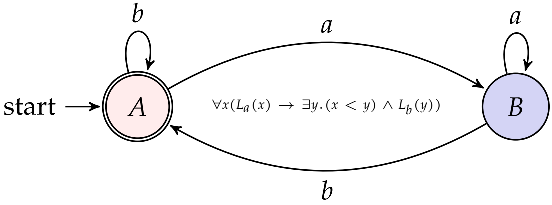
CS 208: Automata Theory and Logic

Lecture 3: Nondeterminism

Ashutosh Trivedi



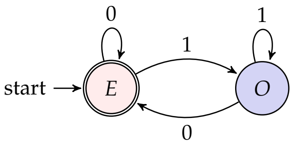
Department of Computer Science and Engineering,
Indian Institute of Technology Bombay.

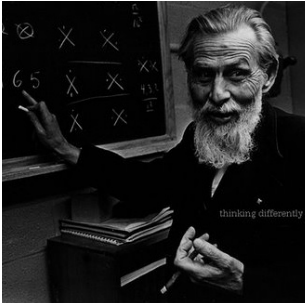
Finite State Automata

Nondeterministic Finite State Automata

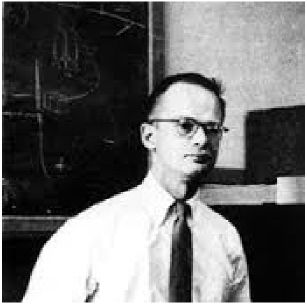
Alternation

Finite State Automata



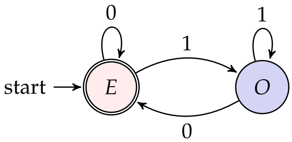


Warren S. McCullough



Walter Pitts

Deterministic Finite State Automata (DFA)



A finite state automaton is a tuple $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$, where:

- S is a finite set called the states;
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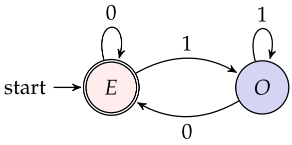

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For a function $\delta : S \times \Sigma \to S$ we define extended transition function $\hat{\delta} : S \times \Sigma^* \to S$ using the following inductive definition:

$$\hat{\delta}(q, w) = \begin{cases} q & \text{if } w = \varepsilon \\ \delta(\hat{\delta}(q, x), a) & \text{if } w = xa \text{ s.t. } x \in \Sigma^* \text{ and } a \in \Sigma. \end{cases}$$

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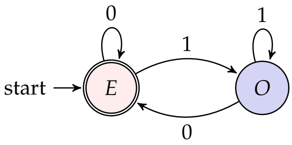

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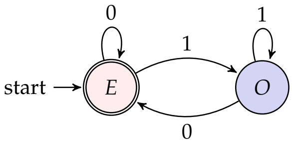
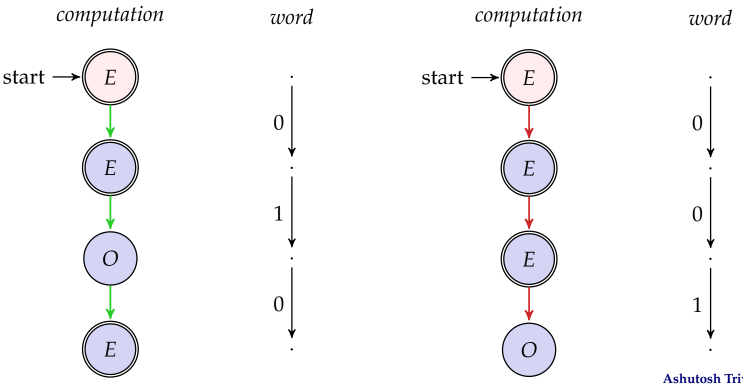
The language $L(\mathcal{A})$ accepted by a DFA $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ is defined as:

$$L(\mathcal{A}) \stackrel{\text{def}}{=} \{w \ : \ \hat{\delta}(w) \in F\}.$$

Computation of a DFA





Deterministic Finite State Automata

Semantics using extended transition function:

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Semantics using accepting computation:

