Sequential Solution Concepts in Cooperative Games with Generalized Characteristic Functions

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Abstract

Motivated by the fact that the worth of a coalition may depend on the order in which agents arrive, Nowak and Radzik (1994) introduced cooperative games with generalized characteristic functions, where the worth function v is defined on sequences of agents π rather than sets S. We call such games *temporal cooperative games (TCGs)*. This order sensitivity necessitates a re-examination of axioms for reward sharing. Prior work proposed several axioms; however, the resulting solution concepts are inherently order-oblivious and closely tied to the Shapley value. In contrast, we focus on sequential solution concepts that explicitly depend on the realized order π. We study reward-sharing mechanisms satisfying *incentive for optimal arrival* (I4OA), which promotes orders that maximize total worth; *online individual rationality* (OIR), which ensures later arrivals do not harm existing agents; and *sequential efficiency* (SE), which requires that the worth of any sequence is fully distributed among its agents. These axioms are intrinsic to TCGs, and we characterize a class of reward-sharing mechanisms uniquely determined by them. The classical Shapley value does not directly extend to this setting. We therefore construct natural Shapley analogs in two worlds: a *sequential* world, where rewards are defined for each sequence–agent pair, and an *extended* world, where rewards are defined per agent, consistent with the framework of Nowak and Radzik (1994). In both cases, the axioms of efficiency, additivity, and null player uniquely characterize the corresponding Shapley analogs. However, these Shapley analogs are disjoint from the class of solutions satisfying the sequential axioms, even for convex and simple TCGs. Our results reveal a fundamental tension in temporal cooperative games: when order matters, solution concepts satisfying natural sequential incentives must differ structurally from Shapley-based allocations, motivating new axiomatic foundations for sequential reward sharing.

1 Introduction

Consider hospitals collaborating to train a medical image classifier for a rare progressive disease via *federated learning*. Hospital A holds early-stage images with subtle texture changes; Hospital B holds mid-stage images with visible structural deformations; Hospital C holds late-stage images where the disease is overt. Each hospital's gradient update shapes the loss landscape that subsequent participants optimize over. If Hospital C joins first, the model anchors on coarse structural signals, and Hospital A's subtle texture updates later struggle to move it. The result is a classifier that misses early-stage diagnoses, which are clinically most valuable. Reverse the order and the model builds rich low-level representations first, with each subsequent participant adding complementary structure.

40th Conference on Neural Information Processing Systems (NeurIPS 2026).

The same three participants, the same data, the same algorithm, but a different joining order yields a substantially better model. The same phenomenon arises in software development pipelines, robotic assembly, and supply chains: the order in which agents join determines the value created, not just who joins.

Classical cooperative games (Maschler et al., 2020) overlook this: their worth[1] functions are defined only for coalitions, ignoring the order of agents. Early works by Nowak and Radzik (1994); Sánchez and Bergantiños (1997) incorporate order through *generalized characteristic functions* (GCF), and follow-up papers (Van den Brink et al., 2014; Feng et al., 2013; Yuan and Driessen, 2013; Li et al., 2024; Michalak et al., 2014; Zou et al., 2020) extend their axioms—but all yield *order-oblivious* solution concepts where rewards do not depend on the arrival order. We focus on *order-sensitive* sequential solution concepts and study three natural axioms for them. *Online individual rationality* (OIR) ensures Hospital A is not penalized after later arrivals improve the model, preserving its incentive to join early. *Incentive for optimal arrival* (I4OA) incentivizes each hospital to join in the order that maximizes diagnostic accuracy. *Sequential efficiency* (SE) ensures the full value of any realized sequence is distributed among its participants. We call cooperative games with generalized characteristic functions *temporal cooperative games* (TCGs).

Ideally, ϕ would also satisfy Shapley fairness by giving each hospital its average marginal contribution across all joining orders, which is the standard notion of equitable reward sharing. However, we prove this is *impossible*: sequential solution concepts satisfying OIR, I4OA, and SE are provably disjoint from those satisfying Shapley-inspired properties such as efficiency, additivity, and the null-player condition (Figure 2), even for convex and simple TCGs. No reward rule can simultaneously protect early participants, incentivize the optimal joining order, and distribute Shapley-fair shares—a fundamental constraint for any mechanism designer in sequential collaborative settings.

1.1 Our Contributions

- We introduce *sequential* solution concepts (Section 2). Unlike existing GCF literature, which yields order-oblivious reward shares, our framework assigns rewards to every (sequence, player) pair, arguing that order-sensitive environments require *order-sensitive* solution concepts.
- We formalize axioms natural to this order-sensitive setting: I4OA, OIR, and SE (Section 2).
- We characterize all sequential solution concepts satisfying I4OA, OIR, and SE via a class of reward-sharing mechanisms SEQSHARE (Algorithm 1), and identify a necessary condition, termed *basis*, for their satisfaction (Lemma 1).
- We formulate order-sensitive analogs of Shapley's axioms for TCGs: efficiency, additivity, and null player (Section 4), and show they uniquely characterize a Shapley-analogous sequential solution concept MARGSOL (Section 5.1); symmetry follows as a consequence (Appendix C).
- To connect with order-oblivious solutions, we define *extended* solution concepts by averaging over sequences. The extended axioms uniquely identify EXT-SHAP (Appendix B.5), aligning exactly with existing GCF literature.
- Finally, we show that the extended solutions induced by SEQSHARE are disjoint from EXT-SHAP, establishing a fundamental incompatibility between desirable sequential properties and Shapley-inspired axioms, even for *convex* and *simple* games (Section 6).

1.2 Related Work

Classical cooperative game theory originated with Von Neumann and Morgenstern (1947), with foundational solution concepts including the core (Gillies, 1959), nucleolus (Schmeidler, 1969), and Shapley value (Shapley, 1953). Characterizations of the Shapley value via variants of its axioms have been studied extensively (Van Den Brink, 2002; Casajus and Huettner, 2013; Hamiache, 2001; Casajus, 2014). Dynamic and online extensions include coalition formation under random arrivals or dissolution (Bullinger and Romen, 2023, 2025), general online coalition formation (Flammini et al., 2021), and cooperative games where worth evolves over time (Lehrer and Scarsini, 2013; Habis and Jean-Jacques Herings, 2010; Kranich et al., 2005). Online cooperative games with sequential arrivals have been studied for monotone games (Ge et al., 2024), cost-sharing (Zhang et al., 2025a), axiomatic frameworks (Aziz et al., 2025), and stable coalition formation (Zhang et al., 2025b); however, all these models define worth over coalitions rather than sequences.

[1]We use the terms *worth* and *characteristic* functions interchangeably, as is common in the literature, and denote them by v.

Generalized characteristic functions were introduced by Nowak and Radzik (1994) and extended by Sánchez and Bergantiños (1997, 1999); Bergantiños and Sánchez (2001). Despite allowing order-dependent worth, the resulting NR and SB solution concepts are sequence-independent. Nowak and Radzik (1994) extended Shapley-style axioms to TCGs and uniquely characterized a solution, while Sánchez and Bergantiños (1997) weakened the null-player property but required an additional symmetry assumption to recover uniqueness (see Section 4), which we do not impose.

Subsequent work characterizes the SB value via bargaining (Yuan and Driessen, 2013) and consistency properties (Feng et al., 2013). Van den Brink et al. (2014) characterize a class of solutions including the SB and NR values as extreme cases. Zou et al. (2020) and Li et al. (2024) study settings with restrictions on agent arrivals. Finally, Michalak et al. (2014) implement generalized games as perfect-information games that admit the NR and SB values as subgame-perfect Nash equilibria.

The fundamental distinction between these works and ours lies in our focus on order-sensitive sequential solution concepts. This perspective leads to two key departures: (1) the axioms of interest differ and are motivated by settings in which order-sensitive solutions are natural, and (2) the required properties are more fine-grained in the Shapley framework, as they must hold for every sequence π. Consequently, our main results are: (1) a characterization of order-sensitive axioms tailored to TCGs (Theorem 1), (2) a characterization of Shapley-like yet order-sensitive axioms (Theorem 2), and (3) a demonstration of the disjointness of these two classes (Section 6).

2 Preliminaries

Consider a set of agents $N = \{1, \ldots, n\}$. Let $\text{PERM}(S)$ denote all permutations of players in $S \subseteq N$. Define all possible sequences of any length by $\Pi := \{\pi : \pi \in \text{PERM}(S), S \in 2^N \setminus \emptyset\}$. We denote the set of agents in a sequence π by $P(\pi)$ and the player at position i in π as $\pi(i)$. We define a characteristic function $v : \Pi \to \mathbb{R}$, which assigns a worth to every sequence of agents. A *temporal cooperative game (TCG)* is thus described by a tuple $\langle N, v\rangle$. We also define $\Pi_{-N'} := \{\pi : \pi \in \text{PERM}(S), S \in 2^{N\setminus N'}\}$, i.e., the set of all agent sequences excluding agents in N'. Denote the last player of π as $\ell(\pi)$.

We call π' a *prefix* of π if the first $|P(\pi')|$ players of π coincide with those of π', i.e., $\pi(i) = \pi'(i)$ for all $i = 1, 2, \ldots, |P(\pi')|$. A prefix π' of π is denoted $\pi' \sqsubset \pi$. Note that $|P(\pi')| < |P(\pi)|$, otherwise they are the same sequence. We assume the characteristic function is *monotone*, i.e., $v(\pi') \leqslant v(\pi)$ for all $\pi' \sqsubset \pi$. We denote the *predecessor* of i in π by π_i, which is the longest prefix of π not containing i. A sequence π is a *full sequence* if $P(\pi) = N$. Denote the set of all monotone characteristic functions by V. The empty sequence $\emptyset$ is not an element of Π; whenever an expression such as $v(\pi_i)$ refers to it (for instance, when i is the first agent of π), we use the convention $v(\emptyset) = 0$. We also assume that worths are nonnegative, $v(\pi) \geqslant 0$ for all $\pi \in \Pi$, as in all of our motivating examples.

The term $\phi_i(\pi, v)$ denotes the reward of agent $i \in P(\pi)$ when arriving in sequence π in the TCG $\langle N, v\rangle$. We denote by $\phi(\pi, v)$ the vector $(\phi_i(\pi, v), i \in N)$, with the convention that $\phi_j(\pi, v) = 0$ for all $j \in N \setminus P(\pi)$. We write $\phi(\pi)$, dropping v when clear from context. A reward-sharing method is *nonnegative* if $\phi_i(\pi, v) \geqslant 0$ for all $\pi \in \Pi$ and $i \in P(\pi)$; that is, no agent is ever asked to pay to participate. For example, in the federated learning setting of the introduction, a hospital contributes data and is never charged for doing so. The optimal sequence $\pi^*(v) \in \operatorname{argmax}_{\pi\in\Pi} v(\pi)$ is defined with a tie-breaking rule such that $P(\pi^*(v)) = N$ (with arbitrary tie-breaking among full-length sequences), since the game is monotone. Our results hold even when $P(\pi^*(v)) \subset N$, and the proofs adapt easily. We summarize the notation in Table 1.

