Theoretical Abstractions in Data Flow Analysis

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Part 1

About These Slides

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These slides constitute the lecture notes for CS618 Program Analysis course at IIT Bombay and have been made available as teaching material accompanying the book:

- Uday Khedker, Amitabha Sanyal, and Bageshri Karkare. *Data Flow Analysis: Theory and Practice*. CRC Press (Taylor and Francis Group). 2009.

  (Indian edition published by Ane Books in 2013)

Apart from the above book, some slides are based on the material from the following books

- M. S. Hecht. *Flow Analysis of Computer Programs*. Elsevier North-Holland Inc. 1977.
- F. Nielson, H. R. Nielson, and C. Hankin. *Principles of Program Analysis*. Springer-Verlag. 1998.

Outline

- The need for a more general setting
- The set of data flow values
- The set of flow functions
- Solutions of data flow analyses
- Algorithms for performing data flow analysis
- Complexity of data flow analysis

Part 2

The Need for a More General Setting

What We Have Seen So Far . . .

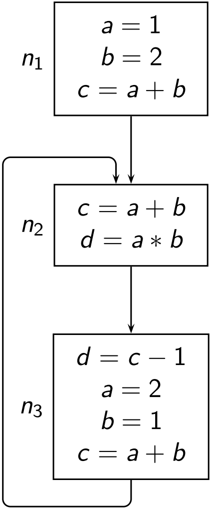
Analysis	Entity	Attribute at p	Paths	
Live variables	Variables	Use	Starting at p	Some
Available expressions	Expressions	Availability	Reaching p	All
Partially available expressions	Expressions	Availability	Reaching p	Some
Anticipable expressions	Expressions	Use	Starting at p	All
Reaching definitions	Definitions	Availability	Reaching p	Some
Partial redundancy elimination	Expressions	Profitable hoistability	Involving p	All

The Need for a More General Setting

- We seem to have covered many variations
- Yet there are analyses that do not fit the same mould of bit vector frameworks
- We use an analysis called *Constant Propagation* to observe the differences

A variable v is a constant with value c at program point p if in every execution instance of p, the value of v is c

An Introduction to Constant Propagation



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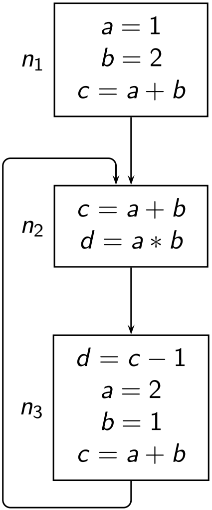

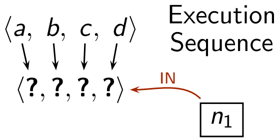

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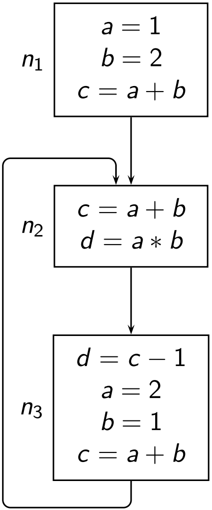

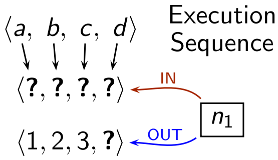

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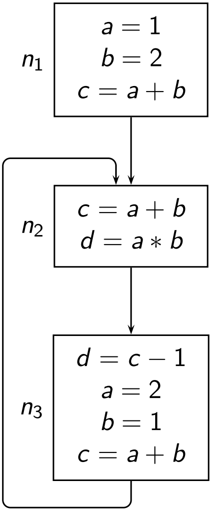

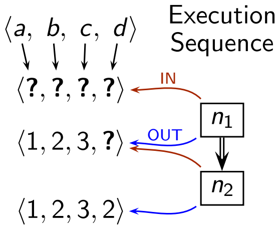

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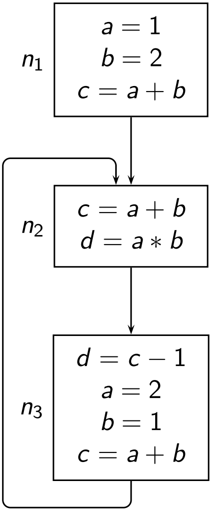

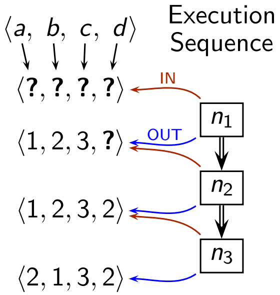

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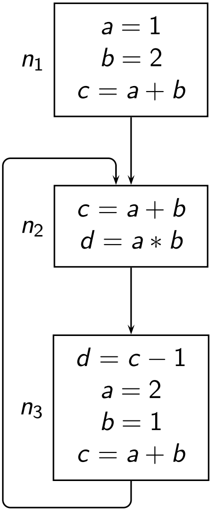

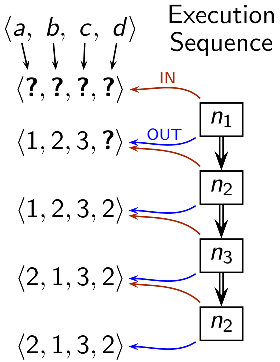

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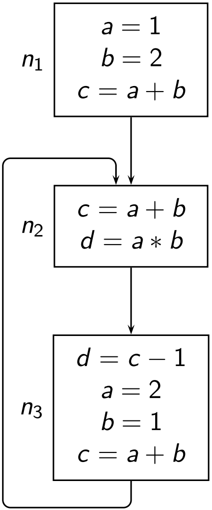

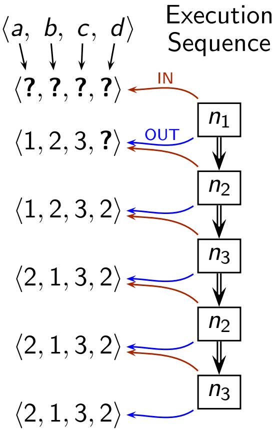

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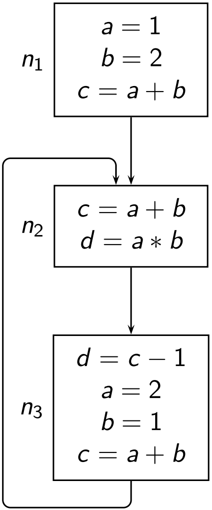

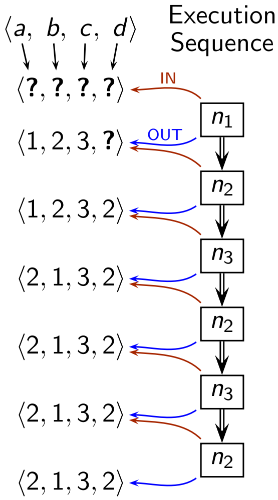

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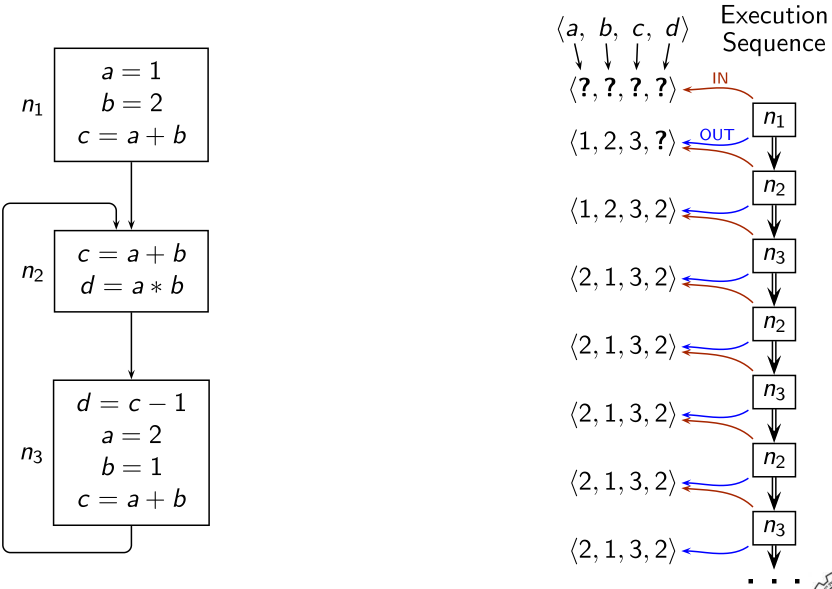

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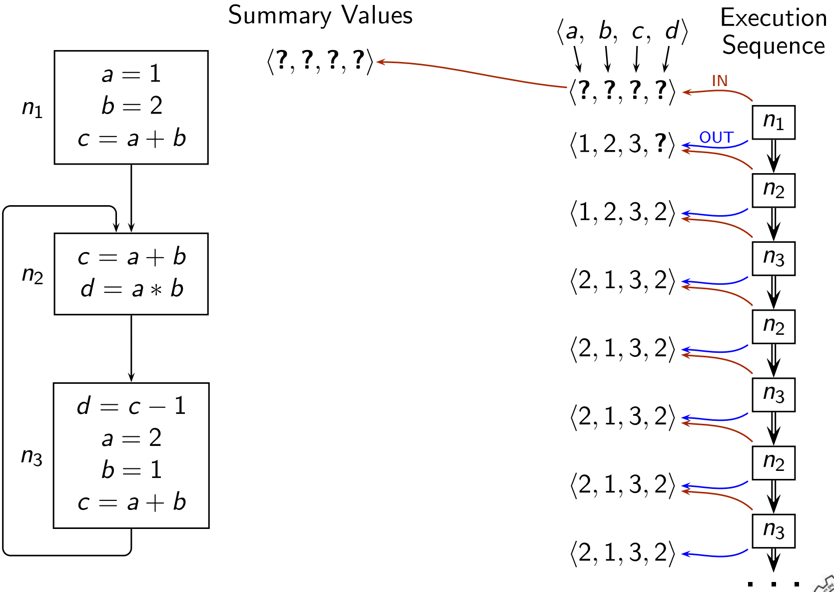