- A computation of a DFA $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ on a word $w = a_0 a_1 \ldots a_{n-1}$ is the finite sequence $s_0, a_1, s_1, a_2, \ldots, a_{n-1}, s_n$, where s_0 is the starting state, and $\delta(s_{i-1}, a_i) = s_{i+1}$.
- A word w is accepted by a DFA $\mathcal{A}$ if the last state of the unique computation of $\mathcal{A}$ on w is an accept state, i.e. $s_n \in F$.
- Language of a DFA $\mathcal{A}$

$$L(\mathcal{A}) = \{w \,:\, \text{word } w \text{ is accepted by DFA } \mathcal{A}\}.$$

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Proposition

Both semantics define the same language. *Proof by induction.*

Regular Languages: Properties

Definition (Regular Languages)

A language is called regular if it is accepted by a finite state automaton.

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Let A and B be languages (remember they are sets). We define the following operations on them:

1. Union: $A \cup B = \{w \,:\, w \in A \text{ or } w \in B\}$
2. Intersection: $A \cap B = \{w \,:\, w \in A \text{ and } w \in B\}$

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4. Concatenation: $AB = \{wv : w \in A \text{ and } v \in B\}$
5. Closure (Kleene Closure, or Star):
   $A^* = \{w_1 w_2 \ldots w_k : k \geq 0 \text{ and } w_i \in A\}$. In other words:

$$A^* = \cup_{i \geq 0} A^i$$

   where $A^0 = \emptyset$, $A^1 = A$, $A^2 = AA$, and so on.

Define the notion of a set being closed under an operation (say, $\mathbb{N}$ and $\times$).

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Theorem

The class of regular languages is closed under union, intersection, complementation, concatenation, andKleene closure.

Closure under Union

Exercise

Construct a DFA accepting the following language:

$L = \{w \in \{0,1\}^* \;:\; \text{number of 1's is even or number of 0's is multiple of 3}\}$

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Or equivalently, $L = L_1 \cup L_2$ where

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"The intuitive idea of simulating both automata together.." Generalize!

Closure under Union

Lemma

The class of regular languages is closed under union.

Proof.

- Let A_1 and A_1 be regular languages.

Closure under Union

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- Let A_1 and A_1 be regular languages.
- Let $M_1 = (S_1, \Sigma, \delta_1, s_1, F_1)$ and $M_2 = (S_2, \Sigma, \delta_2, s_2, F_2)$ be finite automata accepting these languages.

Closure under Union

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- Simulate both automata together!
- Product Construction:
  We define $M = (S, \Sigma, \delta, s, F)$ where
  - $S = S_1 \times S_2$,
  - $\delta : S \times \Sigma \to S$ is such that $\delta((s_1, s_2), a) = (\delta(s_1, a), \delta(s_2, a))$ for all $s_1 \in S_1$, $s_2 \in S_2$, and $a \in \Sigma$,
  - $s = (s_1, s_2)$;
  - $F = (F_1 \times S_2) \cup (S_1 \times F_2)$.
- Prove that M accepts $A_1 \cup A_2$ (How?)
- The language $A_1 \cup A_2$ is regular since it is accepted by a DFA M.

Closure under Intersection

Lemma

The class of regular languages is closed under intersection.

Proof.

- Let A_1 and A_1 be regular languages.
- Let $M_1 = (S_1, \Sigma, \delta_1, s_1, F_1)$ and $M_2 = (S_2, \Sigma, \delta_2, s_2, F_2)$ be finite automata accepting these languages.
- Simulate both automata together!
- Product Construction:
  We define $M = (S, \Sigma, \delta, s, F)$ where
  - $S = S_1 \times S_2$,
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  - $s = (s_1, s_2)$;
  - $F = (F_1 \times F_2)$.
- Prove that M accepts $A_1 \cap A_2$ (How?)
- The language $A_1 \cap A_2$ is regular since it is accepted by a DFA M.

Closure under Complementation

Exercise

Construct a DFA accepting the following language:

$$L = \{w \in \{0,1\}^* \; : \; w \text{ is a word that does not contain } 110\}$$

Closure under Complementation

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Construct a DFA accepting the following language:

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"Find the differences between automata accepting L and L'". Generalize!

Closure under Complementation

Lemma

The class of regular languages is closed under complementation.