Since our focus is on reward-sharing mechanisms that incentivize agents to arrive in the order that maximizes total worth, we first ask: under what conditions can such incentives even be guaranteed? Intuitively, for a mechanism to incentivize every agent to join in the optimal sequence, there must exist a consistent set of constraints on agent rewards and the worth of every sequence. We formalize this requirement as follows.

Definition 1 (Basis Conditions). A TCG $\langle N, v\rangle$ satisfies the *basis conditions* if there exists a vector $x \in \mathbb{R}^n$ such that

$$\text{(i) } \textstyle\sum_{i\in P(\pi)} x_i \geqslant v(\pi),\ \forall \pi \in \Pi, \qquad \text{(ii) } \textstyle\sum_{i\in P(\pi^*(v))} x_i = v(\pi^*(v)). \tag{1}$$

Any such x is called a *basis solution*. A TCG admitting a basis solution is called a *basis-satisfiable TCG*. The basis conditions are analogous to the core of classical cooperative games.

Table 1: *Summary of Notation*

Symbol	Description	Symbol	Description
$N = \{1, \ldots, n\}$	Set of agents	π_i	Predecessor of i: longest prefix of π not containing i
2^N	Power set of N	$v : \Pi \to \mathbb{R}$	Characteristic function
$\text{PERM}(S)$	All permutations of $S \subseteq N$	V	Set of all monotone characteristic functions
Π	All sequences over subsets of N	$\langle N, v \rangle$	A temporal cooperative game (TCG)
π	A sequence of agents	$\Pi_{-N'}$	All sequences over $N \setminus N'$
$P(\pi)$	Agents appearing in π	$\pi^*(v)$	Optimal sequence maximizing $v(\pi)$
$\pi(i)$	Agent at position i in π	$\phi_i(\pi, v)$	Reward of agent i in sequence π under v
$\ell(\pi)$	Last player of π	$\phi(\pi, v)$	Reward vector for all agents in π
$\pi' \sqsubset \pi$	Proper prefix of π		

Before defining the axiomatic properties, we make the modelling assumptions explicit. We consider a complete-information setting: all agents in N are available, and the characteristic function v is known. The sequential solution concept ϕ is announced to all agents prior to any commitments, after which agents choose when to arrive. This is motivated by the *incentive for early arrival* (I4EA) property introduced by Ge et al. (2024): agents are all available (e.g., candidates in a job market) but choose when to commit, and a well-designed ϕ should make the optimal joining order weakly dominant for every agent.

The players arrive sequentially in a TCG, so rewards must also be given as they arrive. When a new player i arrives, extending sequence π to $\pi + i$, the worth generated must be distributed among the players present.

Definition 2 (Online Individual Rationality). A sequential solution concept ϕ satisfies *online individual rationality* (OIR) for a TCG $\langle N, v \rangle$ if $\phi_i(\pi, v) \leqslant \phi_i(\pi', v)$ for all $i \in P(\pi)$ and all $\pi \sqsubset \pi'$.

The sequential nature of the game also creates the possibility that players stop arriving; the next player's arrival is uncertain. Therefore, the sequential solution concept should ensure the worth of any sequence π is fully distributed among its players.

Definition 3 (Sequential Efficiency (SE)). A sequential solution concept ϕ satisfies *sequential efficiency* (SE) for a TCG $\langle N, v \rangle$ if $\sum_{i \in P(\pi)} \phi_i(\pi, v) = v(\pi)$ for all $\pi \in \Pi$.

Definition 4 (Incentive for Optimal Arrival (I4OA)). A sequential solution concept ϕ satisfies *incentive for optimal arrival* (I4OA) for a TCG $\langle N, v \rangle$ if $\phi_i(\pi^*(v), v) \geqslant \phi_i(\pi, v)$ for all $\pi \in \Pi$ and $i \in N$.

This property is closely related to I4EA introduced by Ge et al. (2024), which requires that agents are not penalized for arriving earlier in online coalition formation games. I4OA strengthens this notion to an order-sensitive setting: rather than merely protecting early arrivals, it requires that the optimal joining order constitutes a Nash equilibrium of the arrival game induced by ϕ.

Note that SE and I4OA serve different normative goals. SE guarantees the mechanism is well-defined for every realized sequence, even if agents deviate from the optimal order or some agents do not join at all. I4OA is a design goal that nudges agents toward the optimal order via the reward rule.

In the first part of this paper, we characterize reward-sharing mechanisms satisfying OIR, I4OA, and SE for basis-satisfiable TCGs. In the second part, we define properties analogous to the Shapley properties in classical cooperative games and characterize the reward-sharing schemes satisfying them. Finally, we establish the non-existence of sequential solution concepts satisfying both sets of properties simultaneously. Detailed proofs are deferred to the appendix.

3 The SEQSHARE Class of Mechanisms

In this section, we introduce a class of reward-sharing mechanisms that we later show to uniquely satisfy OIR, I4OA, and SE. We restrict our attention to basis-satisfiable TCGs (Definition 1), and the following lemma justifies why this restriction is necessary.

Lemma 1. *For a TCG $\langle N, v \rangle$, if there exists a sequential solution concept ϕ satisfying I4OA and SE, then $\langle N, v \rangle$ must be basis-satisfiable.*

Proof Sketch. Given a TCG $\langle N, v\rangle$ and a sequential solution concept ϕ satisfying I4OA and SE, setting $x = \phi(\pi^*)$ meets the basis conditions. Condition (ii) follows directly from SE applied to π^*. Condition (i) follows from I4OA, which bounds $\phi_i(\pi)$ by $\phi_i(\pi^*)$ for all i and π, and SE ensures $v(\pi) = \sum_i \phi_i(\pi)$, implying the required inequality. □

Lemma 1 shows that basis-satisfiability is necessary for I4OA and SE to hold together, and Theorem 1 below shows, constructively, that it is also sufficient. Basis-satisfiability is therefore a feasibility condition of the problem itself rather than a restriction of our mechanisms: on a TCG that is not basis-satisfiable, no sequential solution concept satisfies both I4OA and SE. Our mechanisms are thus defined exclusively for basis-satisfiable TCGs, and SEQSHARE returns null otherwise. Unlike the core of a classical cooperative game, the basis can be empty even when the TCG is convex (Appendix D.2).

REMARK. The existence of a basis solution does not imply that every ϕ will automatically satisfy I4OA and SE. In fact, for any two of I4OA, OIR, and SE, there exists a sequential solution concept satisfying both while violating the third, even in games with a non-empty basis. The following example shows this for each pair.

Example 1. Consider the TCG: $v(1) = 1, v(12) = 2, v(2) = 1, v(21) = 3$. It is easy to check that the game has a basis solution. In this example, $\pi^* = 21$, $x_1 = 1, x_2 = 2$. Now, consider the sequential solution concepts below: (i) ϕ satisfies OIR and I4OA but not SE since $v(12) \neq \phi_1(12) + \phi_2(12)$; (ii) ϕ' satisfies SE and I4OA but not OIR as $\phi'_1(1) = 1 > 0 = \phi'_1(12)$; (iii) ϕ'' satisfies SE and OIR but not I4OA since $\phi''_1(12) = 2 > 1 = \phi''_1(21)$. This example also exhibits that no two of these properties imply the third. □

Table 2: *I4OA-SE-OIR independence.*

π	$v(\pi)$	ϕ	ϕ'	ϕ''
1	1	$(0,0)$	$(1,0)$	$(1,0)$
2	1	$(0,0)$	$(0,1)$	$(0,1)$
12	2	$(0,0)$	$(0,2)$	$(2,0)$
21	3	$(1,2)$	$(1,2)$	$(1,2)$

Our desired class of sequential solution concepts starts with the basis conditions to check if a solution exists (and returns NULL if the basis is empty). If the basis conditions hold, Algorithm 1 yields a class of solution concepts satisfying OIR, I4OA, and SE. It is a class because there are multiple possible sequential solution concepts ϕ given by this algorithm. However, we show that each satisfies the three properties, and any sequential solution concept satisfying all three is in this class.

The class SEQSHARE operates in two phases. First, an *offline initialization phase* computes the basis solution via the BASIS subroutine (line 1 of Algorithm 1), which amounts to solving a linear program and is therefore solvable in polynomial time in $|\Pi|$ (Section 3.2). This phase requires global knowledge of v but is performed offline—once, before any agent arrivals—analogously to how a solution concept is computed before play begins in classical cooperative games. Second, the *reward assignment* at each arrival step is online: given the current sequence and the precomputed upper bounds x, SEQSHARE assigns rewards without any knowledge of future arrivals.

SEQSHARE performs the following steps:

1. It checks for a basis solution. The returned value x_i is the upper bound on agent i's reward for all sequences, so $\phi_i(\pi^*)$ is set to x_i.
2. For a sequence π with $\pi(1) = i$, the algorithm sets $\phi_i(i) = v(i)$, which becomes the lower bound for $\phi_i(\pi)$ for every sequence having i as a prefix.
3. It computes the marginal contribution of $\pi(2) = j$ and divides it among i and j so that $\phi_i(ij)$ and $\phi_j(ij)$ lie within the upper and lower bounds. These bounds are then updated for every sequence containing ij as a prefix.
4. The algorithm iterates, assigning rewards within the bounds at each arrival and updating the lower bound to the most recently assigned value.

Figure 1 illustrates one particular run of SEQSHARE on a four-player game; it is an example of how rewards fill the agents' caps. A complete numerical trace of Algorithm 1 on a three-agent TCG, listing the vector returned by BASIS and the vector y returned by IMPROVIZE at every arrival step, is given in Appendix A.4. The non-uniqueness of the sequential solution concept arises from two sources: (i) the basis solution may not be unique, and (ii) the IMPROVIZE function returns a non-unique y vector.

REMARK. The existence of a basis solution is a property of the game $\langle N, v\rangle$, not of a sequential solution concept.