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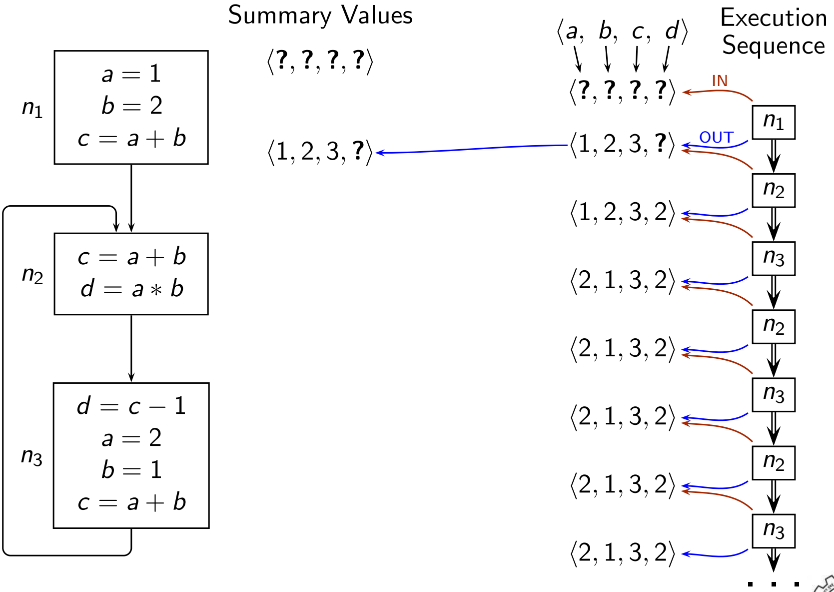

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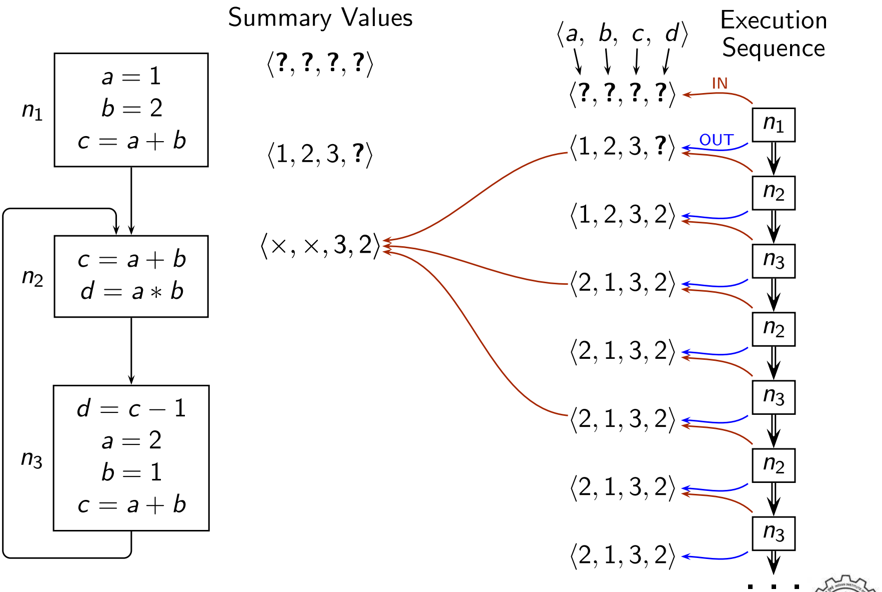

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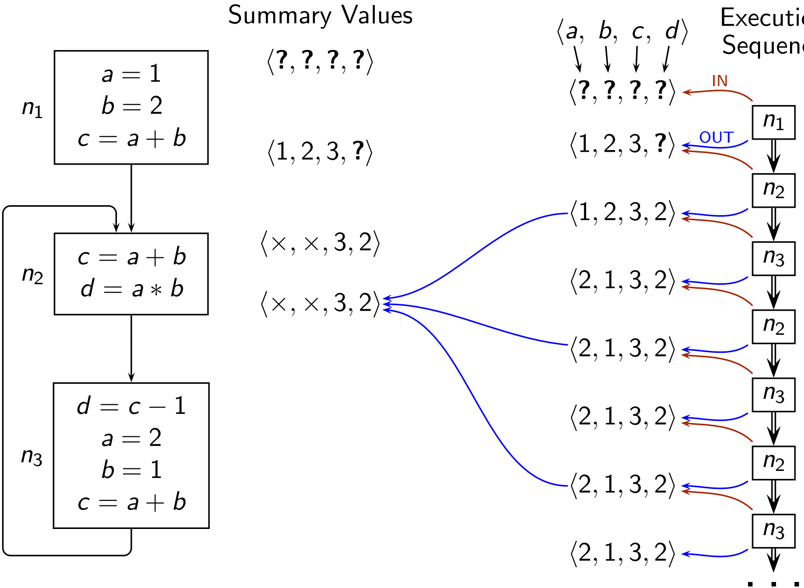

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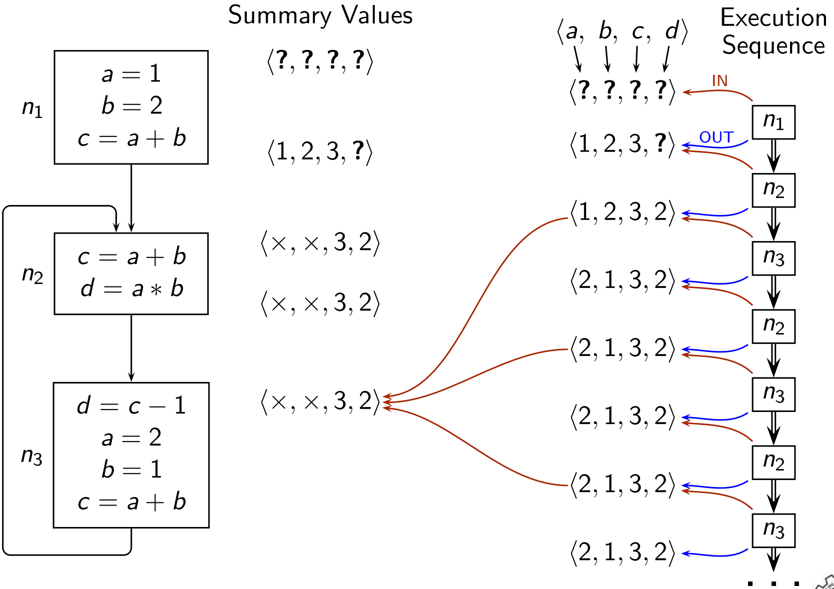

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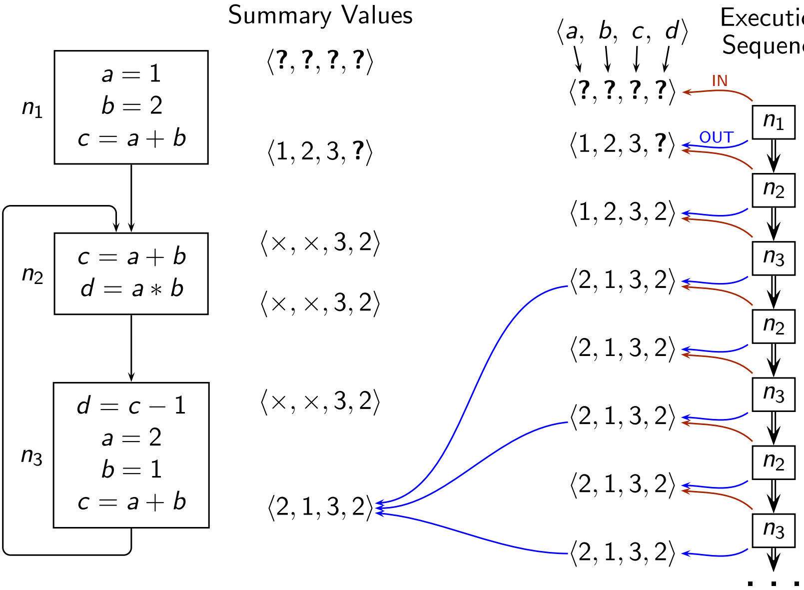
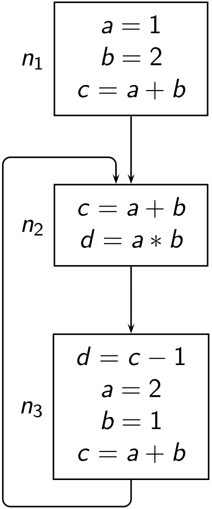


An Introduction to Constant Propagation

Summary Values



$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$

$\langle 1, 2, 3, \mathbf{?} \rangle$

$\langle \times, \times, 3, 2 \rangle$

$\langle \times, \times, 3, 2 \rangle$

$\langle \times, \times, 3, 2 \rangle$

$\langle 2, 1, 3, 2 \rangle$

Desired Solution

Difference #1: Data Flow Values

- Tuples of the form $\langle \eta_1, \eta_2, \ldots, \eta_k \rangle$ where η_i is the data flow value for i^{th} variable

  Unlike bit vector frameworks, value η_i is not 0 or 1 (i.e. true or false). Instead, it is one of the following:

  - **?** indicating that not much is known about the constantness of variable v_i
  - $\times$ indicating that variable v_i does not have a constant value
  - An integer constant c_1 if the value of v_i is known to be c_1 at compile time

Difference #2: Dependence of Data Flow Values Across Entities

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- Given a statement $a = b * c$, can the constantness of a be determined independently of the constantness of b and c?

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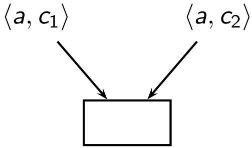
- Given a statement $a = b * c$, can the constantness of a be determined independently of the constantness of b and c?

No

Difference #3: Confluence Operation

- Confluence operation $\langle a, c_1 \rangle \sqcap \langle a, c_2 \rangle$



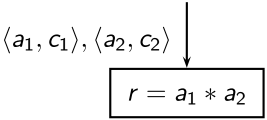
$\sqcap$	$\langle a, \textbf{?} \rangle$	$\langle a, \times \rangle$	$\langle a, c_1 \rangle$
$\langle a, \textbf{?} \rangle$	$\langle a, \textbf{?} \rangle$	$\langle a, \times \rangle$	$\langle a, c_1 \rangle$
$\langle a, \times \rangle$	$\langle a, \times \rangle$	$\langle a, \times \rangle$	$\langle a, \times \rangle$
$\langle a, c_2 \rangle$	$\langle a, c_2 \rangle$	$\langle a, \times \rangle$	If $c_1 = c_2$ $\langle a, c_1 \rangle$ Otherwise $\langle a, \times \rangle$

- This is neither $\cap$ nor $\cup$

  What are its properties?