Proof.

- Let A be a regular language over Σ. Let $\overline{A}$ be the set $\Sigma^* \setminus A$.
- Let $M = (S, \Sigma, \delta, s, F)$ be a DFA accepting A.
- Complementation:
  We define $M' = (S', \Sigma, \delta', s', F')$ where
  - $S' = S$,
  - $\delta' : S' \times \Sigma \to S'$ is such that $\delta'(s, a) = \delta(s, a)$ for all $s \in S$, and $a \in \Sigma$,
  - $s' = s$; and
  - $F = S \setminus F$.
- Prove that M accepts $\overline{A}$. (How?)
- The language $\overline{A}$ is regular since it is accepted by a DFA M.

□

Closure under Concatenation

Exercise

Construct a DFA accepting the following language:

$$L = \{w_1 w_2 \ : \ \text{no. of 1's in } w_1 \text{ is even and no. of 0's is } w_2 \text{ is multiple of } 3\}$$

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"The intuitive idea of ε transitions.."

Closure under Concatenation

Lemma

The class of regular languages is closed under concatenation.

Proof.

(Attempt).

- Let A_1 and A_1 be regular languages.
- Let $M_1 = (S_1, \Sigma, \delta_1, s_1, F_1)$ and $M_2 = (S_2, \Sigma, \delta_2, s_2, F_2)$ be finite automata accepting these languages.
- How can we find an automaton that accepts the concatenation?
- Does this automaton fit our definition of a finite state automaton?
- Determinism vs Non-determinism

□

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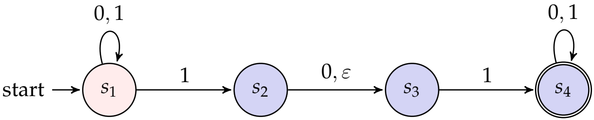
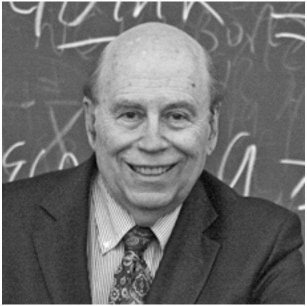
Nondeterministic Finite State Automata

Alternation

Nondeterministic Finite State Automata

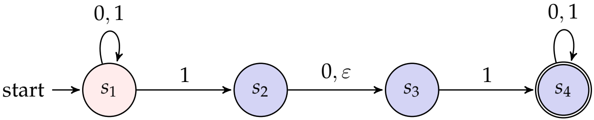




Michael O. Rabin

Dana Scott

Non-deterministic Finite State Automata



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- $\delta : S \times (\Sigma \cup \{\varepsilon\}) \to 2^S$ is the transition function;
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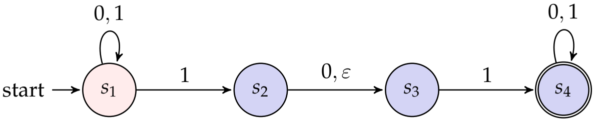
Non-deterministic Finite State Automata



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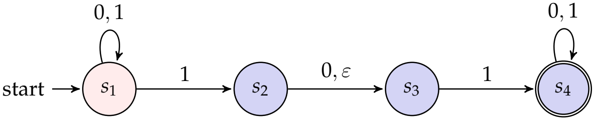
$$\hat{\delta}(q, w) = \begin{cases} \{q\} & \text{if } w = \varepsilon \\ \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a) & \text{if } w = xa \text{ s.t. } x \in \Sigma^* \text{ and } a \in \Sigma. \end{cases}$$