Algorithm 1 SEQSHARE Class of Mechanisms

Require: TCG $\langle N, v \rangle$
Find $\pi^*(v) = \text{argmax}_\pi\, v(\pi)$, $x = \text{BASIS}(N, v, \pi^*(v))$
if x = NULL **then** Output NULL, EXIT
else $\phi(\pi^*(v)) = x$; $\phi(\pi) = \mathbf{0}$, $\forall \pi \neq \pi^*(v)$
end if
for $i \in N$ **do** $\phi_i(\{i\}) = v(\{i\})$
end for
for $k = 2, \ldots, n$ **do**
  for $\pi \in \Pi,\ |\pi| = k - 1$ **do**
    for $j \in N \setminus P(\pi)$ **do**
      $y = \text{IMPROVIZE}(j, \pi, \phi(\pi), \phi(\pi^*(v)))$
      $\phi_i(\pi + j) = \phi_i(\pi) + y_i,\ \forall i \in P(\pi)$
      $\phi_j(\pi + j) = y_j$
    end for
  end for
end for
Output: ϕ

procedure BASIS$(N, v, \pi^*(v))$
  if solution to Equation (1) exists **then**
    return x of Equation (1) w.r.t. $\pi^*(v)$
  else return NULL
  end if
end procedure

procedure IMPROVIZE$(j, \pi, \phi(\pi), \phi(\pi^*))$
  $y_i \in [0, \phi_i(\pi^*) - \phi_i(\pi)],\ \forall i \in P(\pi + j)$
  $\sum_{i \in P(\pi+j)} y_i = v(\pi + j) - v(\pi)$
  return y
end procedure

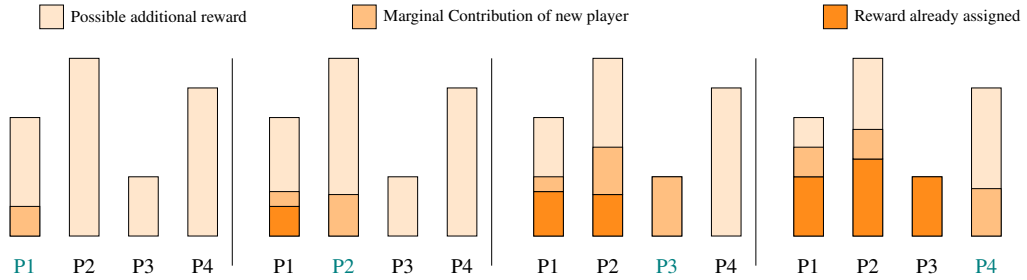


Figure 1: *(a) The first player arrives, and all the value generated is assigned to player P1. (b) The second player P2 arrives, and the marginal value generated by the arrival of this player is distributed among P1 and P2 in a way that ensures the total reward of each agent is less than the upper bound. The total reward for any agent is the sum of the reward in the previous round and the share of the marginal increase assigned in the current round. (c) The third player P3 arrives, and the marginal value generated by P3 is distributed among P1, P2, and P3, keeping total rewards lower than the upper bound. (d) Finally, P4 arrives, and the marginal value generated by this player can only be divided among P1, P2, and P4, as P3 has already reached its maximum possible reward.*

3.1 Properties of SEQSHARE

Lemma 1 states that a basis solution is necessary to satisfy I4OA and SE. The following theorem shows that it is also sufficient, and that any solution satisfying all three properties belongs to SEQSHARE.

Theorem 1. *Given a basis-satisfiable TCG $\langle N, v \rangle$, a nonnegative sequential solution concept ϕ satisfies OIR, I4OA, and SE for $\langle N, v \rangle$ if and only if it belongs to the class* SEQSHARE.

Proof Sketch. If $\langle N, v \rangle$ violates the basis conditions, then by Lemma 1 no sequential solution concept can satisfy I4OA and SE, and SEQSHARE returns NULL. Hence, assume the basis conditions hold.

We first prove the reverse direction constructively using Algorithm 1. The algorithm initializes the allocation using BASIS on $\pi^*(v)$ and assigns zero elsewhere. The construction proceeds by induction on sequence length.

For length-one sequences, SE holds trivially, I4OA follows from the basis conditions, and OIR holds by monotonicity. Assume the properties hold for all sequences of length $k - 1$. For a sequence $\pi + i$ of length k, the function IMPROVIZE allocates the marginal reward so that payoffs are non-decreasing (OIR), bounded above by $\phi(\pi^*)$ (I4OA), and sum to $v(\pi + i)$ (SE).

For the forward direction, suppose ϕ satisfies I4OA, SE, and OIR but $\phi \notin$ SEQSHARE. For all $\phi' \in$ SEQSHARE, let π' be the shortest sequence where ϕ' and ϕ differ; among these, choose the ϕ' corresponding to the longest such sequence. Call this sequence $\pi_{\text{longshort}}$. Using the properties of ϕ,

we construct a solution in SEQSHARE that matches ϕ on all sequences up to and including $|\pi_{\text{longshort}}|$, contradicting the choice of $\pi_{\text{longshort}}$. Hence, every solution satisfying I4OA, SE, and OIR belongs to SEQSHARE. □

3.2 Running Time of SEQSHARE

The input to SEQSHARE is the full table of worths $\{v(\pi)\}_{\pi\in\Pi}$, so its natural size measure is $|\Pi|$. Counting sequences by their length k, there are $n!/(n-k)!$ sequences of length k, and therefore

$$|\Pi| \;=\; \sum_{k=1}^{n} \frac{n!}{(n-k)!} \;=\; \lfloor e \cdot n! \rfloor - 1. \tag{2}$$

For example, $|\Pi| = 15$ when $n = 3$ and $|\Pi| = 64$ when $n = 4$.

Proposition 1. *For every TCG $\langle N, v\rangle$, Algorithm 1 can be implemented to run in time $T_{\mathrm{LP}}(n, |\Pi|) + O(n\,|\Pi|)$, where $T_{\mathrm{LP}}(n, m)$ is the time to solve a linear feasibility problem with n variables and m constraints. In particular, the running time is polynomial in $|\Pi|$. Once the basis vector x is computed, the reward of every agent at each arrival step can be assigned online in $O(n)$ time.*

Proof Sketch. The BASIS LP has n variables and $|\Pi| + 1$ constraints, and the loop calls IMPROVIZE once per sequence of length at least two, each call being a greedy $O(n)$ filling of slacks that never fails by the basis conditions (Appendix A.3). □

Since $|\Pi| \geqslant n!$, the running time is exponential in the number of agents n. This is the same barrier faced by the exact computation of the classical Shapley value, which is #P-complete for compactly represented games such as weighted voting games (Deng and Papadimitriou, 1994), and it is unavoidable for any method that reads the entire worth table. As with the Shapley value, we view the axiomatic foundations as the first step; tractable special classes of TCGs and sampling-based approximations in the spirit of Castro et al. (2009) are natural directions, which we discuss in Section 7.

4 Properties Inspired by Shapley in TCGs

In classical cooperative games, the characteristic function v depends on a coalition, whereas in TCGs, it depends on a player sequence. To compare solution concepts, we distinguish between *sequential solution concepts* $\phi \in \Phi = \{\phi : \Pi \times V \to \mathbb{R}^n\}$, which depend on a sequence π, and *extended solution concepts* $\psi \in \Psi = \{\psi : V \to \mathbb{R}^n\}$, which do not, where V is the set of monotone characteristic functions. Intuitively, $\phi_i(\pi)$ represents agent i's payoff given a specific formation order π (e.g., Hospital A's reward when joining in order π), whereas ψ_i provides a single, sequence-independent payoff for agent i. Any sequential solution concept ϕ can be *reduced* to an extended solution concept $\overline{\phi}$ by averaging $\phi(\pi)$ over all full sequences, as made precise in Definition 5.

Definition 5. Given a TCG $\langle N, v\rangle$ and $\phi \in \Phi$, the reduction of ϕ to $\overline{\phi} \in \Psi$ is defined as $\overline{\phi}_i = \frac{1}{n!} \sum_{\pi\in\Pi : P(\pi)=N} \phi_i(\pi)$.

The map $\phi \mapsto \overline{\phi}$ is many-to-one and has no meaningful inverse. An extended solution concept assigns exactly one reward per agent, independent of the realized order, so it cannot recover the order-specific information carried by a sequential solution concept. None of our results requires a reverse construction, since the incompatibility in Section 6 only asks whether the guarantees of SEQSHARE survive the forward reduction.

In parallel with the Shapley properties, we now define additivity, efficiency, and null player for both types of solution concepts. Properties for the sequential solution concepts are called *sequential* properties; those for extended solution concepts are called *extended* properties. Sequential efficiency is already defined in Definition 3.

Definition 6 (Sequential Additivity (SA)). A sequential solution concept $\phi \in \Phi$ is *sequentially additive* if for every pair of TCGs $\langle N, v\rangle$ and $\langle N, u\rangle$, $\phi_i(\pi, u) + \phi_i(\pi, v) = \phi_i(\pi, u + v)$ for all $\pi \in \Pi$ and $i \in N$.

Definition 7 (Sequential Null Player (SNP)). A sequential solution concept $\phi \in \Phi$ satisfies *sequential null player* if for every TCG $\langle N, v\rangle$, $v(\pi + i) = v(\pi)\ \forall \pi \in \Pi_{-i} \implies \phi_i(\pi) = 0\ \forall \pi \in \Pi$.

Definition 8 (Extended Efficiency (EE)). An extended solution concept $\psi \in \Psi$ satisfies *extended efficiency* if for every TCG $\langle N, v\rangle$, $\sum_{i\in N} \psi_i(v) = \frac{1}{n!}\sum_{\pi\in\Pi:P(\pi)=N} v(\pi)$.

Definition 9 (Extended Null Player (ENP)). An extended solution concept $\psi \in \Psi$ satisfies *extended null player* if for every TCG $\langle N, v\rangle$, $v(\pi + i) = v(\pi)\ \forall \pi \in \Pi_{-i} \implies \psi_i(v) = 0$.

This definition of extended null player is stronger than that in Sánchez and Bergantiños (1997), where a player is null only if inserting it at any position of a sequence does not improve worth. Our null player only requires the agent to be added at the end. Since the antecedent of our definition is weaker, ENP is a stronger property. Their characterization requires four properties, including symmetry, while we need only three (Section 5). We show in Example 2 that even using their definitions of the Shapley properties, the TCG-appropriate properties conflict with the Shapley-inspired ones.

Example 2. Consider the TCG with $v(1) = 1, v(12) = 1, v(2) = 1, v(21) = 2$. The only solution in SEQSHARE is $\phi(1) = (1, 0), \phi(2) = (0, 1), \phi(12) = (1, 0), \phi(21) = (1, 1)$, with reduction $\overline{\phi} = (1, 0.5)$. The solution of Sánchez and Bergantiños (1997) for this game is $\psi = (0.75, 0.75)$. Hence there exist games for which no $\phi \in$ SEQSHARE can be reduced to the solution of Sánchez and Bergantiños (1997). □

Definition 10 (Extended Additivity (EA)). An extended solution concept $\psi \in \Psi$ satisfies *extended additivity* if for every pair of TCGs $\langle N, v\rangle$ and $\langle N, u\rangle$, $\psi_i(u) + \psi_i(v) = \psi_i(u + v)$ for all $i \in N$.