Difference #4: Flow Functions for Constant Propagation

- Flow function for $r = a_1 * a_2$



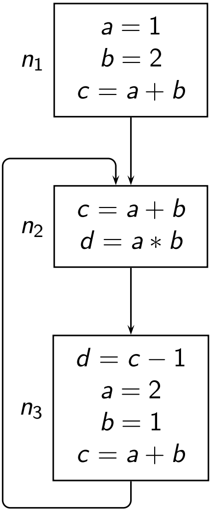
mult	$\langle a_1, \mathbf{?} \rangle$	$\langle a_1, \times \rangle$	$\langle a_1, c_1 \rangle$
$\langle a_2, \mathbf{?} \rangle$	$\langle r, \mathbf{?} \rangle$	$\langle r, \times \rangle$	$\langle r, \mathbf{?} \rangle$
$\langle a_2, \times \rangle$	$\langle r, \times \rangle$	$\langle r, \times \rangle$	$\langle r, \times \rangle$
$\langle a_2, c_2 \rangle$	$\langle r, \mathbf{?} \rangle$	$\langle r, \times \rangle$	$\langle r, (c_1 * c_2) \rangle$

- This cannot be expressed in the form

$$f_n(X) = \text{Gen}_n \cup (X - \text{Kill}_n)$$

where Gen_n and Kill_n are constant effects of block n

Difference #5: Solution Computed by Iterative Method



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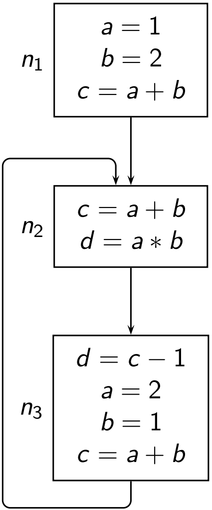

Iteration #1

$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$

$\langle 1, 2, 3, \mathbf{?} \rangle$

$\langle 1, 2, 3, \mathbf{?} \rangle$

$\langle 1, 2, 3, 2 \rangle$

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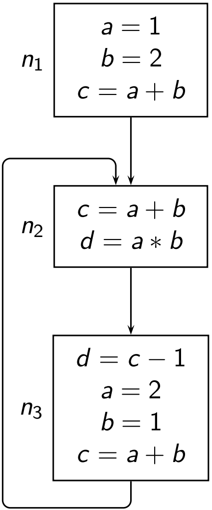
$\langle 2, 1, 3, 2 \rangle$

Difference #5: Solution Computed by Iterative Method



Iteration #1	Iteration #2
$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$	$\langle \mathbf{?}, \mathbf{?}, \mathbf{?}, \mathbf{?} \rangle$
$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle 1, 2, 3, \mathbf{?} \rangle$
$\langle 1, 2, 3, \mathbf{?} \rangle$	$\langle \times, \times, 3, 2 \rangle$
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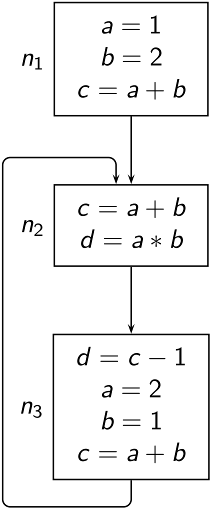

Iteration #1	Iteration #2	Iteration #3
$\langle \textbf{?}, \textbf{?}, \textbf{?}, \textbf{?} \rangle$	$\langle \textbf{?}, \textbf{?}, \textbf{?}, \textbf{?} \rangle$	$\langle \textbf{?}, \textbf{?}, \textbf{?}, \textbf{?} \rangle$
$\langle 1, 2, 3, \textbf{?} \rangle$	$\langle 1, 2, 3, \textbf{?} \rangle$	$\langle 1, 2, 3, \textbf{?} \rangle$
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$\langle 1, 2, 3, 2 \rangle$	$\langle \times, \times, \times, \times \rangle$	$\langle \times, \times, \times, \times \rangle$
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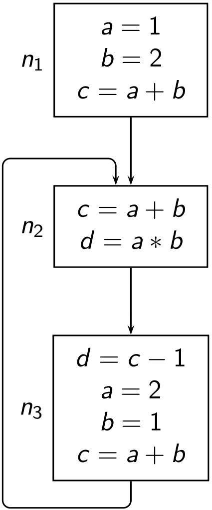

Iteration #1	Iteration #2	Iteration #3	Desired solution
$\langle ?, ?, ?, ? \rangle$	$\langle ?, ?, ?, ? \rangle$	$\langle ?, ?, ?, ? \rangle$	$\langle ?, ?, ?, ? \rangle$
$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$
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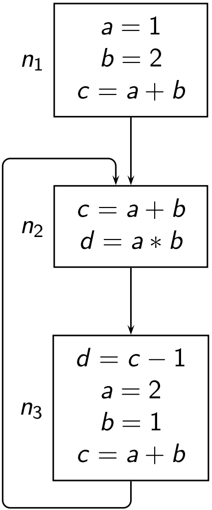

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$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$	$\langle 1, 2, 3, ? \rangle$
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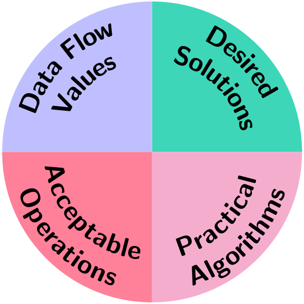
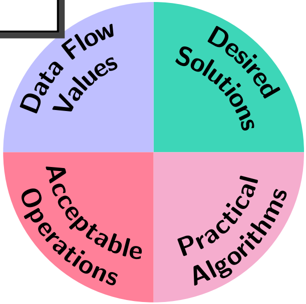
Issues in Data Flow Analysis



Issues in Data Flow Analysis

- Representation
- Approximation: Partial Order, Lattices



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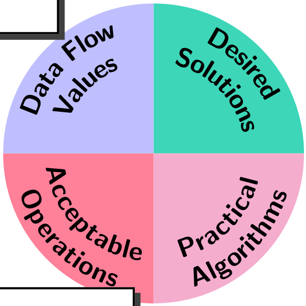
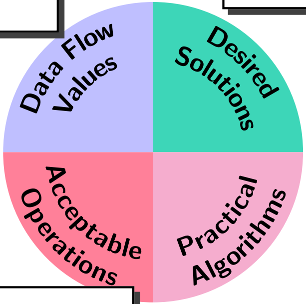


- Merge: Commutativity, Associativity, Idempotence
- Flow Functions: Monotonicity, Distributivity, Boundedness, Separability

Issues in Data Flow Analysis

- Representation
- Approximation: Partial Order, Lattices

- Existence, Computability
- Soundness, Precision



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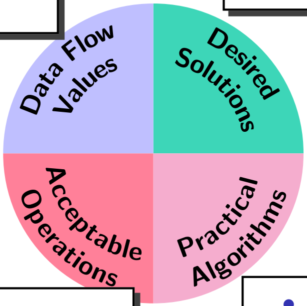


- Merge: Commutativity, Associativity, Idempotence
- Flow Functions: Monotonicity, Distributivity, Boundedness, Separability

- Complexity, efficiency
- Convergence
- Initialization

Part 3

Data Flow Values: An Overview

Data Flow Values: An Outline of Our Discussion

- The need to define the notion of abstraction
- Lattices, variants of lattices
- Relevance of lattices for data flow analysis
  - Partial order relation as approximation of data flow values
  - Meet operations as confluence of data flow values
- Constructing lattices
- Example of lattices

Part 4

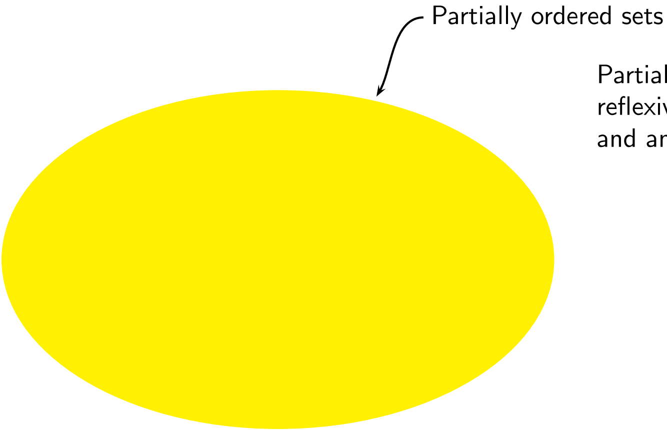
A Digression on Lattices

Partially Ordered Sets

Sets in which elements can be compared and ordered

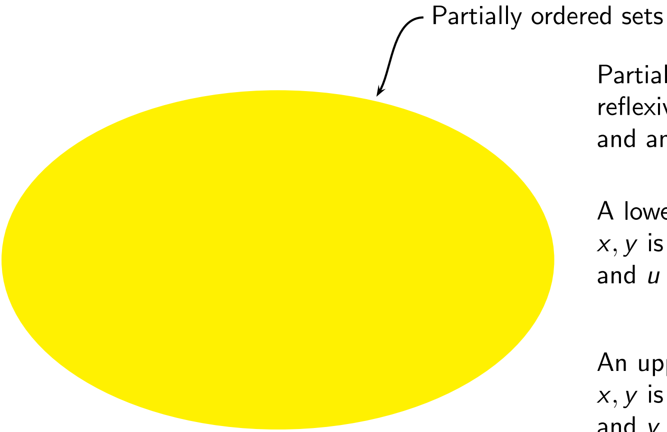
- *Total order*. Every element in comparable with every element (including itself)
- *Discrete order*. Every element is comparable only with itself but not with any other element
- *Partial order*. An element is comparable with some but not necessarily all elements