Non-deterministic Finite State Automata



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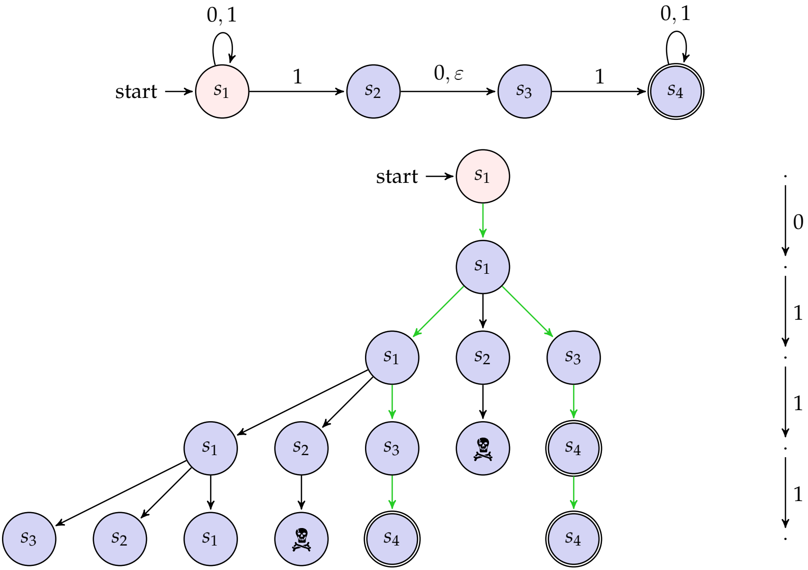
Computation of an NFA



Non-deterministic Finite State Automata

Semantics using extended transition function:

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Semantics using accepting computation:

- A computation of an NFA on a word $w = a_0 a_1 \ldots a_{n-1}$ is a finite sequence

$$s_0, r_1, s_1, r_2, \ldots, r_{k-1}, s_n$$

  where s_0 is the starting state, and $s_{i+1} \in \delta(s_{i-1}, r_i)$ and $r_0 r_1 \ldots r_{k-1} = a_0 a_1 \ldots a_{n-1}$.
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- Language of an NFA $\mathcal{A}$

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Proposition

Both semantics define the same language. *Proof by induction.*

Why study NFA?

NFAs are often more convenient to design than DFAs, e.g.:

- $\{w : w$ contains 1 in the third last position$\}$.
- $\{w : w$ is a multiple of 2 or a multiple of 3$\}$.
- Union and intersection of two DFAs as an NFA
- Exponentially succinct than DFA
  - Consider the language of words having n-th symbol from the end is 1.
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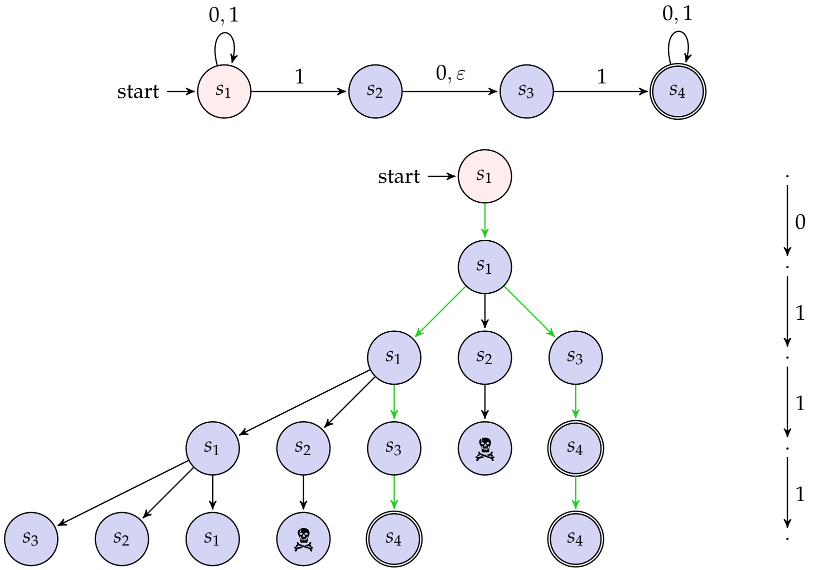
And, surprisingly perhaps:

Theorem (DFA=NFA)

Every non-deterministic finite automaton has an equivalent (accepting the same language) deterministic finite automaton. Subset construction.

Computation of an NFA: An observation



ε-free NFA = DFA

Let $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ be an ε-free NFA. Consider the DFA $Det(\mathcal{A}) = (S', \Sigma', \delta', s'_0, F')$ where

- $S' = 2^S$,
- $\Sigma' = \Sigma$,
- $\delta' : 2^S \times \Sigma \to 2^S$ such that $\delta'(P, a) = \bigcup_{s \in P} \delta(s, a)$,
- $s'_0 = \{s_0\}$, and
- $F' \subseteq S'$ is such that $F' = \{P \, : \, P \cap F \neq \emptyset\}$.