One important property for the Shapley value is *symmetry*. A *swap* of agents i and j in sequence π is defined as: (a) i replaced by j if only $i \in P(\pi)$, (b) j replaced by i if only $j \in P(\pi)$, or (c) both positions swapped if $i, j \in P(\pi)$. This is denoted by $\pi_{\text{swap}(i,j)}$.

Definition 11 (Sequential Symmetry (SS)). A sequential solution concept $\phi \in \Phi$ satisfies *sequential symmetry* if for every TCG $\langle N, v\rangle$, $v(\pi) = v(\pi_{\text{swap}(i,j)})\ \forall \pi \in \Pi \implies \phi_i(\pi) = \phi_j(\pi_{\text{swap}(i,j)})\ \forall \pi \in \Pi$.

Definition 12 (Extended Symmetry (ES)). An extended solution concept $\psi \in \Psi$ satisfies *extended symmetry* if for every TCG $\langle N, v\rangle$, $v(\pi) = v(\pi_{\text{swap}(i,j)})\ \forall \pi \in \Pi \implies \psi_i = \psi_j$.

However, we will see in Appendix C that each of these symmetry properties is a consequence of the other three properties in the respective solution spaces.

5 Solutions Satisfying Shapley-inspired Properties

In this section, we introduce two solution concepts, one in Φ and one in Ψ, and show that each uniquely satisfies a corresponding set of properties.

We define the sequential solution concept MARGSOL $\in \Phi$ inspired by marginal contributions in classical cooperative games:

$$\text{MARGSOL}_i(\pi) = v(\pi_i + i) - v(\pi_i), \quad \forall \pi \in \Pi,\ i \in N. \tag{3}$$

Define the *Extended Shapley Value* EXT-SHAP by:

$$\text{EXT-SHAP}_i = \frac{1}{n!}\sum_{\pi\in\Pi:|\pi|=n} \text{MARGSOL}_i(\pi). \tag{4}$$

Observe that EXT-SHAP $= \overline{\text{MARGSOL}}$, i.e., EXT-SHAP is the reduced version of MARGSOL. The following lemma shows that sequential properties are stronger than their extended counterparts.

Lemma 2. *For every TCG $\langle N, v\rangle$ and every $\phi \in \Phi$: (1) ϕ satisfies SE $\implies$ $\overline{\phi}$ satisfies EE; (2) ϕ satisfies SA $\implies$ $\overline{\phi}$ satisfies EA; (3) ϕ satisfies SNP $\implies$ $\overline{\phi}$ satisfies ENP.*

Proof Sketch. Averaging the sequential property over all full-length sequences gives the corresponding extended property. □

Our next goal is to show that the three properties are uniquely satisfied by MARGSOL. We use a construction analogous to the Shapley uniqueness proof (Maschler et al., 2020, Chapter 19). We define the *carrier games* in the context of TCGs as follows.

Definition 13 (Carrier Game). For any sequence $\pi \in \Pi$, the carrier game over π is the simple game[2] $\langle N, u_\pi \rangle$ where $u_\pi(\pi') = 1$ if $\pi \sqsubset \pi'$ and 0 otherwise.

Lemma 3. *Every TCG $\langle N, v \rangle$ is a linear combination of carrier games.*

Proof Sketch. The set of all carrier games forms a linearly independent set of size $|\Pi|$, forming a spanning set for the vector space of all TCGs. □

5.1 Uniqueness of MARGSOL

Lemma 4. *Consider a TCG $\langle N, u_{\pi,\alpha} \rangle$ where $u_{\pi,\alpha}(\pi') = \alpha$ if $\pi \sqsubset \pi'$ and 0 otherwise. If $\phi \in \Phi$ satisfies SE and SNP, then $\phi_i(\pi', u_{\pi,\alpha}) = \alpha$ if $i = \ell(\pi)$ and $\pi \sqsubset \pi'$, and 0 otherwise.*

Proof Sketch. Intuitively, all players except $\ell(\pi)$ are null players in $\langle N, u_{\pi,\alpha} \rangle$, so by efficiency the last player must receive the entire reward α. □

Theorem 2. *For every TCG $\langle N, v \rangle$, a sequential solution concept $\phi \in \Phi$ satisfies SE, SA, and SNP if and only if $\phi \equiv$ MARGSOL.*

Proof Sketch. ($\Leftarrow$) If player i is a sequential null player, then $\text{MARGSOL}_i(\pi) = 0$ for all π. Additivity (SA) holds by linearity of marginal contributions, and efficiency (SE) follows by telescoping the sum of incremental values across the sequence.

($\Rightarrow$) Any TCG can be written as a sum of carrier games. For each carrier game, MARGSOL is the unique solution satisfying SE and SNP. If another solution ϕ also satisfies SE, SA, and SNP, it must agree with MARGSOL on all carrier games. By linearity and the basis decomposition, ϕ must equal MARGSOL on the entire game. □

Although EXT-SHAP is the averaged version of MARGSOL and MARGSOL uniquely satisfies SE, SA, and SNP (Theorem 2), this does not automatically imply that EXT-SHAP uniquely satisfies EE, EA, and ENP. Nowak and Radzik (1994, Theorem 1) established that EXT-SHAP indeed uniquely satisfies EE, EA, and ENP. We restate this theorem and present an alternative proof in Appendix B.5.

REMARK. Unlike the classical Shapley value, MARGSOL and EXT-SHAP are uniquely characterized by three axioms rather than four: symmetry is redundant because in a TCG carrier game only the last player is non-null, and receives the entire worth, so symmetry is not required. We show in Appendix C that symmetry follows from the other three properties.

6 SEQSHARE and Shapley-compliant: Best of Both Worlds

The SEQSHARE class characterizes three desirable properties for TCGs: OIR, SE, and I4OA, embodying the spirit of *sequential incentives*—rewards are shaped by the realized arrival order, protecting early participants and aligning individual incentives with the socially optimal sequence. EXT-SHAP, on the other hand, uniquely characterizes three Shapley-equivalent properties: EA, EE, and ENP, yielding Shapley-based allocations where each agent receives a reward reflecting their incremental worth averaged over all possible sequences. A solution concept satisfying both sets of properties simultaneously would be the ideal candidate. The central result of this section shows that these two principles carve out provably disjoint regions of the solution space, and that this holds even for convex and simple TCGs.

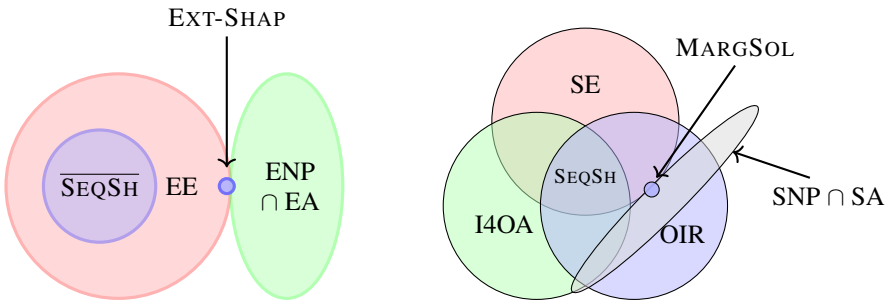
To show the result for general basis-satisfiable TCGs, we first make the following observation.

Lemma 5. *For a TCG $\langle N, v \rangle$ and $\phi \in$ SEQSHARE, if $\overline{\phi} =$ EXT-SHAP then for all $i \in N$:*

$$\phi_i(\pi^*(v)) \geqslant \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \big(v(\pi_i + i) - v(\pi_i)\big) = \text{EXT-SHAP}_i. \tag{5}$$

Proof Sketch. Suppose $\phi_i(\pi^*(v))$ is strictly less than the average marginal contribution. By I4OA, $\phi_i(\pi) \leq \phi_i(\pi^*(v))$ for all π, so the average $\overline{\phi}_i$ is also strictly less than EXT-SHAP_i, contradicting $\overline{\phi} =$ EXT-SHAP. □

[2]Following classical cooperative game terminology: a simple game is a TCG where the worth of any sequence is either zero or one.



(a) *Solutions in the extended space.* **(b)** *Solutions in the sequential space.*

Figure 2: *Impossibility of Shapley-inspired properties and solutions in* SEQSHARE *(*SEQSH*).*

Hence, Equation (5) is necessary for a solution in SEQSHARE to reduce to EXT-SHAP. The following example shows a game that violates this condition.

Example 3. Consider the TCG, $v(1) = 1, v(12) = 3, v(123) = a, v(13) = 4, v(132) = 6, v(2) = b, v(21) = 4, v(213) = 8, v(23) = 5, v(231) = 7, v(3) = 4, v(31) = 5, v(312) = a, v(32) = 5, v(321) = 7$ with $a = 7, b = 3$. Clearly, $v(\pi^*) = 8$. The basis solution (Equation (1)) dictates that $x_1 \geqslant v(1) = 1, x_2 \geqslant v(2) = 3, x_3 \geqslant v(3) = 4$ and $x_1 + x_2 + x_3 = 8$, which implies that the basis solution must be $x_1 = 1, x_2 = 3, x_3 = 4$. This is the only possible basis solution and therefore any $\phi \in$ SEQSHARE assigns $\phi_i(\pi^*) = x_i$. Routine calculations for Player 1 give EXT-SHAP$_1 = \frac{8}{6}$. However, $\phi_1(\pi^*) = 1 < \frac{8}{6} =$ EXT-SHAP$_1$, which violates Equation (5). □

REMARK. Since no solution in SEQSHARE reduces to EXT-SHAP, no solution in SEQSHARE can equal MARGSOL. MARGSOL satisfies SE and OIR (marginal contributions are non-negative for monotone games) but violates I4OA, confirming that no sequential solution concept can satisfy both sets of properties simultaneously.

Having established that Shapley-inspired properties and I4OA define provably disjoint axiomatic worlds for general TCGs, we now ask whether additional structure on the game can bring these two worlds closer. In Appendices D.2 and D.3 we consider convex and simple TCGs, respectively, and conclude that even in these restricted classes the barrier remains.

7 Conclusions and Future Work

Many AI systems involve agents contributing to a shared objective sequentially, where the order of participation shapes the outcome. This paper develops the axiomatic foundations for reward sharing in such settings. We characterized all reward-sharing mechanisms satisfying three natural sequential properties via the SEQSHARE class, and showed that a Shapley-analog uniquely satisfies the order-sensitive extensions of efficiency, additivity, and null player. Our central result is a sharp impossibility: no mechanism can simultaneously satisfy sequential incentive properties and Shapley-based fairness, even for convex and simple TCGs.