Partially Ordered Sets and Lattices



Partial order $\sqsubseteq$ is reflexive, transitive, and antisymmetric

Partially Ordered Sets and Lattices

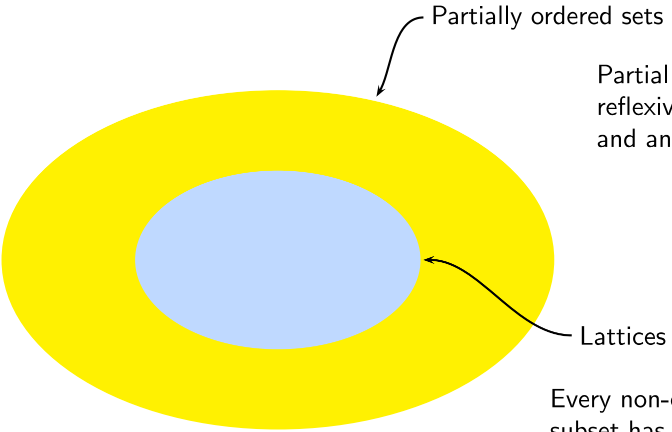


Partial order $\sqsubseteq$ is reflexive, transitive, and antisymmetric

A lower bound of x, y is u s.t. $u \sqsubseteq x$ and $u \sqsubseteq y$

An upper bound of x, y is u s.t. $x \sqsubseteq u$ and $y \sqsubseteq u$

Partially Ordered Sets and Lattices



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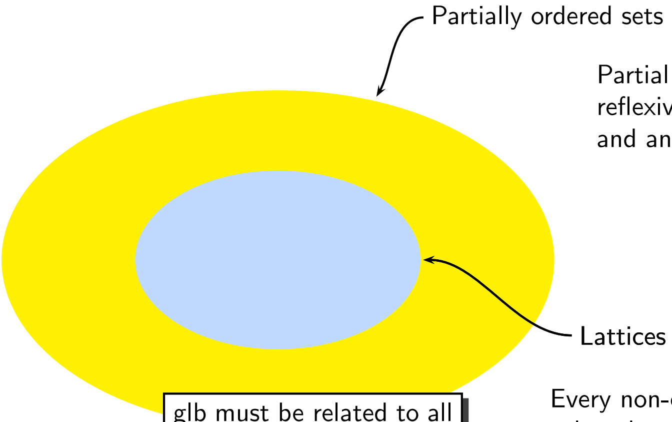
Every non-empty finite subset has a greatest lower bound (glb) and a least upper bound (lub)

Partially Ordered Sets and Lattices



Partial order $\sqsubseteq$ is reflexive, transitive, and antisymmetric

Every non-empty finite subset has a greatest lower bound (glb) and a least upper bound (lub)

glb must be related to all other lower bounds. Hence it must be unique

Partially Ordered Sets

Set $\{1, 2, 3, 4, 6, 9, 12\}$ with $\sqsubseteq$ relation as "divides" (i.e. $a \sqsubseteq b$ iff a divides b)

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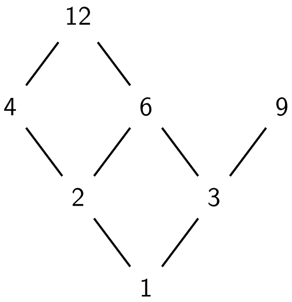

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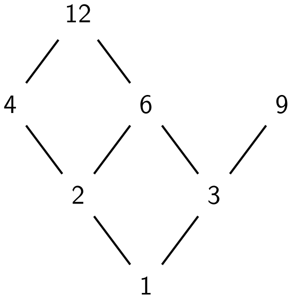
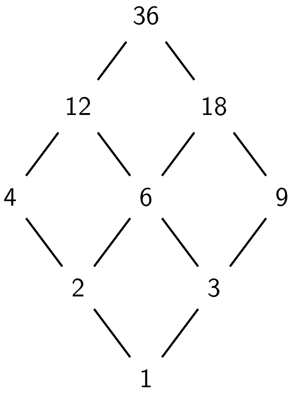


Subset $\{4, 9, 6\}$ and $\{12, 9\}$ do not have an upper bound in the set

Lattice

Set $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ with $\sqsubseteq$ relation as "divides"



Complete Lattice

- Lattice: A partially ordered set such that every non-empty finite subset has a glb and a lub

  Example: Lattice $\mathbb{Z}$ of integers under "less-than-equal-to" ($\leq$) relation

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  - ∞ is the top element denoted $\top$: $\forall i \in \mathbb{Z},\ i \leq \top$
  - $-\infty$ is the bottom element denoted $\bot$: $\forall i \in \mathbb{Z},\ \bot \leq i$

$\mathbb{Z} \cup \{\infty, -\infty\}$ is a Complete Lattice

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    OR

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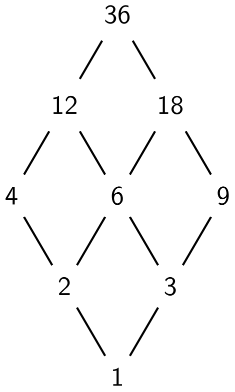
  - lub($\emptyset$) is $\bot$

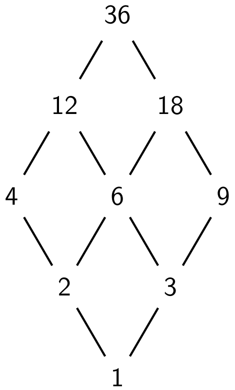
Operations on Lattices

- Meet ($\sqcap$) and Join ($\sqcup$)



Operations on Lattices

- Meet ($\sqcap$) and Join ($\sqcup$)
  - $x \sqcap y$ computes the glb of x and y
    $z = x \sqcap y \Rightarrow z \sqsubseteq x \wedge z \sqsubseteq y$



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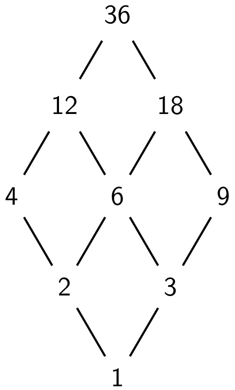

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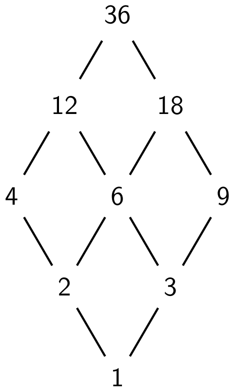

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$$
\begin{aligned}
\forall x \in L,\ x \sqcap \top &= x \\
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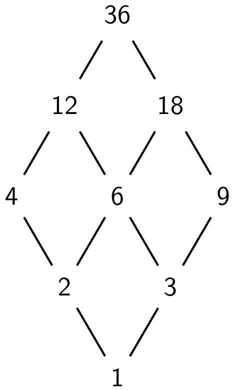
$$



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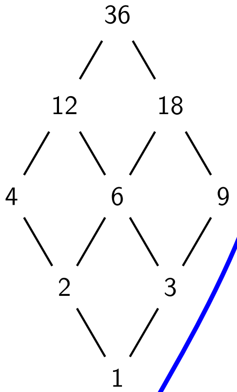
Greatest common divisor



$x \sqcap y = gcd(x, y)$

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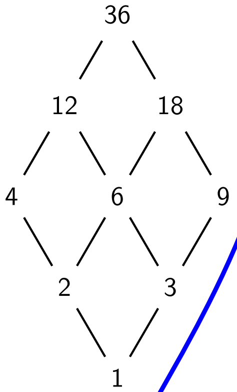


$x \sqcap y = gcd(x, y)$

Lowest common multiple

$x \sqcup y = lcm(x, y)$

Partial Order and Operations

- For a lattice $\sqsubseteq$ induces $\sqcap$ and $\sqcup$ and vice-versa
- The choices of $\sqsubseteq$, $\sqcap$, and $\sqcup$ cannot be arbitrary

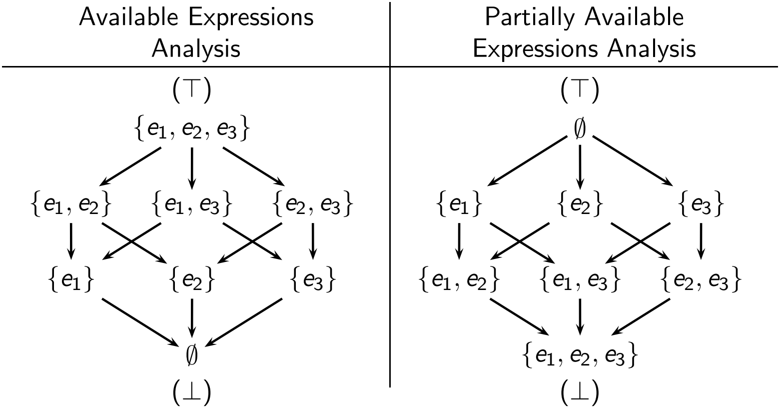
  They have to be

  - consistent with each other, and
  - definable in terms of each other
- For some variants of lattices, $\sqcap$ or $\sqcup$ may not exist

  Yet the requirement of its consistency with $\sqsubseteq$ cannot be violated

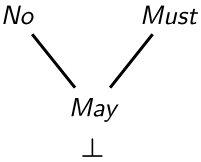
Finite Lattices are Complete

- Any given set of elements has a glb and a lub



Lattice for May-Must Analysis

- There is no ⊤ among the natural values



Interpreting data flow values

- *No.* Information does not hold along any path
- *Must.* Information must hold along all paths
- *May.* Information may hold along some path

- An artificial ⊤ can be added

Some Variants of Lattices

A poset L is

- A lattice iff each non-empty finite subset of L has a glb and lub
- A complete lattice iff each subset of L has a glb and lub
- A meet semilattice iff each non-empty finite subset of L has a glb
- A join semilattice iff each non-empty finite subset of L has a lub
- A bounded lattice iff L is a lattice and has $\top$ and $\bot$ elements

A Bounded Lattice Need Not be Complete (1)

- Let A be all finite subsets of $\mathbb{Z}$
  Then, A is an infinite set
- The poset $L = (A \cup \{\mathbb{Z}\}, \subseteq)$ is a bounded lattice with $\top = \mathbb{Z}$ and $\bot = \emptyset$
  The join $\sqcup$ of this lattice is $\cup$
- To see why, consider a set S containing those subsets of L that do not contain the number 1
  There are two possibilities:
  - S contains only a finite number of sets that not contain 1 (say S_f)
    $\Rightarrow S_f$ is a finite set
  - S contains *all* finite sets that do not contain 1 (say S_∞)
    $\Rightarrow S_\infty$ is a infinite set

A Bounded Lattice Need Not be Complete (2)

- S_f contains only a finite number of sets each of which does not contain 1
  - The union of all its member sets is a finite set excluding 1
  - Thus S_f has a lub in L

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- S_∞ contains *all* finite sets that do not contain 1
  - Since the number of such sets is infinite, their union is an infinite set
  - $\mathbb{Z} - \{1\}$ is not contained in L (the only infinite set in L is $\mathbb{Z}$)
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Hence L is not complete

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- It may be tempting to assume that $\mathbb{Z}$ is the lub of S_∞ because it is an upper bound of S_∞ and no other upper bound of S_∞ in the lattice is weaker $\mathbb{Z}$
- However, the join operation $\cup$ of L does not compute $\mathbb{Z}$ as the lub of S_∞ (because it must exclude 1)
- The join operation $\cup$ is inconsistent with the partial order $\supseteq$ of L. Hence we say that join does not exist for S_∞

A Bounded Lattice Need Not be Complete (2)

- A bounded lattice L has a glb and lub of L in L
- A complete lattice L should have glb and lub of *all* subsets of L
- A lattice L should have glb and lub of *all* finite non-empty subsets of L