Theorem (ε-free NFA = DFA)

$L(\mathcal{A}) = L(Det(\mathcal{A}))$. *By induction, hint $\hat{\delta}(s_0, w) = \hat{\delta}'(\{s_0\}, w)$.*

Exercise (3.1)

Extend the proof for NFA with ε transitions. *hint: ε-closure*

Proof of correctness: $L(\mathcal{A}) = L(Det(\mathcal{A}))$.

The proof follows from the observation that $\hat{\delta}(s_0, w) = \hat{\delta}'(\{s_0\}, w)$. We prove it by induction on the length of w.

- Base case: Let the size of w be 0, i.e. $w = \varepsilon$. The base case follows immediately from the definition of extended transition functions:

$$\hat{\delta}(s_0, \varepsilon) = \varepsilon \text{ and } \hat{\delta}'(\{s_0\}, w) = \varepsilon.$$

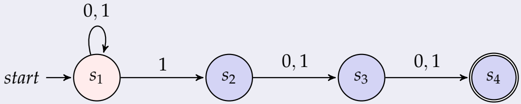
- Induction Hypothesis: Assume that for all words $w \in \Sigma^*$ of size n we have that $\hat{\delta}(s_0, w) = \hat{\delta}'(\{s_0\}, w)$.
- Induction Step: Let $w = xa$ where $x \in \Sigma^*$ and $a \in \Sigma$ be a word of size $n + 1$, and hence x is of size n. Now observe,

$$\begin{aligned}
\hat{\delta}(s_0, xa) &= \bigcup_{s \in \hat{\delta}(s_0, x)} \delta(s, a), \text{by definition of } \hat{\delta}. \\
&= \bigcup_{s \in \hat{\delta}'(\{s_0\}, x)} \delta(s, a), \text{from inductive hypothesis.} \\
&= \delta'(\hat{\delta}'(\{s_0\}, x), a), \text{ from definition } \delta'(P, a) = \bigcup_{s \in P} \delta(s, a). \\
&= \hat{\delta}'(\{s_0\}, xa), \text{ by definition of } \hat{\delta}'.
\end{aligned}$$

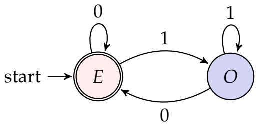
Equivalence of NFA and DFA

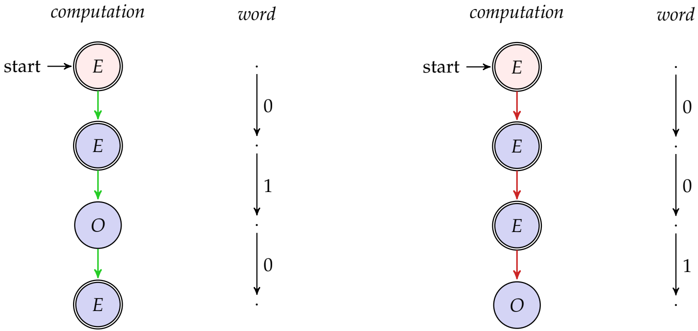
Exercise (In class)

Determinize the following automaton:



Complementation of the Language of a DFA





Hint: Simply swap the accepting and non-accepting states!

Complementation of a DFA

Theorem

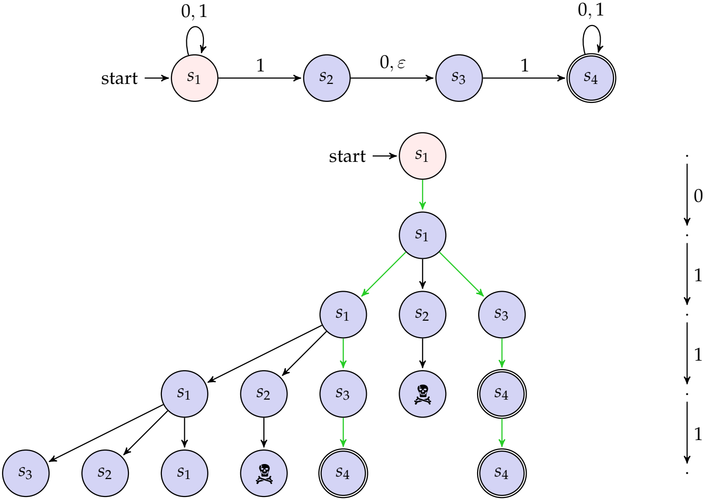
Complementation of the language of a DFA $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ is the language accepted by the DFA $\mathcal{A}' = (S, \Sigma, \delta, s_0, S \setminus F)$.