This paper has the following assumptions and limitations. *(i) Complete information.* The worth function v is known to all before anyone commits. *(ii) Basis-satisfiability.* The positive characterization (Theorem 1) applies only to basis-satisfiable TCGs. By Lemma 1 this is not an artifact of SEQSHARE, since no sequential solution concept can satisfy I4OA and SE on other games; but it does mean that in such games a designer must give up one of the two properties, and, as Appendix D.2 shows, even convexity does not rule this case out. *(iii) Monotone, nonnegative worths and nonnegative rewards.* We assume v is monotone along prefixes and nonnegative, and we restrict attention to nonnegative sequential solution concepts. Games in which a late arrival can destroy value, or mechanisms that charge agents, are outside our scope. *(iv) Computation.* SEQSHARE runs in time polynomial in $|\Pi|$ but exponential in n (Proposition 1), so it suits a small consortium, such as a few hospitals, rather than large multi-agent systems; the paper is purely theoretical and contains no empirical evaluation.

Open directions are to single out a unique mechanism within SEQSHARE, to find subclasses of TCGs where sequential and Shapley-like axioms co-exist or where SEQSHARE is computable in time polynomial in n, and to design sampling-based approximations in the spirit of Castro et al. (2009).

Acknowledgments and Disclosure of Funding

SN acknowledges support from an Amazon Research Grant (2D-09926960) and two grants from the Anusandhan National Research Foundation (ANRF), Government of India (CRG/2023/001442 and ANRF/ARG/2025/008746/QSS). DD acknowledges support from the Asha Navani Fellowship.

The authors declare no competing interests.

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A Appendix to Section 3

A.1 Proof of Lemma 1

Proof. Consider a TCG $\langle N, v\rangle$ and a sequential solution concept ϕ that satisfies I4OA and SE in the game. We claim that $x = \phi(\pi^*)$ satisfies the basis condition.

- Condition (ii) of Equation (1) is satisfied by applying SE of ϕ for π^*.
- Since ϕ also satisfies I4OA, $\phi_i(\pi) \leqslant \phi_i(\pi^*), \forall \pi \in \Pi, \forall i \in N$. Hence, $\sum_{i\in P(\pi)} \phi_i(\pi) \leqslant \sum_{i\in P(\pi)} \phi_i(\pi^*)$. By SE of ϕ, $v(\pi) = \sum_{i\in P(\pi)} \phi_i(\pi)$. Therefore, we get $\sum_{i\in P(\pi)} \phi_i(\pi^*) \geqslant v(\pi)$, which is condition (i) of Equation (1).

Hence $\phi(\pi^*)$ satisfies basis conditions. □

A.2 Proof of Theorem 1

Proof. If the game $\langle N, v\rangle$ does not satisfy the basis conditions, then from Lemma 1 we know that no ϕ can satisfy I4OA and SE. In that case, SEQSHARE returns NULL and hence the theorem holds. For the rest of the proof, we will assume that the game satisfies the basis conditions.

First, we prove the reverse direction, constructively via Algorithm 1. The function BASIS returns a valid *partial* allocation x, which is assigned to the value of the sequential solution concept for the specific sequence $\pi^*(v)$, i.e., $\phi(\pi^*(v))$, and ϕ is initialized to zero for all other sequences. The rest of the proof is by induction on the length of the sequence π, with the singleton sequences as the base case. Since the empty sequence is not an element of Π, no statement about it is needed.

Base case. Consider a singleton sequence (i). Algorithm 1 sets $\phi_i(i) = v(i)$, so SE holds for (i) by construction. Property I4OA holds because condition (i) of the basis conditions (Equation (1)), applied to the sequence (i), gives $x_i \geqslant v(i)$, and hence $\phi_i(i) = v(i) \leqslant x_i = \phi_i(\pi^*(v))$. Nonnegativity holds because we assumed $v(i) \geqslant 0$. Finally, OIR places no constraint on a singleton taken by itself, because a singleton has no proper prefix in Π; its role as a lower bound for longer sequences is handled in the inductive step.

Hence, by the induction hypothesis, assume that the forward implication of the theorem is true for all π such that $|\pi| = k-1, k \geqslant 2$. We show that the implication is true for every $\pi + i$ of length k as well. Consider the function IMPROVIZE. For every π of length $k-1$ and the existing partial share $\phi(\pi)$, IMPROVIZE allocates the reward to agents in such a way that it is at least $\phi_i(\pi)$ (required by OIR) and at most $\phi_i(\pi^*)$ (required by I4OA); the newly arrived agent receives $y_i \geqslant 0$, so the rule stays nonnegative. It remains to check that IMPROVIZE always has a feasible output, that is, that the total available slack covers the marginal worth. Summing the slacks over $P(\pi + i)$ and using $\phi_i(\pi) = 0$ for the agent $i \notin P(\pi)$, we obtain

$$\sum_{j \in P(\pi+i)} \big(\phi_j(\pi^*) - \phi_j(\pi)\big) = \sum_{j \in P(\pi+i)} x_j \;-\; \sum_{j \in P(\pi)} \phi_j(\pi) \;\geqslant\; v(\pi+i) - v(\pi).$$

Here the inequality uses two facts. The first term is at least $v(\pi + i)$ by condition (i) of the basis conditions applied to the sequence $\pi + i$, and the second term equals $v(\pi)$ by SE for π, which holds by the induction hypothesis. Moreover, each individual slack is nonnegative by the induction hypothesis (I4OA for π), and the marginal $v(\pi + i) - v(\pi)$ is nonnegative because v is monotone; this is where monotonicity is used. Hence a vector y with $0 \leqslant y_j \leqslant \phi_j(\pi^*) - \phi_j(\pi)$ and the required sum always exists. Therefore, I4OA and OIR are satisfied for each $i \in N$. Also, the construction of the y vectors (reward shares) is such that the sum of rewards of all agents $j \in \pi + i$ equals $v(\pi + i)$, satisfying SE.

Next, we prove the forward direction. We prove this via contradiction. Suppose there is a sequential solution concept ϕ for a given game $\langle N, v\rangle$ that satisfies I4OA, SE, and OIR but is not from the class SEQSHARE. Then we will construct a sequential solution concept $\tilde{\phi} \in$ SEQSHARE which matches ϕ for every sequence and for every player, leading to the contradiction that $\phi \in$ SEQSHARE.

For every solution ϕ' from class SEQSHARE, there is a shortest length sequence $\pi \in \Pi$ such that $\phi'_i(\pi) \neq \phi_i(\pi)$ for some $i \in P(\pi)$ while $\phi'_i(\pi') = \phi_i(\pi')$ for all $i \in P(\pi')$ and $\pi' \in \Pi : |\pi'| < |\pi|$, i.e., π has the minimum length where the sequential solution concepts ϕ and ϕ' differ.

Choose the ϕ' where the length of the sequence π is the longest over all ϕ's (breaking ties arbitrarily). Call this longest (over sequential solution concepts) shortest (over sequences) sequence to be $\pi_{\text{longshort}}$. Note that, by assumption, no $\phi'' \in$ SEQSHARE can satisfy $\phi''_i(\pi) = \phi_i(\pi)$ for all $i \in P(\pi)$ where $\pi \in \Pi$ and $|\pi| \leqslant |\pi_{\text{longshort}}|$. We will construct a $\tilde{\phi}$ that achieves this and belongs to SEQSHARE to show a contradiction.

Start with a game where the required properties are satisfiable. Since this game has a sequential solution concept ϕ that satisfies I4OA and SE, BASIS must be non-empty (Lemma 1). Also, $\phi(\pi^*)$ will be a solution to the basis by Lemma 1. Hence, set $\tilde{\phi}$ such that $\tilde{\phi}(\pi^*) = \phi(\pi^*)$.

Construct $\tilde{\phi} \in$ SEQSHARE as follows: $\tilde{\phi}_i(\pi') = \phi_i(\pi'), \forall i \in P(\pi'), \forall \pi' : |\pi'| < |\pi_{\text{longshort}}|$. Clearly, this is feasible, since there are $\phi'' \in$ SEQSHARE that satisfy these conditions and the construction of ϕ'' is always from shorter to longer sequences.

Define $y_i = \phi_i(\pi_{\text{longshort}}) - \phi_i(\pi_{\text{longshort}} - \ell(\pi_{\text{longshort}}))$ for all $i \in P(\pi_{\text{longshort}})$. For every agent i already present in the prefix, $y_i \geqslant 0$ because ϕ satisfies OIR. For the newly arrived agent $\ell(\pi_{\text{longshort}})$,

the prefix does not contain it, so $y_{\ell(\pi_{\text{longshort}})} = \phi_{\ell(\pi_{\text{longshort}})}(\pi_{\text{longshort}}) \geqslant 0$ because ϕ is nonnegative; this is exactly where the nonnegativity assumption of Theorem 1 is used, and without it a rule could pay a negative share to an arriving agent and fall outside SEQSHARE. Here $\ell(\pi)$ is the last agent of the sequence π and the subtraction $\pi - \ell(\pi)$ denotes that the last agent of π is removed from the end of that sequence. Since ϕ satisfies I4OA, $\phi_i(\pi_{\text{longshort}}) \leqslant \phi_i(\pi^*)$ for all $i \in P(\pi_{\text{longshort}})$. Now, set $\forall i \in P(\pi_{\text{longshort}})$

$$\begin{aligned}\tilde{\phi}_i(\pi_{\text{longshort}}) &= \tilde{\phi}_i(\pi_{\text{longshort}} - \ell(\pi_{\text{longshort}})) + y_i \\ &= \phi_i(\pi_{\text{longshort}} - \ell(\pi_{\text{longshort}})) + y_i \\ &= \phi_i(\pi_{\text{longshort}}).\end{aligned}$$

Note that this retains the sequential solution concept $\tilde{\phi}$ in SEQSHARE since it adds y_i on top of the previously allocated reward to the sequence $\pi_{\text{longshort}} - \ell(\pi_{\text{longshort}})$, since the value that y_i can take is according to the function IMPROVIZE. After adding $y_{\ell(\pi_{\text{longshort}})}$ it gives $\phi_{\ell(\pi_{\text{longshort}})}(\pi_{\text{longshort}})$ which is at most $\phi_{\ell(\pi_{\text{longshort}})}(\pi^*) = \tilde{\phi}_{\ell(\pi_{\text{longshort}})}(\pi^*)$. The second equality above comes because by construction $\tilde{\phi}$ matches ϕ for all sequences of length less than that of $\pi_{\text{longshort}}$.