Ascending and Descending Chains

- Strictly ascending chain $x \sqsubset y \sqsubset \cdots \sqsubset z$
- Strictly descending chain $x \sqsupset y \sqsupset \cdots \sqsupset z$
- DCC: Descending Chain Condition
  All strictly descending chains are finite
- ACC: Ascending Chain Condition
  All strictly ascending chains are finite

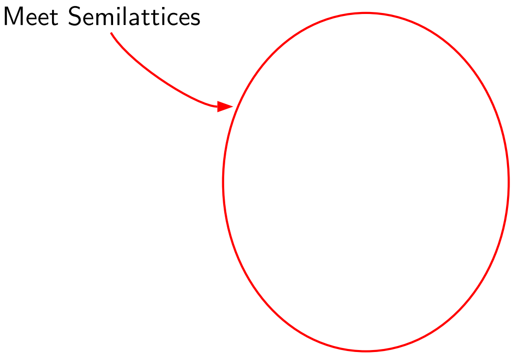
Complete Lattice and Ascending and Descending Chains

- If L satisfies acc and dcc, then
  - L has finite height, and
  - L is complete
- A complete lattice need not have finite height (i.e. strict chains may not be finite)

  Example:

  Lattice of integers under $\leq$ relation with ∞ as $\top$ and $-\infty$ as $\bot$

Variants of Lattices



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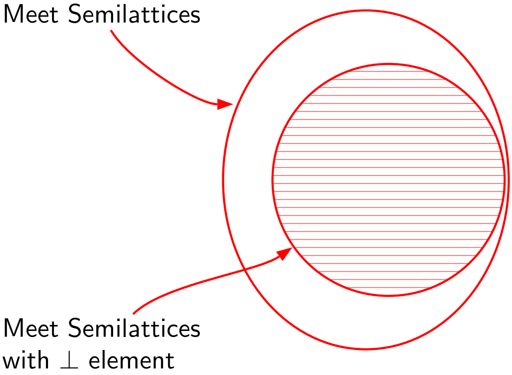

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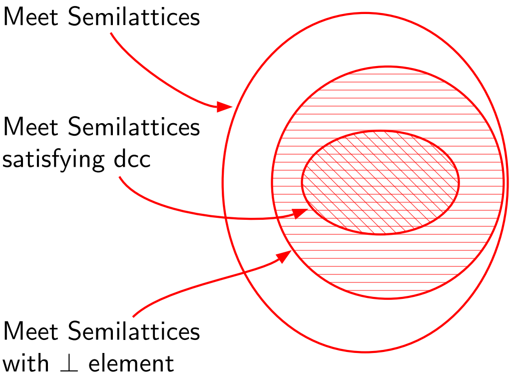

- dcc: descending chain condition

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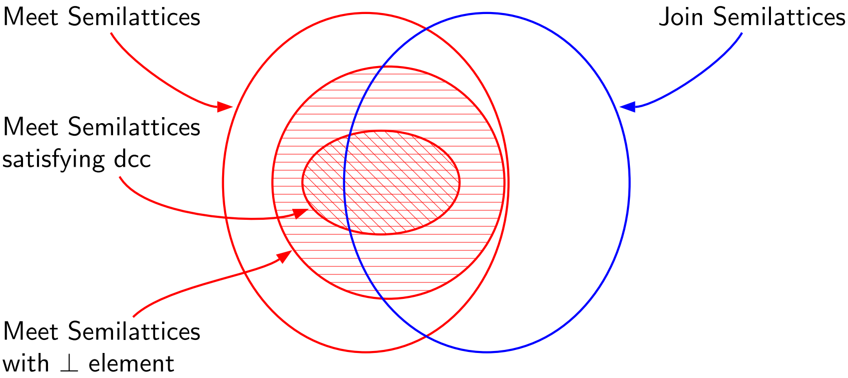

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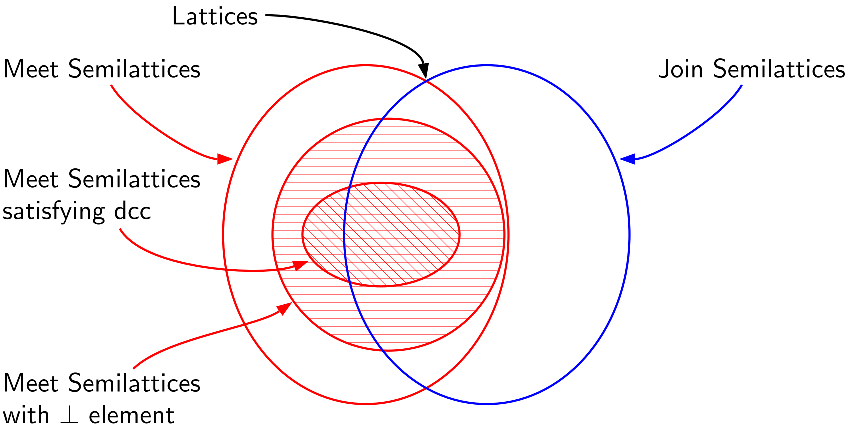

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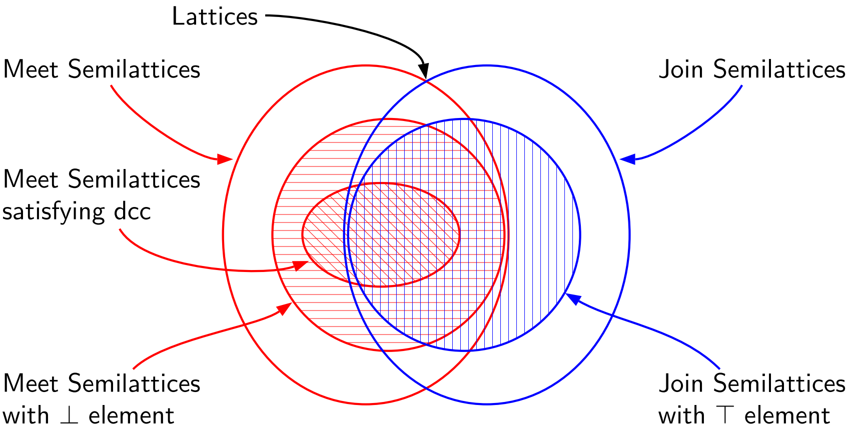

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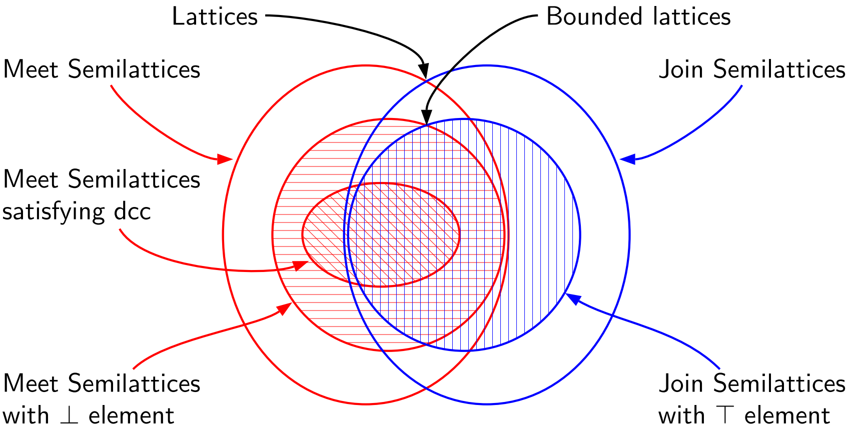

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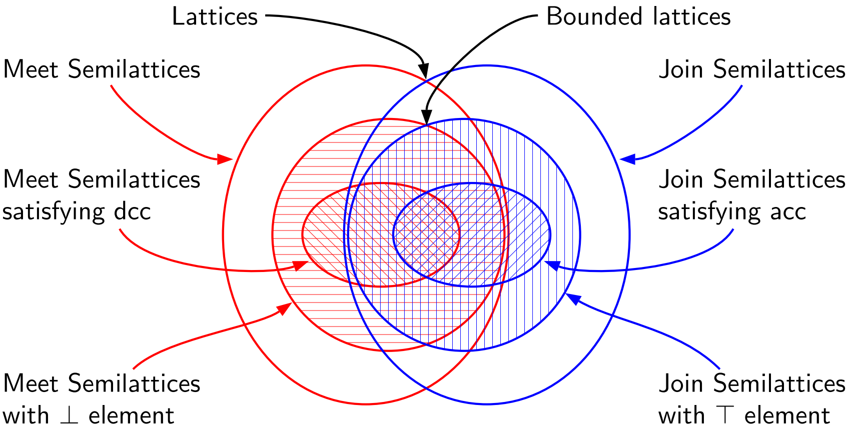

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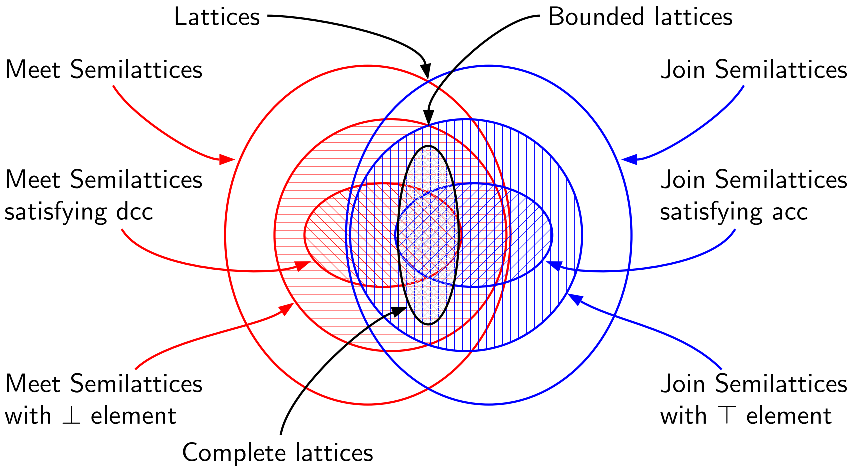
Variants of Lattices



