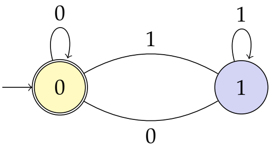
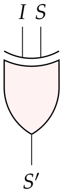
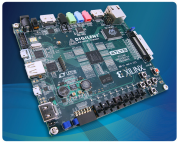
CS 226: Digital Logic Design

Lecture 2: Binary Numbers

Ashutosh Trivedi







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Number-Base Conversions

Binary Arithmetic

Recap: Decimal Numbers

- **digits** $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

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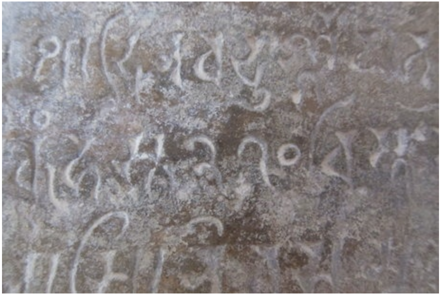
- Place-value system



The number 270 from a 9th century inscription in Gwalior, India [source]

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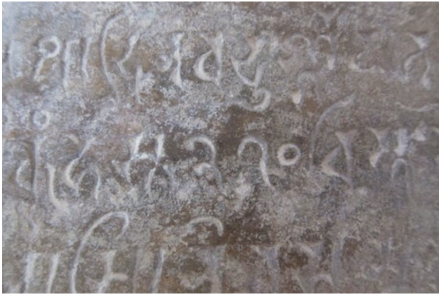


The number 270 from a 9th century inscription in Gwalior, India [source]

- Examples: 270, and 7392, and 7392.56.

Place-Value System

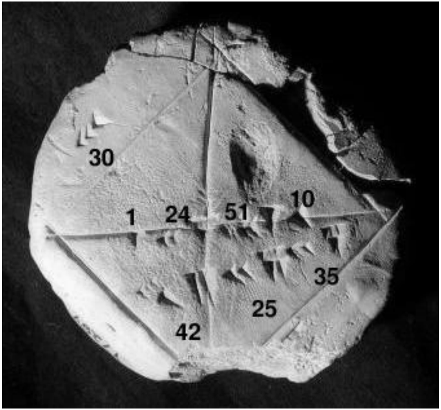
$$\begin{aligned} 7392.56 &= 7 * 1000 + 3 * 100 + 9 * 10 + 2 * 1 + 5 * \frac{1}{10} + 6 * \frac{1}{100} \\ &= 7 * 10^3 + 3 * 10^2 + 9 * 10^1 + 2 * 10^0 + 5 * 10^{-1} + 6 * 10^{-2}. \end{aligned}$$

Discussion:

- Is there something special about having 10 digits?
- Can we define arbitrary large numbers using fewer or more digits?
- Examples:
  1. **binary-digits** $= \{0, 1\}$
  2. **octal-digits** $= \{0, 1, 2, 3, 4, 5, 6, 7\}$
  3. **hexadecimal-digits** $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$
  4. **Sexagesimal-digits** [1]

[1] Used as early as 3000 BC by Babylonians!

Babylonian clay tablet YBC 7289



Babylonian clay tablet YBC 7289 with annotations. The diagonal displays an approximation of the square root of 2 in four sexagesimal figures, 1 24 51 10, which is good to about six decimal digits. [source]

Base-r Systems

Let the digits of a base-r system be $\mathcal{B} = \{0, 1, 2, \ldots, r-1\}$.
A base-r number

$$(a_n a_{n-1} \cdots a_0 . a_{-1} a_{-2} \cdots a_{-m})_r$$

where $a_i \in \mathcal{B}$ is equal to decimal number:

$$a_n * r^n + a_{n-1} * r^{n-1} + \cdots + a_1 * r + a_0 + a_{-1} r^{-1} + a_{-2} * r^{-2} + \cdots + a_{-m} * r^{-m}.$$

The following number-systems are important for this course.

1. Decimal System with **decimal-digits** $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
2. Binary System with **binary-digits** $= \{0, 1\}$
3. Octal System with **octal-digits** $= \{0, 1, 2, 3, 4, 5, 6, 7\}$
4. Hexadecimal System with **hexadecimal-digits** $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}$

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Let's convert various numbers in different bases to decimal.

- $(4021.2)_5$
- $(123.4)_8$
- $(B44B)_16$
- $(110101)_2$

How to do the converse?

Question: Given a number in Decimal convert it into base-r.

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Examples:

– What is 11 in binary?

$$\begin{aligned} 11 &= (10+1) \\ &= ((5*2)+1) \\ &= (((2*2+1)*2)+1) \end{aligned}$$

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Question: Given a number in Decimal convert it into base-r.
Examples:

– What is 11 in binary?

$$
\begin{aligned}
11 &= (10 + 1) \\
&= ((5 * 2) + 1) \\
&= (((2 * 2 + 1) * 2) + 1) \\
&= ((((1 * 2) * 2 + 1) * 2) + 1)
\end{aligned}
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&= (((((0 * 2 + 1) * 2) * 2 + 1) * 2) + 1) \\
&= 1 * 2^3 + 1 * 2^1 + 1 \\
&= (1011)_2.
\end{aligned}
$$

- What is 111 in octal?
- General algorithm?

How to do the converse?

Examples:

- What is 0.6875 in binary?

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– What is $(0.513)_{10}$ in octal?

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&= \cdots \\
&= \frac{1}{2}(1 + \frac{1}{2}(0 + \frac{1}{2}(1 + \frac{1}{2}(1 + 0)))) \\
&= (0.1011)_2.
\end{aligned}
$$

- What is $(0.513)_{10}$ in octal?
- What is $(153.513)_1 0$ in octal?
- General algorithm?

Octal and Hexadecimal Numbers

Decimal	Binary	Octal	Hexadecimal
00	0000	00	0
01	0001	01	1
02	0010	02	2
03	0011	03	3
04	0100	04	4
05	0101	05	5
06	0110	06	6
07	0111	07	7
08	1000	10	8
09	1001	11	9
10	1010	12	A = 10
11	1011	13	B = 11
12	1100	14	C = 12
13	1101	15	D = 13
14	1110	16	E = 14
15	1111	17	F = 15

– Notice that $2^3 = 8$ and $2^4 = 16$.
– Converting between Octal and Binary, and Hex and Binary. Examples.

Number-Base Conversions

Binary Arithmetic

Let's generalize Decimal Arithmetic

- Addition
  - What do you need to remember?
  - What is the algorithm?
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- Multiplication
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- Multiplication
  - What do you need to remember?
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- Division
  - What do you need to remember?
  - What is the algorithm?
  - How to extend that in Binary?