Proof.

- $L(\mathcal{A}) = \{w \in \Sigma^* \ : \ \hat{\delta}(s_0, w) \in F\}$,
- $\Sigma^* \setminus L(\mathcal{A}) = \{w \in \Sigma^* \ : \ \hat{\delta}(s_0, w) \not\in F\}$,
- $L(\mathcal{A}') = \{w \in \Sigma^* \ : \ \hat{\delta}(s_0, w) \in S \setminus F\}$, and
- transition function is total.

$\square$

Complementation of the language of an NFA



Question: Can we simply swap the accepting and non-accepting states?

Complementation of the language of a NFA

Question: Can we simply swap the accepting and non-accepting states?

Let the NFA $\mathcal{A}$ be $(S, \Sigma, \delta, s_0, F)$ and let the NFA $\mathcal{A}'$ be $(S, \Sigma, \delta, s_0, S \setminus F)$ the NFA after swapping the accepting states.

- $L(\mathcal{A}) = \{w \in \Sigma^* \ : \ \hat{\delta}(s_0, w) \cap F \neq \emptyset\}$,
- $L(\mathcal{A}') = \{w \in \Sigma^* \ : \ \hat{\delta}(s_0, w) \cap (S \setminus F) \neq \emptyset\}$.
- Consider, the complement language of $\mathcal{A}$

$$\begin{aligned} \Sigma^* \setminus L(\mathcal{A}) &= \{w \in \Sigma^* \ : \ \hat{\delta}(s_0, w) \cap F = \emptyset\} \\ &= \{w \in \Sigma^* \ : \ \hat{\delta}(s_0, w) \subseteq S \setminus F\}. \end{aligned}$$

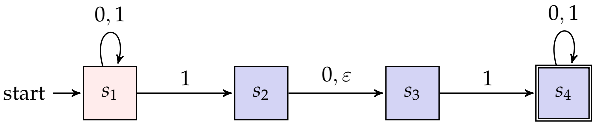
- Hence $L(\mathcal{A}')$ does not quite capture the complement. Moreover, the condition for $\Sigma^* \setminus L(\mathcal{A})$ is not quite captured by either DFA or NFA.

Finite State Automata

Nondeterministic Finite State Automata

Alternation

Universal Non-deterministic Finite Automata



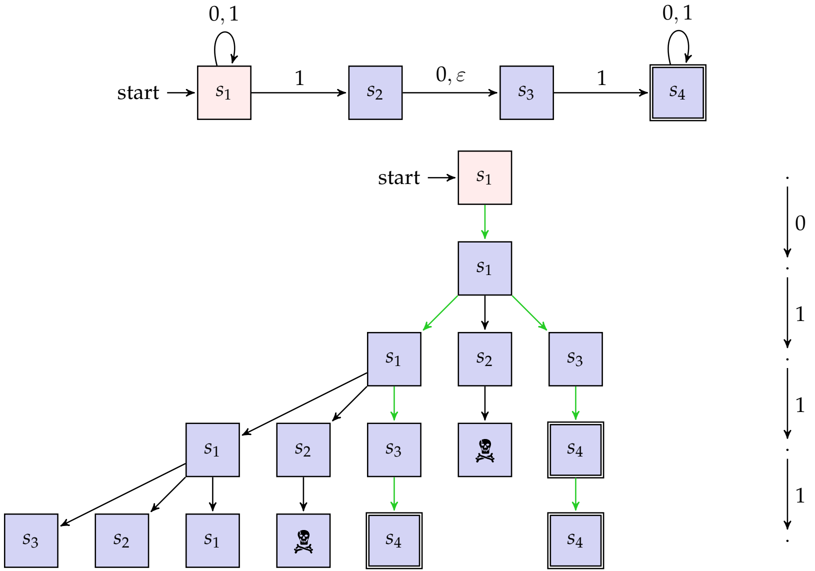
A universal non-deterministic finite state automaton (UNFA) is a tuple $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$, where:

- S is a finite set called the states;
- Σ is a finite set called the alphabet;
- $\delta : S \times (\Sigma \cup \{\varepsilon\}) \to 2^S$ is the transition function;
- $s_0 \in S$ is the start state; and
- $F \subseteq S$ is the set of accept states.