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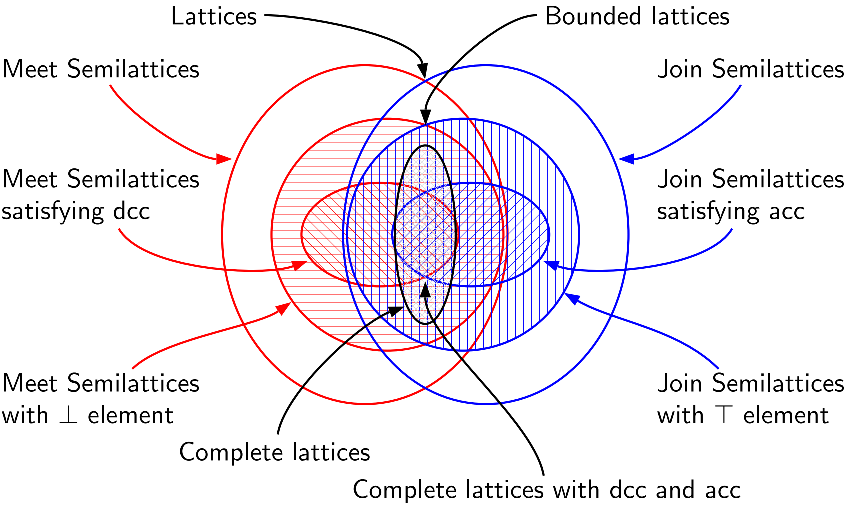
Variants of Lattices



- dcc: descending chain condition
- acc: ascending chain condition

An Example of Lattices: Maintaining Like Counts on Cloud

Maintain n servers and divide the traffic
– Each server maintains an n-tuple for each page
– Updates the counters for its own slot

	Server Blue			Server Red			Server Green		
Page 1	0	0	0	0	0	0	0	0	0
Page 2	0	0	0	0	0	0	0	0	0
Page 3	0	0	0	0	0	0	0	0	0

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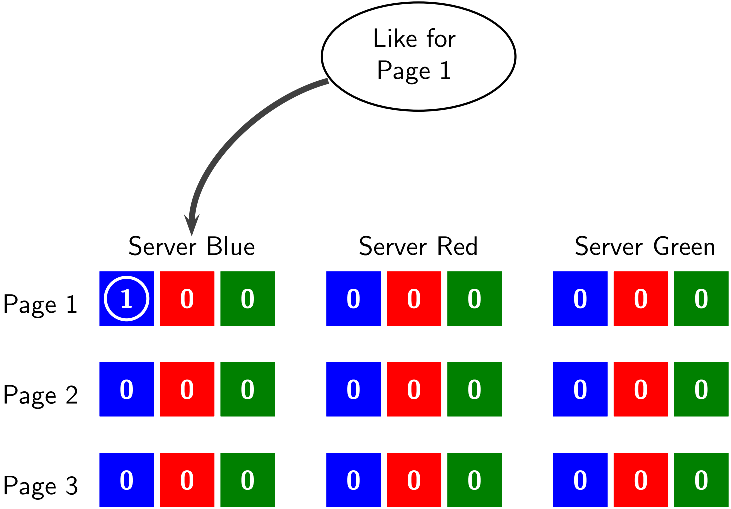

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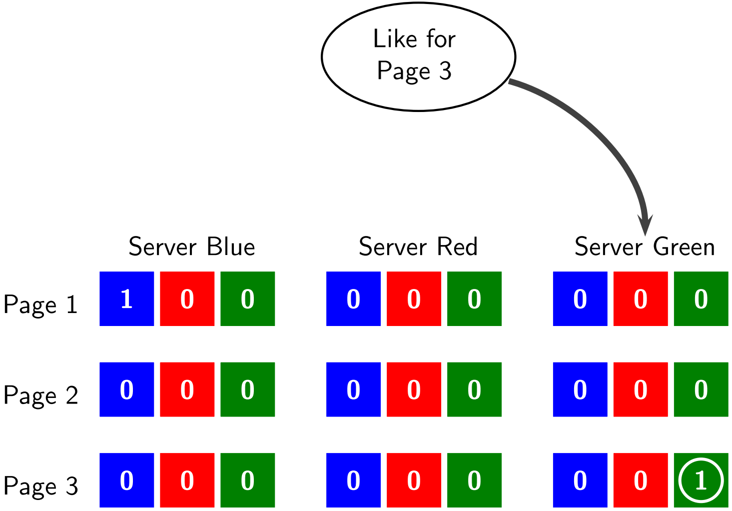

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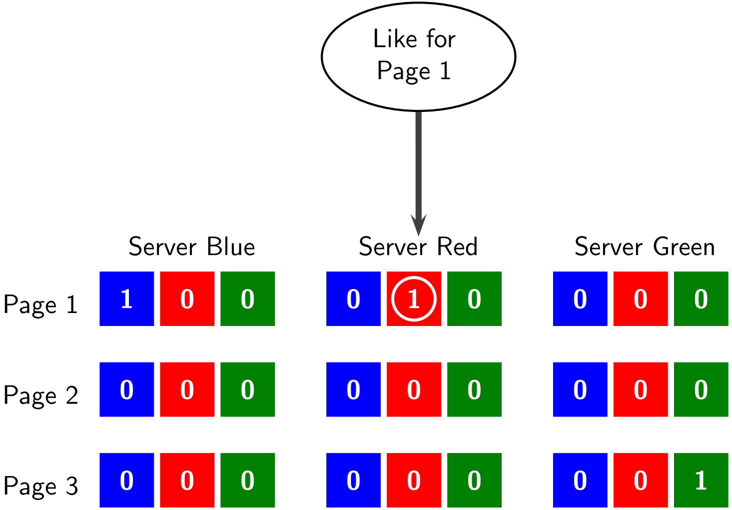

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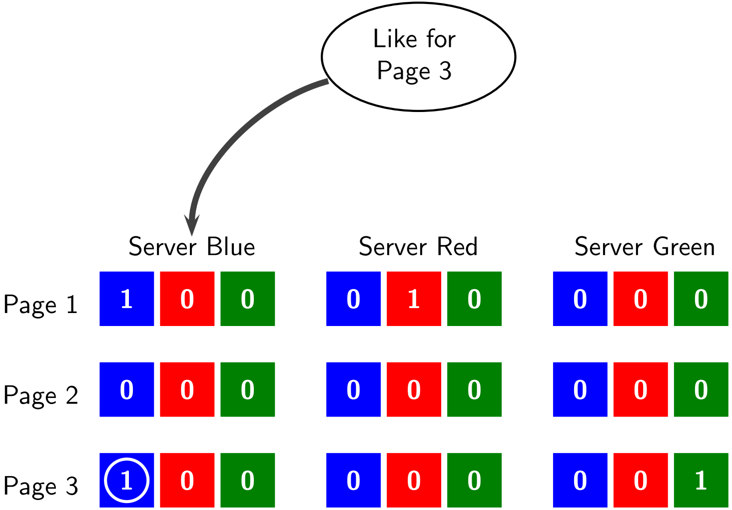

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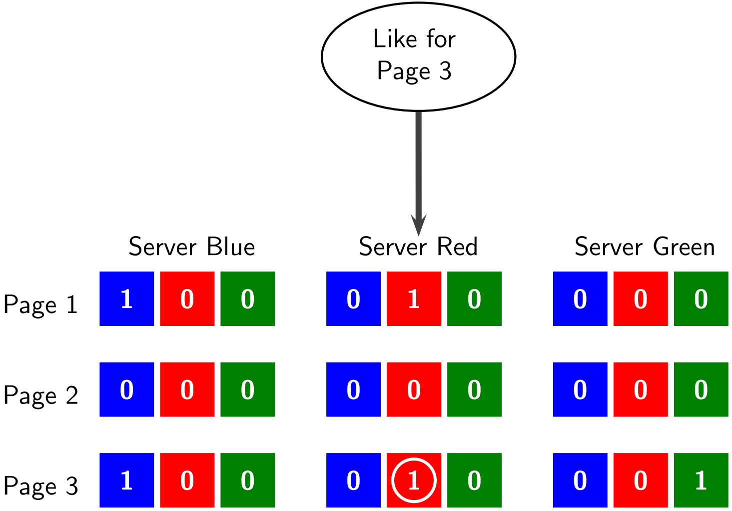

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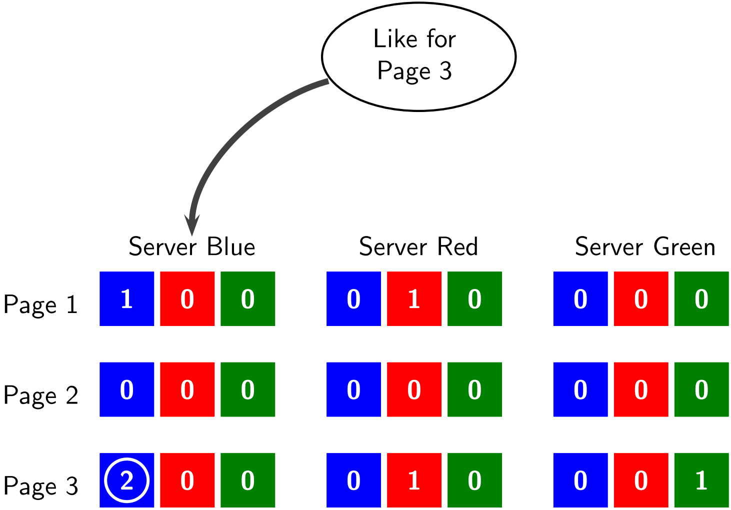

An Example of Lattices: Maintaining Like Counts on Cloud

Synchronize:
– Send the data to other servers
– Update the counters using point-wise max

	Server Blue			Server Red			Server Green		
Page 1	1	0	0	0	1	0	0	0	0
Page 2	0	0	0	0	0	0	0	0	0
Page 3	2	0	0	0	1	0	0	0	1

An Example of Lattices: Maintaining Like Counts on Cloud

Synchronize:
– Send the data to other servers
– Update the counters using point-wise max

- Lattice of n-tuples using point-wise $\geq$ as the partial order

$$\langle x_1, x_2, \ldots, x_n \rangle \sqsubseteq \langle y_1, y_2, \ldots, y_n \rangle = (x_1 \geq y_1) \wedge (x_2 \geq y_2) \ldots \wedge (x_n \geq y_n)$$

- Tuples merged with max operation

$$\langle x_1, x_2, \ldots, x_n \rangle \sqcap \langle y_1, y_2, \ldots, y_n \rangle = \langle \max(x_1, y_1), \max(x_2, y_2), \ldots, \max(x_n, y_n) \rangle$$