Since $\pi_{\text{longshort}}$ was chosen arbitrarily among all possible longest shortest sequences, and this process of assigning $\tilde{\phi}_i(\pi_{\text{longshort}})$ does not affect other sequences of the same length, we can do the same for all other sequences of the same length simultaneously. Hence, $\tilde{\phi}$ matches ϕ for all sequences of that length and lower. This completes the proof. □

A.3 Proof of Proposition 1

Proof. We bound the cost of each part of Algorithm 1 separately and then add the bounds. Throughout, we assume that the worth table $\{v(\pi)\}_{\pi \in \Pi}$ is stored so that $v(\pi)$ can be read in constant time, and that $\phi(\pi)$ is stored as a vector of length n.

Step 1: the size of Π. A sequence of length k is an ordered choice of k distinct agents out of n, and there are $n!/(n-k)!$ such choices. Summing over $k = 1, \ldots, n$ and substituting $m = n - k$ gives

$$|\Pi| \;=\; \sum_{k=1}^{n} \frac{n!}{(n-k)!} \;=\; \sum_{m=0}^{n-1} \frac{n!}{m!} \;=\; \Big(\sum_{m=0}^{n} \frac{n!}{m!}\Big) - 1.$$

By the classical identity $\sum_{m=0}^{n} n!/m! = \lfloor e \cdot n! \rfloor$ for $n \geqslant 1$, which follows from the Taylor series of e because the tail $\sum_{m>n} n!/m!$ lies strictly between 0 and 1, this proves Equation (2). In particular $|\Pi| \geqslant n!$, which is why every bound below that is polynomial in $|\Pi|$ is exponential in n.

Step 2: finding $\pi^*(v)$. Line 1 of Algorithm 1 computes a maximizer of v over Π. Because v is monotone (Section 2), some full sequence attains the maximum, so it suffices to scan the $n!$ full sequences, and in any case a single scan of the whole table costs $O(|\Pi|)$ time.

Step 3: the BASIS *subroutine.* The basis conditions (Equation (1)) consist of one inequality for every $\pi \in \Pi$ and one equality for $\pi^*(v)$, all in the n unknowns $x_1, \ldots, x_n$. Deciding feasibility and returning a feasible x is therefore a linear feasibility problem with n variables and $|\Pi| + 1$ constraints, whose cost we denote by $T_{\mathrm{LP}}(n, |\Pi|)$. By the polynomial-time solvability of linear programming (for example, by interior-point methods), $T_{\mathrm{LP}}(n, |\Pi|)$ is polynomial in n and $|\Pi|$, and hence polynomial in $|\Pi|$ since $n \leqslant |\Pi|$.

Step 4: initialization. Setting $\phi(\pi) = \mathbf{0}$ for every $\pi \neq \pi^*(v)$, copying x into $\phi(\pi^*(v))$, and setting $\phi_i(i) = v(i)$ for every $i \in N$ touches each stored vector once and costs $O(n\,|\Pi|)$ time.

Step 5: the arrival loop. The loop visits every pair (π, j) with $|\pi| \leqslant n - 1$ and $j \in N \setminus P(\pi)$, and each such pair corresponds to exactly one sequence $\pi + j$ of length at least two. Conversely, every sequence of length at least two arises from exactly one such pair, namely by removing its last agent. Hence the loop body runs exactly $|\Pi| - n$ times.

It remains to bound the cost of one call to IMPROVIZE. We implement it by a greedy rule. Let $M = v(\pi+j) - v(\pi)$ be the marginal worth, and let $s_i = \phi_i(\pi^*) - \phi_i(\pi)$ be the slack of agent $i \in P(\pi+j)$. Visiting the agents of $P(\pi+j)$ in any fixed order, the rule sets $y_i = \min\{s_i,\ M - \sum_{i' \text{ visited before } i} y_{i'}\}$. Each y_i is computed in constant time from a running sum, so one call costs $O(n)$ time, and updating

$\phi(\pi + j)$ from $\phi(\pi)$ and y costs another $O(n)$. The greedy rule returns a valid output whenever one exists, because it stops only after the running sum reaches M or after all slack is used. The feasibility argument in the proof of Theorem 1 shows that the total slack $\sum_i s_i$ is at least M, by condition (i) of the basis conditions together with SE for π, so the running sum always reaches M. Each s_i is nonnegative by I4OA for π, and M is nonnegative by monotonicity of v, so every y_i lies in $[0, s_i]$ as IMPROVIZE requires. The total cost of the loop is therefore $O(n\,(|\Pi| - n)) = O(n\,|\Pi|)$.

Combining the steps. Adding the costs of Steps 2 to 5 gives a total of $T_{\mathrm{LP}}(n, |\Pi|) + O(|\Pi|) + O(n\,|\Pi|) + O(n\,|\Pi|)$, which is $T_{\mathrm{LP}}(n, |\Pi|) + O(n\,|\Pi|)$, as claimed. Finally, for the online statement, note that once x has been computed in the offline phase, the reward vector after each new arrival depends only on the current $\phi(\pi)$, on x, and on the two worths $v(\pi)$ and $v(\pi + j)$. It is therefore produced by a single greedy call in $O(n)$ time, without enumerating any other sequence. □

A.4 A Worked Example of SEQSHARE

Figure 1 illustrates the reward-assignment logic of SEQSHARE on a single four-player run. To complement it, we give here a complete numerical trace of Algorithm 1 on a three-agent TCG, showing the vector returned by BASIS and the vector y returned by IMPROVIZE at every arrival step.

The game. Let $N = \{1, 2, 3\}$ with the characteristic function in Table 3.

Table 3: *A three-agent TCG. Worth of every sequence $\pi \in \Pi$.*

π	1	2	3	12	13	21	23	31	32	123	132	213	231	312	321
$v(\pi)$	2	1	1	4	3	3	2	3	2	6	5	5	4	5	4

The game is monotone: $v(\pi') \le v(\pi)$ for every prefix $\pi' \sqsubset \pi$. Order matters throughout; for instance $v(12) = 4$ while $v(21) = 3$, and $v(123) = 6$ while $v(321) = 4$.

Line 1: computing π^* and the basis. The optimal sequence is $\pi^*(v) = 123$ with $v(\pi^*) = 6$. The basis conditions of Equation (1) require $x \in \mathbb{R}^3$ with $x_1 + x_2 + x_3 = 6$ and $\sum_{i \in P(\pi)} x_i \ge v(\pi)$ for all $\pi \in \Pi$. Collecting the binding constraints:

$$x_1 \ge 2, \quad x_2 \ge 1, \quad x_3 \ge 1, \qquad x_1 + x_2 \ge 4, \quad x_1 + x_3 \ge 3, \quad x_2 + x_3 \ge 2.$$

Note that each pairwise constraint is generated by the *larger* of the two orderings of that pair, e.g. $x_1 + x_2 \ge \max\{v(12), v(21)\} = 4$.

The feasible set is non-empty, so the game is basis-satisfiable. It is also not a singleton: $x = (3, 2, 1)$, $x = (2, 2, 2)$, and $x = (4, 1, 1)$ are all valid basis solutions. This is the first source of non-uniqueness of SEQSHARE noted in Section 3. We fix $x = (3, 2, 1)$ for the remainder of the trace, so $\phi(\pi^*) = \phi(123) = (3, 2, 1)$.

Line 5: length-one sequences. Each agent arriving alone receives the full worth it generates, $\phi_i(i) = v(i)$:

$$\phi(1) = (2, 0, 0), \qquad \phi(2) = (0, 1, 0), \qquad \phi(3) = (0, 0, 1).$$

Each is already within its cap, since $2 \le x_1$, $1 \le x_2$, $1 \le x_3$. Observe that agent 3 is *exactly* at its cap, $\phi_3(3) = 1 = x_3$. Its remaining slack is zero, so agent 3 receives nothing further in any sequence beginning with 3. This is precisely the situation depicted in Figure 1(d).

Lines 7–15: the arrival loop. At each step, IMPROVIZE returns a y satisfying $y_i \in [0,\ \phi_i(\pi^*) - \phi_i(\pi)]$ and $\sum_{i \in P(\pi + j)} y_i = v(\pi + j) - v(\pi)$, and the new allocation is $\phi_i(\pi + j) = \phi_i(\pi) + y_i$. Table 4 lists every call.

Three steps in Table 4 are worth highlighting.

(i) Zero slack forces exclusion. At $\pi = 13$, $j = 2$, agent 3 has slack $x_3 - \phi_3(13) = 0$, so $y_3 = 0$ is forced and the marginal 2 is split between agents 1 and 2 only. The same occurs at $\pi = 31$ and $\pi = 32$. This is the mechanism behind Figure 1(d).

(ii) The final step into π^* is uniquely determined. At $\pi = 12$, $j = 3$, the slack is $(0, 1, 1)$, whose components sum to $v(\pi^*) - v(12) = 2$, which is exactly the required marginal. Hence $y = (0, 1, 1)$ is

Table 4: *Trace of Algorithm 1 with $x = (3, 2, 1)$. The "slack" column is the componentwise upper bound $x - \phi(\pi)$ available to* IMPROVIZE*; the marginal is $v(\pi + j) - v(\pi)$.*

π	j	$\phi(\pi)$	slack $x - \phi(\pi)$	marginal	y
$k = 2$					
1	2	$(2, 0, 0)$	$(1, 2, 1)$	2	$(1, 1, 0)$
1	3	$(2, 0, 0)$	$(1, 2, 1)$	1	$(0, 0, 1)$
2	1	$(0, 1, 0)$	$(3, 1, 1)$	2	$(2, 0, 0)$
2	3	$(0, 1, 0)$	$(3, 1, 1)$	1	$(0, 1, 0)$
3	1	$(0, 0, 1)$	$(3, 2, 0)$	2	$(2, 0, 0)$
3	2	$(0, 0, 1)$	$(3, 2, 0)$	1	$(0, 1, 0)$
$k = 3$					
12	3	$(3, 1, 0)$	$(0, 1, 1)$	2	$(0, 1, 1)$
13	2	$(2, 0, 1)$	$(1, 2, 0)$	2	$(1, 1, 0)$
21	3	$(2, 1, 0)$	$(1, 1, 1)$	2	$(1, 0, 1)$
23	1	$(0, 2, 0)$	$(3, 0, 1)$	2	$(2, 0, 0)$
31	2	$(2, 0, 1)$	$(1, 2, 0)$	2	$(1, 1, 0)$
32	1	$(0, 1, 1)$	$(3, 1, 0)$	2	$(2, 0, 0)$

the only admissible choice. This holds in general: sequential efficiency forces $\sum_i \phi_i(\pi) = v(\pi)$, so the total slack on entering π^* always equals the final marginal, and IMPROVIZE necessarily recovers $\phi(\pi^*) = x$. The initialization on line 3 and the loop are therefore consistent, never conflicting.

(iii) Freedom elsewhere. At $\pi = 1$, $j = 2$ the slack is $(1, 2, 1)$ and the marginal is 2, so $y = (0, 2, 0)$ is equally admissible and yields $\phi(12) = (2, 2, 0)$ instead of $(3, 1, 0)$. This is the second source of non-uniqueness: IMPROVIZE returns a set, not a point.