The language $L(\mathcal{A})$ accepted by a UNFA $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ is defined as:

$$L(\mathcal{A}) \stackrel{\text{def}}{=} \{w \;:\; \hat{\delta}(w) \subseteq F\}.$$

Computation of an UNFA



Universal Non-deterministic Finite Automata

Semantics using extended transition function:

- The language $L(\mathcal{A})$ accepted by an NFA $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ is defined as:

$$L(\mathcal{A}) \stackrel{\text{def}}{=} \{w \;:\; \hat{\delta}(w) \subseteq F\}.$$

Semantics using accepting computation:

- A computation of an NFA on a word $w = a_0 a_1 \ldots a_{n-1}$ is a finite sequence

$$s_0, r_1, s_1, r_2, \ldots, r_{k-1}, s_n$$

  where s_0 is the starting state, and $s_{i+1} \in \delta(s_{i-1}, r_i)$ and $r_0 r_1 \ldots r_{k-1} = a_0 a_1 \ldots a_{n-1}$.
- A word w is accepted by an NFA $\mathcal{A}$ if the last state of all computations of $\mathcal{A}$ on w is an accept state $s_n \in F$.
- Language of an NFA $\mathcal{A}$

$$L(\mathcal{A}) = \{w \;:\; \text{word } w \text{ is accepted by NFA } \mathcal{A}\}.$$

Proposition

Both semantics define the same language. *Proof by induction.*

ε-free UNFA = DFA

Let $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ be an ε-free UNFA. Consider the DFA $Det(\mathcal{A}) = (S', \Sigma', \delta', s'_0, F')$ where

- $S' = 2^S$,
- $\Sigma' = \Sigma$,
- $\delta' : 2^S \times \Sigma \to 2^S$ such that $\delta'(P, a) = \bigcup_{s \in P} \delta(s, a)$,
- $s'_0 = \{s_0\}$, and
- $F' \subseteq S'$ is such that $F' = \{P \ : \ P \subseteq F\}$.

Theorem (ε-free UNFA = DFA)

$L(\mathcal{A}) = L(Det(\mathcal{A}))$. *By induction, hint $\hat{\delta}(s_0, w) = \hat{\delta}'(s_0, w)$.*

ε-free UNFA = DFA

Let $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ be an ε-free UNFA. Consider the DFA $Det(\mathcal{A}) = (S', \Sigma', \delta', s'_0, F')$ where

- $S' = 2^S$,
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- $s'_0 = \{s_0\}$, and
- $F' \subseteq S'$ is such that $F' = \{P : P \subseteq F\}$.

Theorem (ε-free UNFA = DFA)

$L(\mathcal{A}) = L(Det(\mathcal{A}))$. *By induction, hint $\hat{\delta}(s_0, w) = \hat{\delta}'(s_0, w)$.*

Exercise (3.2)

Extend the proof for UNFA with ε transitions.

Complementation of an NFA

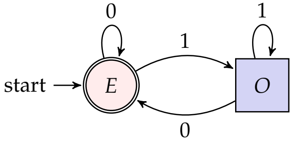
Theorem

Complementation of the language of an NFA $\mathcal{A} = (S, \Sigma, \delta, s_0, F)$ *is the language accepted by the UNFA* $\mathcal{A}' = (S, \Sigma, \delta, s_0, S \setminus F)$.

Exercise (3.3)

Write a formal proof for this theorem.