Page 3


An Example of Lattices: Maintaining Like Counts on Cloud

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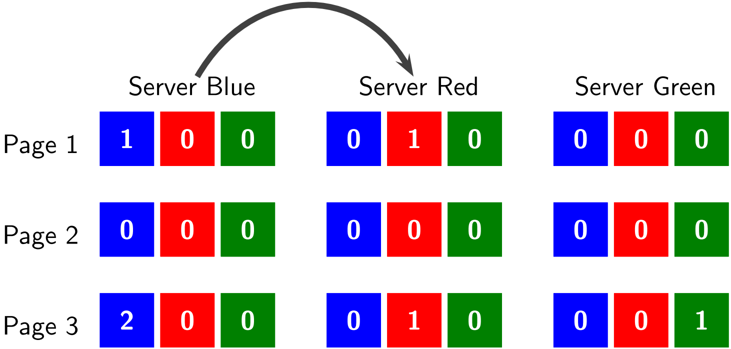

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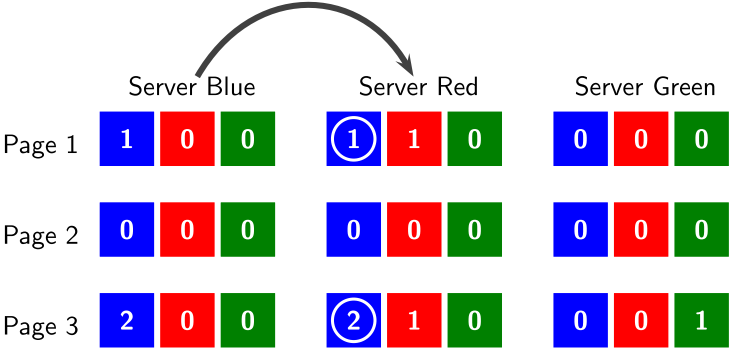

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Page 1	1	0	0	1	1	0	0	0	0
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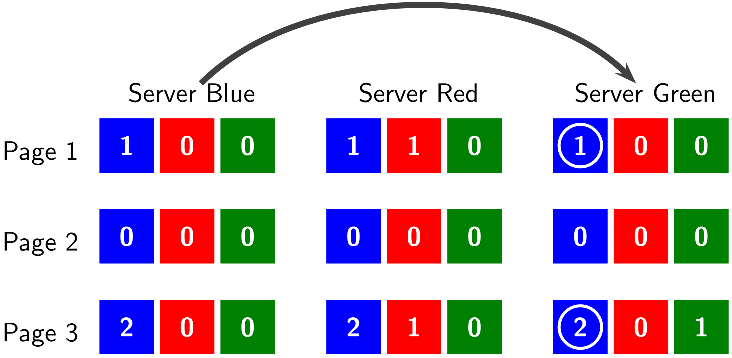

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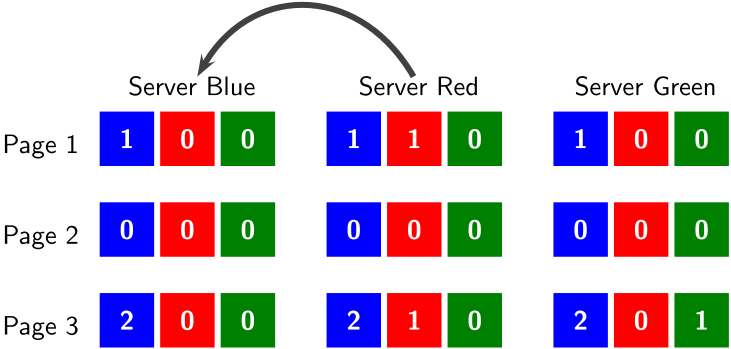

An Example of Lattices: Maintaining Like Counts on Cloud

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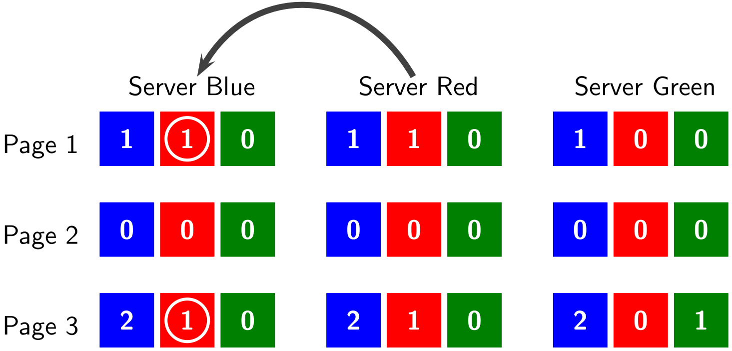

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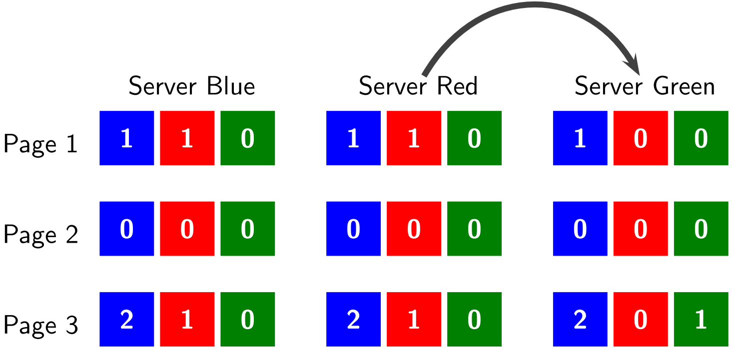

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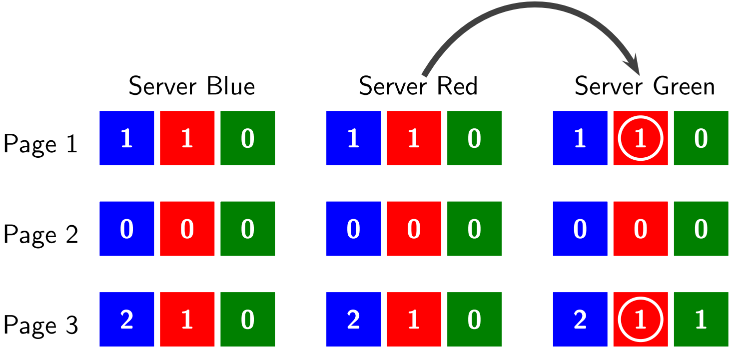

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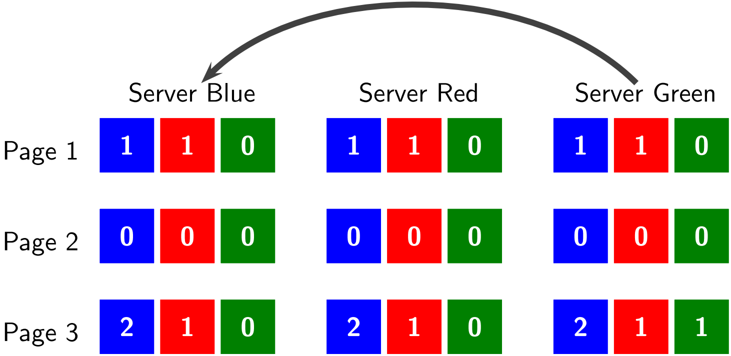

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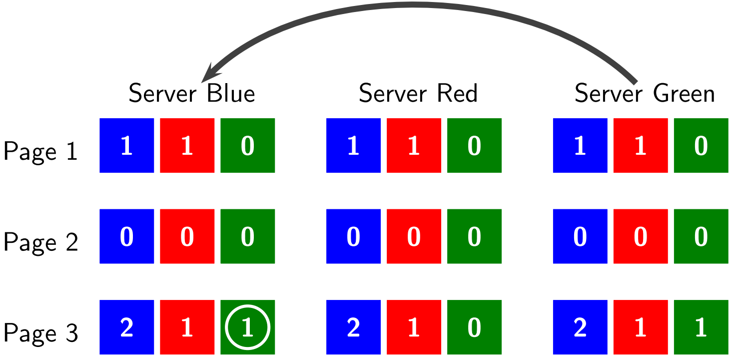

An Example of Lattices: Maintaining Like Counts on Cloud

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	Server Blue			Server Red			Server Green		
Page 1	1	1	0	1	1	0	1	1	0
Page 2	0	0	0	0	0	0	0	0	0
Page 3	2	1	1	2	1	0	2	1	1

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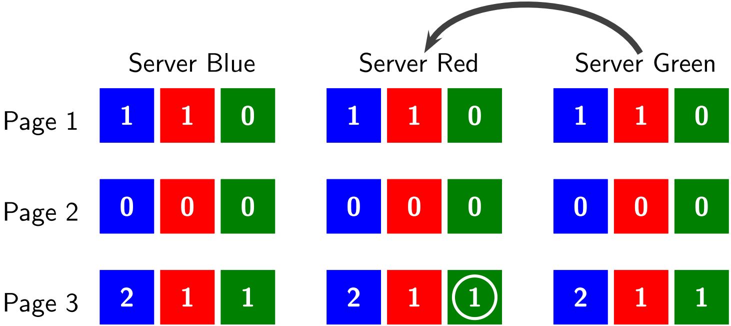

An Example of Lattices: Maintaining Like Counts on Cloud

After synchronization, all servers have the same data
Count for a page:
– Take sum of all counts at any server for the page

	Server Blue			Server Red			Server Green		
Page 1	1	1	0	1	1	0	1	1	0
Page 2	0	0	0	0	0	0	0	0	0
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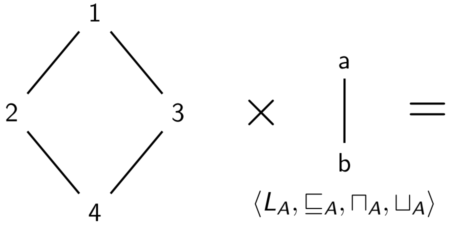
Constructing Lattices

- Powerset construction with subset or superset relation
- Products of lattices
  - Cartesian product
  - Lexicographic product
  - Interval product
  - Set of mappings
- Lattices on sequences using prefix or suffix as partial orders

Cartesian Product of Lattice



$\langle L_N, \sqsubseteq_N, \sqcap_N, \sqcup_N \rangle$ $\times$ $\langle L_A, \sqsubseteq_A, \sqcap_A, \sqcup_A \rangle$ $=$

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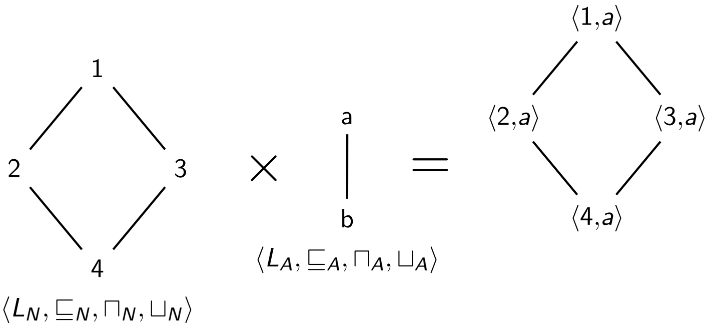