The resulting sequential solution concept. Table 5 gives the full ϕ produced by this run.

Table 5: *The sequential solution concept $\phi \in$* SEQSHARE *obtained above.*

π	1	2	3	12	13	21	23	31
$\phi(\pi)$	$(2, 0, 0)$	$(0, 1, 0)$	$(0, 0, 1)$	$(3, 1, 0)$	$(2, 0, 1)$	$(2, 1, 0)$	$(0, 2, 0)$	$(2, 0, 1)$
π	32	123	132	213	231	312	321	
$\phi(\pi)$	$(0, 1, 1)$	$(3, 2, 1)$	$(3, 1, 1)$	$(3, 1, 1)$	$(2, 2, 0)$	$(3, 1, 1)$	$(2, 1, 1)$	

All three axioms can be read off directly. **SE:** the entries of $\phi(\pi)$ sum to $v(\pi)$ for each of the fifteen sequences. **I4OA:** every coordinate is bounded by $x = (3, 2, 1)$; agent 1 never exceeds 3, agent 2 never exceeds 2, agent 3 never exceeds 1. **OIR:** along every prefix chain rewards are non-decreasing, e.g. $\phi_1(1) = 2 \le \phi_1(12) = 3 \le \phi_1(123) = 3$ and $\phi_2(2) = 1 \le \phi_2(23) = 2 \le \phi_2(231) = 2$.

B Appendix to Section 5

B.1 Proof of Lemma 2

Proof. Part (a): since ϕ satisfies SE, we have $\sum_{i \in P(\pi)} \phi_i(\pi, v) = v(\pi), \forall \pi \in \Pi$. Averaging this over all full-length sequences, we get $\forall \pi \in \Pi$

$$\begin{aligned}
\frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} v(\pi) &= \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \sum_{i \in P(\pi)} \phi_i(\pi, v) \\
&= \sum_{i \in N} \left(\frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \phi_i(\pi, v) \right) \\
&= \sum_{i \in N} \overline{\phi}_i(\pi, v)
\end{aligned}$$

The first equality comes via changing the order of the summation, and the second equality is from the definition of the reduced solution concept. Hence, $\overline{\phi}$ satisfies EE.

Part (b): since ϕ satisfies SA, we have $\phi_i(\pi, u) + \phi_i(\pi, w) = \phi_i(\pi, u + w), \forall i \in N$. Averaging this over all full length sequences, we get

$$\begin{aligned}
&\frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} (\phi_i(\pi, u) + \phi_i(\pi, w)) \\
&= \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \phi_i(\pi, u + w), \ \forall \pi \in \Pi, i \in N \\
&\implies \overline{\phi}_i(u) + \overline{\phi}_i(w) = \overline{\phi}_i(u + w), \forall i \in N
\end{aligned}$$

The implication holds from the definition of the reduced solution concept. Hence, $\overline{\phi}$ satisfies EA.

Part (c): since ϕ satisfies SNP, we have $v(\pi + i) = v(\pi), \ \forall \pi \in \Pi : i \notin P(\pi) \implies \phi_i(\pi, v) = 0, \forall \pi \in \Pi$. For every such player i, averaging over all possible full length sequences, we get $\frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \phi_i(\pi, v) = 0$, which is $\overline{\phi}_i(v) = 0$ by definition. Hence,

$$v(\pi + i) = v(\pi), \ \forall \pi \in \Pi : i \notin P(\pi) \implies \overline{\phi}_i(v) = 0.$$

Hence, $\overline{\phi}$ satisfies ENP. □

B.2 Proof of Lemma 3

Proof. Consider an arbitrary TCG $\langle N, v \rangle$. We show that every v can be written as a linear combination of u_π (Definition 13).

Part 1: we show that $u_\pi, \pi \in \Pi$ are linearly independent. Suppose not, then $\exists \{\alpha_\pi : \pi \in \Pi\}$, not all 0 such that $\sum_{\pi \in \Pi} \alpha_\pi u_\pi(\pi') = 0, \forall \pi' \in \Pi$.

Let $T = \{\alpha_\pi : \pi \in \Pi, \alpha_\pi \neq 0\}$ be the set of all non-zero α_π's. Let $\tau = \{\pi : \pi \in \Pi, \alpha_\pi \in T\}$ be the indices of α in T. Observe that, any $\pi \notin \tau$ must have $\alpha_\pi = 0$. Consider the permutation $\pi_0 \in \tau$, such that no prefix of π_0 is in τ. We call this a *minimal sequence* of τ. We can write

$$\begin{aligned}
\sum_{\pi \in \Pi} \alpha_\pi u_\pi(\pi_0) = &\sum_{\pi \in \Pi, \pi \sqsubset \pi_0} \alpha_\pi u_\pi(\pi_0) + \alpha_{\pi_0} u_{\pi_0}(\pi_0) \\
&+ \sum_{\pi \in \Pi, \pi \not\sqsubset \pi_0} \alpha_\pi u_\pi(\pi_0) = 0.
\end{aligned}$$

Note that the first sum in the expanded form does not have any $\pi \in \tau$; so all $\alpha_\pi = 0$ for that sum. In the third sum, π is not a prefix of π_0. Hence by definition of u_π, $u_\pi(\pi_0) = 0$ for them. So we conclude that $\alpha_{\pi_0} u_{\pi_0}(\pi_0) = 0$. But $u_{\pi_0}(\pi_0) = 1$ by definition, and $\alpha_{\pi_0} \neq 0$ since $\pi_0 \in \tau$ which is a contradiction. Hence, u_π's are linearly independent.

Part 2: we next prove that u_π span the entire space of $\mathbb{R}^{|\Pi|}$. Every sequence $\pi \in \Pi$ denotes a unique carrier game $\langle N, u_\pi \rangle$. Hence, the number of unique u_π is $|\Pi|$.

$u_\pi : \Pi \to \mathbb{R}$, so if they are linearly independent, they must span the entire vector space of $|\Pi|$ dimensions, i.e., $\mathbb{R}^{|\Pi|}$.

Note that from parts 1 and 2, the lemma is immediate, since for any arbitrary TCG $\langle N, v \rangle$, the worth function v lives in $\mathbb{R}^{|\Pi|}$, since $v : \Pi \to \mathbb{R}$. For a fixed N, Π is finite. Parts 1 and 2 showed that $u_\pi, \pi \in \Pi$ forms a basis of $\mathbb{R}^{|\Pi|}$, and hence any v can be written as a linear combination of u_π. Therefore, every TCG $\langle N, v \rangle$ is a linear combination of the carrier games. □

B.3 Proof of Lemma 4

Proof. Suppose $i \in N \setminus \{\ell(\pi)\}$ is not the last player of π. In $\langle N, u_{\pi,\alpha} \rangle$, every such player satisfies $u_{\pi,\alpha}(\pi' + i) = u_{\pi,\alpha}(\pi'), \ \forall \pi' \in \Pi_{-i}$, i.e., each such i is a sequential null player. Until player $\ell(\pi)$

arrives in the sequence π, no marginal contribution is generated in this TCG by any other player arriving after. Hence, by SNP, all other players must get $\phi_i(\pi', u_{\pi,\alpha}) = 0, \forall \pi' \in \Pi$. The entire worth of α then must go to $\ell(\pi)$ due to SE. □

B.4 Proof of Theorem 2

Proof. ($\Leftarrow$) It is trivial to see that if i is a sequential null player, $\text{MARGSOL}_i(\pi) = 0, \forall \pi \in \Pi$. Also, $\text{MARGSOL}_i(\pi, u) + \text{MARGSOL}_i(\pi, v) = u(\pi_i + i) + v(\pi_i + i) - u(\pi_i) - v(\pi_i) = \text{MARGSOL}_i(\pi, u + v)$. Hence MARGSOL satisfies SA and SNP. To see that it also satisfies SE, note that

$$\begin{aligned}
&\sum_{i \in P(\pi)} \text{MARGSOL}_i(\pi) = \sum_{i \in P(\pi)} (v(\pi_i + i) - v(\pi_i)) \\
&= (v(\pi(1)) - v(\emptyset)) + (v(\pi(1)\pi(2)) - v(\pi(1))) \\
&+ (v(\pi(1)\pi(2)\pi(3)) - v(\pi(1)\pi(2))) + \\
&\ldots + (v(\pi(1)\pi(2)\pi(3)\cdots\pi(|P(\pi)|)) \\
&- v(\pi(1)\pi(2)\cdots\pi(|P(\pi)|-1))) = v(\pi).
\end{aligned}$$

Here, $\pi(k)$ is the agent at the k^{th} position of π, and $v(\emptyset) = 0$ by the convention of Section 2.

($\Rightarrow$) Consider an arbitrary TCG $\langle N, v \rangle$. Write it as a sum of its carrier games which is possible due to Lemma 3:

$$v(\pi') = \sum_{\pi \in \Pi} \alpha_\pi u_\pi(\pi') = \sum_{\pi \in \Pi} u_{\pi,\alpha_\pi}(\pi'). \tag{6}$$

For each carrier game, we know the structure of the unique sequential solution concept that satisfies SE and SNP from Lemma 4. Note that MARGSOL satisfies all three properties for every game $\langle N, v \rangle$, hence it must satisfy them for the carrier games as well. Suppose there exists a different sequential solution concept ϕ that also satisfies the same three properties. We will prove that ϕ must be MARGSOL.

From Lemma 4 and the discussion above, we conclude that ϕ must be the same as MARGSOL for the carrier game $\langle N, u_{\pi,\alpha} \rangle$. Hence we have $\forall i \in N, \forall \pi, \pi' \in \Pi$

$$\begin{aligned}
&\text{MARGSOL}_i(\pi', u_{\pi,\alpha_\pi}) = \phi_i(\pi', u_{\pi,\alpha_\pi}), \\
&\Rightarrow \sum_{\pi \in \Pi} \text{MARGSOL}_i(\pi', u_{\pi,\alpha_\pi}) = \sum_{\pi \in \Pi} \phi_i(\pi', u_{\pi,\alpha_\pi}), \\
&\Rightarrow \text{MARGSOL}_i\left(\pi', \sum_{\pi \in \Pi} u_{\pi,\alpha_\pi}\right) = \phi_i\left(\pi', \sum_{\pi \in \Pi} u_{\pi,\alpha_\pi}\right), \\
&\Rightarrow \text{MARGSOL}_i\left(\pi', v\right) = \phi_i\left(\pi', v\right).
\end{aligned}$$

The first implication holds by summing over all $\pi \in \Pi$. The second implication is by SA for both sequential solution concepts. The third implication is by Equation (6). Since v was arbitrary, we conclude that ϕ is the same as MARGSOL. □

B.5 Uniqueness of EXT-SHAP

EXT-SHAP is an averaged version of MARGSOL (Equation (4)). Although MARGSOL uniquely satisfies SE, SA, and SNP (Theorem 2), this does not automatically imply that EXT-SHAP uniquely satisfies EE, EA, and ENP. However, Nowak and Radzik (1994, Theorem 1) established that EXT-SHAP indeed uniquely satisfies EE, EA, and ENP. We restate this theorem and present an alternative proof using our framework.