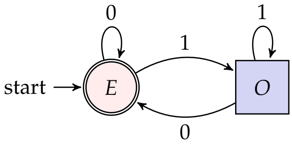
Alternating Finite State Automata



Ashok K. Chandra

Larry J. Stockmeyer

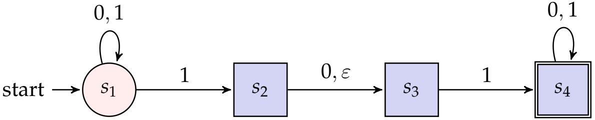
Alternating Finite State Automata

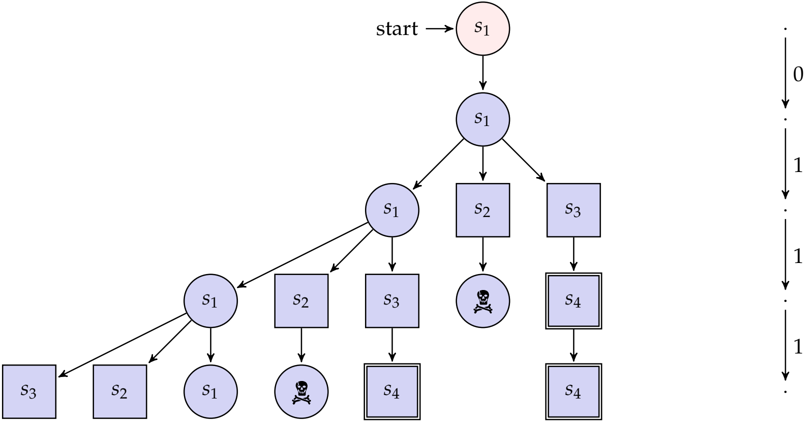


An alternating finite state automaton (AFA) is a tuple $\mathcal{A} = (S, S_\exists, S_\forall, \Sigma, \delta, s_0, F)$, where:

- S is a finite set called the states with a partition $S_\exists$ and $S_\forall$;
- Σ is a finite set called the alphabet;
- $\delta : S \times (\Sigma \cup \{\varepsilon\}) \to 2^S$ is the transition function;
- $s_0 \in S$ is the start state; and
- $F \subseteq S$ is the set of accept states.

Computation of an AFA





Universal Non-deterministic Finite Automata

- A computation of an AFA on a word $w = a_0 a_1 \dots a_{n-1}$ is a game graph $\mathcal{G}(\mathcal{A}, w) = (S \times \{0, 1, 2, \dots, n-1\}, E)$ where:
  - Nodes in $S_\exists \times \{0, 1, 2, \dots, n-1\}$ are controlled by Eva and nodes in $S_\forall \times \{0, 1, 2, \dots, n\}$ are controlled by Adam; and
  - $((s, i), (s', i+1)) \in E$ if $s' \in \delta(s, a_i)$.
- Initially a token is in $(s_0, 0)$ node, and at every step the controller of the current node chooses the successor node.
- Eva wins if the node reached at level i is an accepting state node, otherwise Adam wins.
- We say that Eva has a winning strategy if she can make her decisions no matter how Adam plays.
- A word w is accepted by an AFA $\mathcal{A}$ if Eva has a winning strategy in the graph $\mathcal{G}(\mathcal{A}, w)$.
- Language of an AFA $\mathcal{A}$ $L(\mathcal{A}) = \{w \,:\, \text{word } w \text{ is accepted by AFA } \mathcal{A}\}$.
- Example.

ε-free AFA = NFA

Let $\mathcal{A} = (S, S_\exists, S_\forall, \Sigma, \delta, s_0, F)$ be an ε-free AFA. Consider the NFA $NDet(\mathcal{A}) = (S', \Sigma', \delta', s_0', F')$ where

- $S' = 2^S$,
- $\Sigma' = \Sigma$,
- $\delta' : 2^S \times \Sigma \to 2^{2^S}$ such that $Q \in \delta'(P, a)$ if
  - for all universal states $p \in P \cap S_\forall$ we have that $\delta(p, a) \subseteq Q$ and
  - for all existential states $p \in P \cap S_\exists$ we have that $\delta(p, a) \cap Q \neq \emptyset$,
- $s_0' = \{s_0\}$, and
- $F' \subseteq S'$ is such that $F' = 2^F \setminus \emptyset$.

Theorem (ε-free AFA = NFA)

$L(\mathcal{A}) = L(Det(\mathcal{A}))$.