that for all $\forall \pi, \pi' \in \Pi$ such that $\pi \sqsubset \pi'$, it holds that $v(\pi) = 1 \implies v(\pi') = 1$.

The following example shows that even for simple games there may be no solution in SEQSHARE that reduces to EXT-SHAP.

Example 7. Consider the simple TCG given in Figure 7. The unique basis solution (Equation (1)) is $x_1 = 1, x_2 = 0, x_3 = 0$. Therefore any $\phi \in \text{SEQSHARE}$ assigns $\phi_i(\pi^*) = x_i$. Routine calculations for Player 2 gives $\text{EXT-SHAP}_2 = \frac{1}{6}$. However, $\phi_2(\pi^*) = 0 < \frac{1}{6} = \text{EXT-SHAP}_2$, which violates Equation (13). $\square$

1	2	3		1	3	2	
$v(1) = 1$	$v(12) = 1$	$v(123) = 1$		$v(1) = 1$	$v(13) = 1$	$v(132) = 1$	
2	1	3		2	3	1	
$v(2) = 0$	$v(21) = 0$	$v(213) = 1$		$v(2) = 0$	$v(23) = 1$	$v(231) = 1$	
3	1	2		3	2	1	
$v(3) = 0$	$v(31) = 0$	$v(312) = 1$		$v(3) = 0$	$v(32) = 0$	$v(321) = 1$	

Figure 7: A simple TCG for which no solution in SEQSHARE reduces to EXT-SHAP.

Even for simple TCGs, we cannot hope to have a unique basis solution nor a solution in SEQSHARE that reduces to EXT-SHAP.

8 Conclusion

In this work, we considered the framework of temporal cooperative games (TCGs), where the worth of a coalition depends on the order of agent arrivals. This setting departs from classical cooperative games and requires rethinking fundamental axioms. We proposed three natural temporal properties, namely incentive for optimal arrival (I4OA), online individual rationality (OIR), and sequential efficiency (SE), and showed that together they uniquely characterize a class of reward-sharing mechanisms. We further developed two Shapley-like analogs, one in the sequential world and one in the extended world, and established that they are uniquely characterized by efficiency, additivity, and the null player property. Importantly, we demonstrated a fundamental incompatibility: the Shapley analogs and the mechanisms satisfying I4OA, OIR, and SE are disjoint, and this conflict persists even in restricted settings such as convex and simple TCGs.

Our results highlight a key tension in extending classical cooperative game theory to temporal environments: desirable temporal properties cannot be reconciled with Shapley-like

axioms. As future work, an important direction is to investigate whether a unique solution concept can be obtained by imposing additional properties beyond I4OA, OIR, and SE. Another avenue is to identify special classes of temporal cooperative games where Shapley-like properties and temporal properties such as I4OA can coexist. These questions open new opportunities for developing principled solution concepts for cooperative games in sequential environments.

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