Theorem 3 (Nowak and Radzik (1994))**.** *For every TCG $\langle N, v \rangle$, an extended solution concept $\psi \in \Psi$ satisfies EE, EA, and ENP if and only if $\psi =$* EXT-SHAP.

Lemma 6. *Consider $\langle N, u_{\pi,\alpha} \rangle$ as in Lemma 4. If $\psi \in \Psi$ satisfies EE and ENP, then $\psi_i(u_{\pi,\alpha}) = \frac{(n-|P(\pi)|)!}{n!}\alpha$ if $i = \ell(\pi)$ and 0 otherwise.*

Proof. We know from Lemma 4 that every $i \in N \setminus \{\ell(\pi)\}$ is a null player and should get zero reward share by any extended solution concept ψ by ENP. Hence, the only positive reward should go to $\ell(\pi)$.

By EE, we get

$$\sum_{i \in P(\pi')} \psi_i(u_{\pi,\alpha}) = \frac{1}{n!} \sum_{\pi' \in \Pi : P(\pi') = N} u_{\pi,\alpha}(\pi').$$

Given the earlier arguments, this reward will go entirely to $\ell(\pi)$, i.e., for $i = \ell(\pi)$, the last player of π, we get that $\psi_{\ell(\pi)}(u_{\pi,\alpha}) = \frac{1}{n!} \sum_{\pi' \in \Pi : P(\pi') = N} u_{\pi,\alpha}(\pi')$.

By definition of $u_{\pi,\alpha}$, as long as π is a prefix of a given sequence of players π', the remaining $N \setminus P(\pi)$ players can appear in any sequence. So, there are $(n - |P(\pi)|)!$ different $\pi' \in \Pi$ such that π is a prefix of π'. Therefore, $u_{\pi,\alpha}(\pi') = \alpha$ for all those π'. Hence, $\psi_{\ell(\pi)}(u_{\pi,\alpha}) = \frac{(n-|P(\pi)|)!}{n!}\alpha$. □

Proof of Theorem 3. (⇐) We have already proved in Theorem 2 that MARGSOL satisfies SE, SA, and SNP, and by Lemma 2, its reduced solution concept EXT-SHAP satisfies EE, EA, and ENP.

(⇒) This direction of the proof follows in similar lines to that of Theorem 2. Assume that there exists an extended solution concept $\psi \neq$ EXT-SHAP that also satisfies the three given properties.

For each carrier game, we know the structure of the unique extended solution concept that satisfies EE and ENP from Lemma 6. Note that EXT-SHAP satisfies all three properties for every game $\langle N, v \rangle$, hence it must satisfy them for the carrier games as well. Hence ψ must be same as EXT-SHAP for the carrier game $\langle N, u_{\pi,\alpha} \rangle$. Hence we have

$$\begin{aligned}
& \text{EXT-SHAP}_i(u_{\pi,\alpha_\pi}) = \psi_i(u_{\pi,\alpha_\pi}),\ \forall i \in N, \forall \pi \in \Pi, \\
\Rightarrow \quad & \sum_{\pi \in \Pi} \text{EXT-SHAP}_i(u_{\pi,\alpha_\pi}) = \sum_{\pi \in \Pi} \psi_i(u_{\pi,\alpha_\pi}),\ \forall i \in N, \\
\Rightarrow \quad & \text{EXT-SHAP}_i\left(\sum_{\pi \in \Pi} u_{\pi,\alpha_\pi}\right) = \psi_i\left(\sum_{\pi \in \Pi} u_{\pi,\alpha_\pi}\right),\ \forall i \in N, \\
\Rightarrow \quad & \text{EXT-SHAP}_i\left(v\right) = \psi_i\left(v\right),\ \forall i \in N.
\end{aligned}$$

The first implication holds by summing over all $\pi \in \Pi$. The second implication is by EA for both extended solution concepts. The third implication is by Equation (6). Since v was arbitrary, we conclude that ψ is the same as EXT-SHAP. □

C Why Symmetry Is Not Necessary

Properties in the TCG framework are more restrictive, as they make claims for all *sequences* rather than all *sets* as in classical cooperative games. Hence, three properties suffice to ensure unique solution concepts. In the following, we show that MARGSOL and EXT-SHAP satisfy SS (Definition 11) and ES (Definition 12) respectively.

Theorem 4. MARGSOL *and* EXT-SHAP *satisfy SS and ES, respectively.*

*Proof. Part 1 (*MARGSOL*)*: Consider a TCG $\langle N, v \rangle$, s.t. $v(\pi) = v(\pi_{\text{swap}(i,j)}),\ \forall \pi \in \Pi$. We show the following $\forall \pi \in \Pi, \forall i \in N$

$$\begin{aligned}
& \text{MARGSOL}_i(\pi) \\
& = v(\pi_i + i) - v(\pi_i) \\
& = v((\pi_i)_{\text{swap}(i,j)} + j) - v((\pi_i)_{\text{swap}(i,j)}) \\
& = v((\pi_{\text{swap}(i,j)})_j + j) - v((\pi_{\text{swap}(i,j)})_j) \\
& = \text{MARGSOL}_j(\pi_{\text{swap}(i,j)}).
\end{aligned}$$

The first equality follows from definition. The second equality comes from the 'if' condition of the SS property. The third equality follows from the fact that $(\pi_i)_{\text{swap}(i,j)} = (\pi_{\text{swap}(i,j)})_j$. The last equality is again by definition. Hence, we show that MARGSOL satisfies SS.

*Part 2 (*EXT-SHAP*)*: Since $\text{EXT-SHAP}_i = \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \text{MARGSOL}_i(\pi)$, and we have already shown in Part 1 that $\text{MARGSOL}_i(\pi) = \text{MARGSOL}_j(\pi_{\text{swap}(i,j)})$, it is straightforward to show that

$$\text{EXT-SHAP}_i$$

$$
\begin{aligned}
&= \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \textsc{MargSol}_i(\pi) \\
&= \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \textsc{MargSol}_j(\pi_{\text{swap}(i,j)}) \\
&= \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \textsc{MargSol}_j(\pi) \\
&= \textsc{Ext-Shap}_j.
\end{aligned}
$$

The third equality holds since $\pi_{\text{swap}(i,j)}$ just changes the order of summation in the set $\{\pi \in \Pi : P(\pi) = N\}$. Hence, EXT-SHAP satisfies ES. □

REMARK. In classical cooperative games, symmetry is required to make the Shapley value unique because a carrier game is defined with respect to a critical *set* of players, and dividing worth among multiple non-null players requires symmetry. In TCGs, a carrier game is defined with respect to a critical *sequence*: only the last player is non-null and receives the entire worth, so symmetry is not required.

D Appendix to Section 6

D.1 Proof of Lemma 5

Proof. Consider a sequential solution concept $\phi \in \textsc{SeqShare}$ such that $\overline{\phi} = \textsc{Ext-Shap}$. Suppose $\exists i \in N$ such that $\phi_i(\pi^*(v)) < \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} v(\pi_i + i) - v(\pi_i)$. Since $\phi \in \textsc{SeqShare}$, it satisfies I4OA, i.e.,

$$
\begin{aligned}
&\phi_i(\pi) \leqslant \phi_i(\pi^*(v)),\ \forall \pi \in \Pi \\
\implies &\phi_i(\pi) < \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} v(\pi_i + i) - v(\pi_i),\ \forall \pi \in \Pi.
\end{aligned}
$$

Averaging over all possible $\pi \in \Pi$ such that $P(\pi) = N$, we get

$$
\begin{aligned}
\overline{\phi}_i &= \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} \phi_i(\pi) && (7) \\
&< \frac{1}{n!} \sum_{\pi \in \Pi : P(\pi) = N} v(\pi_i + i) - v(\pi_i) = \textsc{Ext-Shap}_i. && (8)
\end{aligned}
$$

This contradicts the hypothesis that $\overline{\phi} = \textsc{Ext-Shap}$. □

D.2 Convex Games

In classical cooperative games, the Shapley value always lies in the core for convex games. Since the basis conditions (Equation (1)) are analogous to the core conditions, we check for a similar result in TCGs.

Definition 14 (Convex Temporal Cooperative Games). A TCG $\langle N, v \rangle$ is *convex* if $v(\pi + i) - v(\pi) \leqslant v(\pi' + i) - v(\pi')$ for all $\pi, \pi' \in \Pi$ and $i \in N$ such that $i \notin P(\pi')$ and $P(\pi) \subseteq P(\pi')$.

The answer is unfortunately negative. First, the following example illustrates a convex TCG with no basis solution, unlike convex classical games where the Shapley value always lies in the core.

Example 4. Consider the convex TCG from Example 3 with $a = 7, b = 4$. The basis solution must satisfy $x_1 \geqslant 1, x_2 \geqslant 4, x_3 \geqslant 4$ and $x_1 + x_2 + x_3 = 8$, which is impossible. Hence this convex TCG has no basis solution. □

Example 5. The TCG from Example 3 (with $a = 7, b = 3$) is convex and violates Equation (5). □

D.3 Simple Games

Definition 15 (Simple Temporal Cooperative Games). A TCG $\langle N, v\rangle$ with $v : \Pi \to \{0, 1\}$ is called a *simple* temporal cooperative game.

Since we consider monotone games, $v(\pi) = 1 \implies v(\pi') = 1$ for all $\pi \sqsubset \pi'$. The following example shows that even for simple games, there may be no solution in SEQSHARE reducing to EXT-SHAP.

Example 6. Consider the simple TCG, $v(1) = v(12) = v(123) = v(13) = v(132) = v(213) = v(231) = v(312) = v(321) = 1, v(\pi) = 0$ for all others, in particular $v(23) = 0$. The unique basis solution (Equation (1)) is $x_1 = 1, x_2 = 0, x_3 = 0$. Therefore any $\phi \in$ SEQSHARE assigns $\phi_i(\pi^*) = x_i$. Routine calculations for Player 2 give EXT-SHAP$_2 = \frac{1}{6}$. However, $\phi_2(\pi^*) = 0 < \frac{1}{6} =$ EXT-SHAP$_2$, which violates Equation (5). □

Even for simple TCGs, we cannot hope to have a solution in SEQSHARE that reduces to EXT-SHAP.