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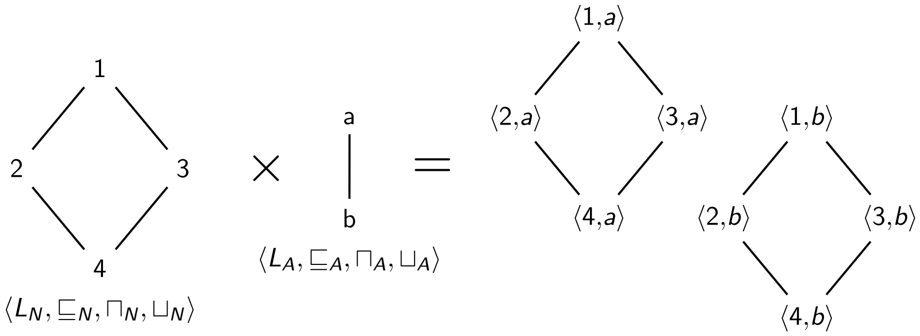

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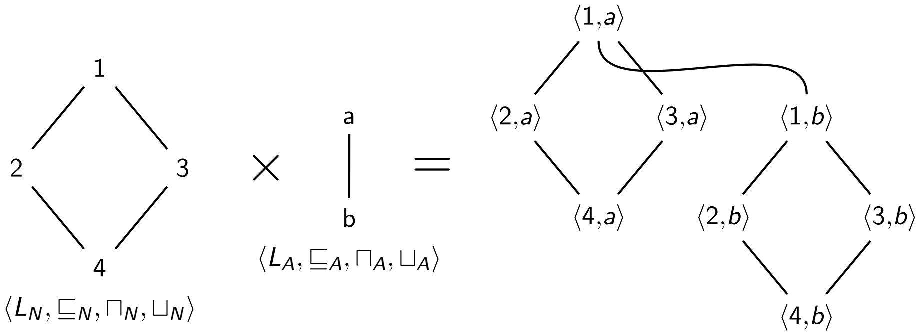

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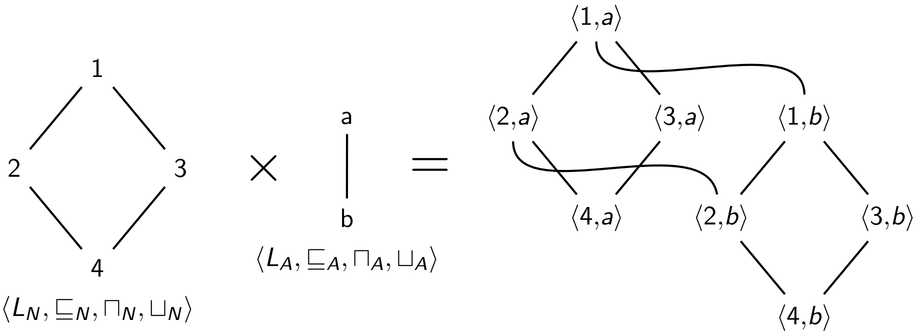

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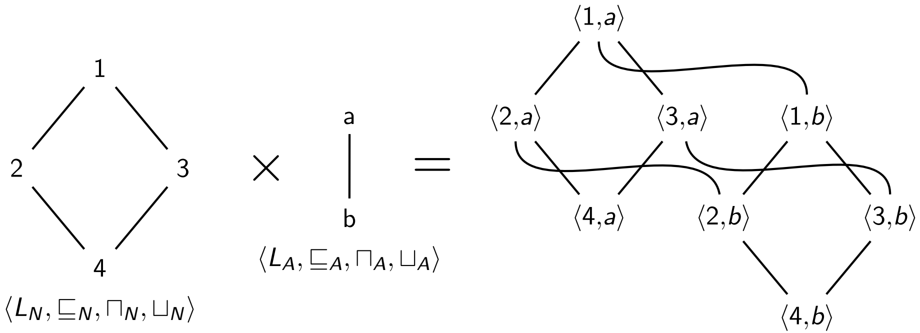

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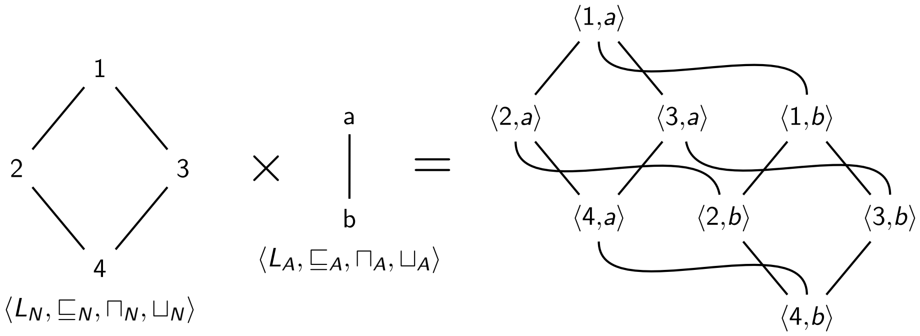

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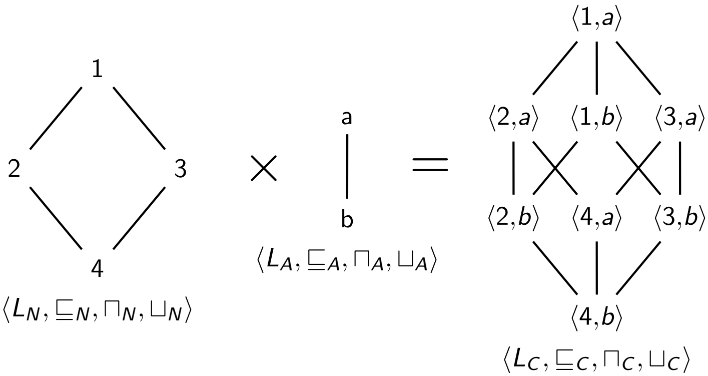

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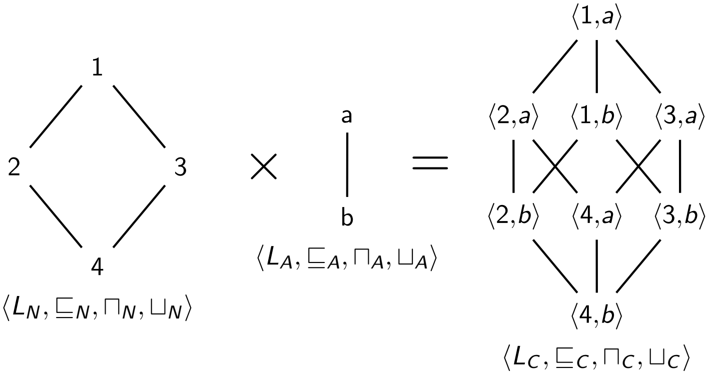

$$\langle x_1, y_1 \rangle \sqsubseteq_C \langle x_2, y_2 \rangle \quad \Leftrightarrow \quad x_1 \sqsubseteq_N x_2 \wedge y_1 \sqsubseteq_A y_2$$

$$\langle x_1, y_1 \rangle \sqcap_C \langle x_2, y_2 \rangle \quad = \quad \langle x_1 \sqcap_N x_2, y_1 \sqcap_A y_2 \rangle$$

$$\langle x_1, y_1 \rangle \sqcup_C \langle x_2, y_2 \rangle \quad = \quad \langle x_1 \sqcup_N x_2, y_1 \sqcup_A y_2 \rangle$$

Example of Cartesian Product: Concept Lattices

- *Context* of concepts. A collection of objects and their attributes
- *Concepts*. Sets of attributes as exhibited by specific objects
  - A concept C is a pair (O, A) where
    O is a set of objects exhibiting attributes in the set A
  - Every object in O has every attribute in A
- Partial order. $(O_2, A_2) \sqsubseteq (O_1, A_1) \Leftrightarrow O_2 \subseteq O_1$
  - Very few objects have all properties
  - Since A is the set of attributes common to all objects in O,

$$O_2 \subseteq O_1 \Rightarrow A_2 \supseteq A_1$$

    As the number of chosen objects decreases, the number of common attributes increases

Example of Concept Lattice (1)

From *Introduction to Lattices and Order* by Davey and Priestley [2002]

		Size			Distance from Sun		Moon?	
		Small (ss)	Medium (sm)	Large (sl)	Near (dn)	Far (df)	Yes (my)	No (mn)
Mercury	Me	x			x			x
Venus	V	x			x			x
Earth	E	x			x		x	
Mars	Ma	x			x		x	
Jupiter	J			x		x	x	
Saturn	S			x		x	x	
Uranus	U		x			x	x	
Neptune	N		x			x	x	
Pluto	P	x				x	x	

Example of Concept Lattice (2)

We write (O, A) as $\dfrac{O}{A}$

$$\frac{\{Me, V, E, Ma, J, S, U, N, P\}}{\{\}}$$

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$$\frac{\{Me, V, E, Ma\}}{\{ss, dn\}} \qquad \frac{\{E, Ma\}}{\{ss, my\}} \qquad \frac{\{J, S, U, N, P\}}{\{df, my\}}$$

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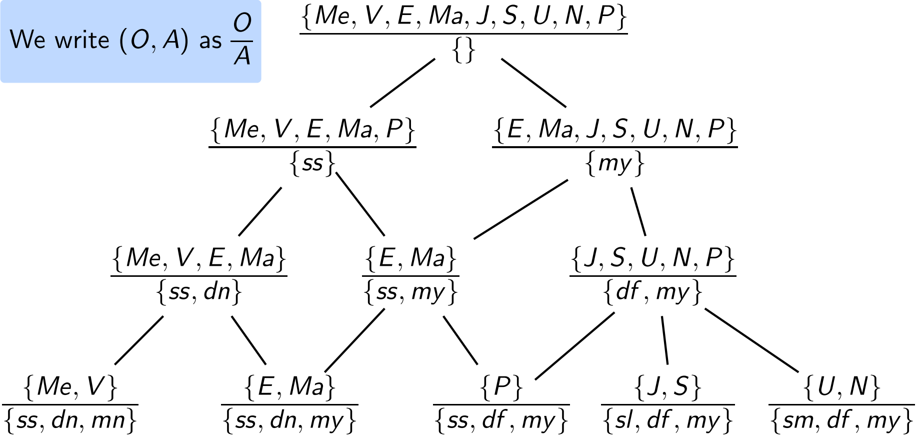

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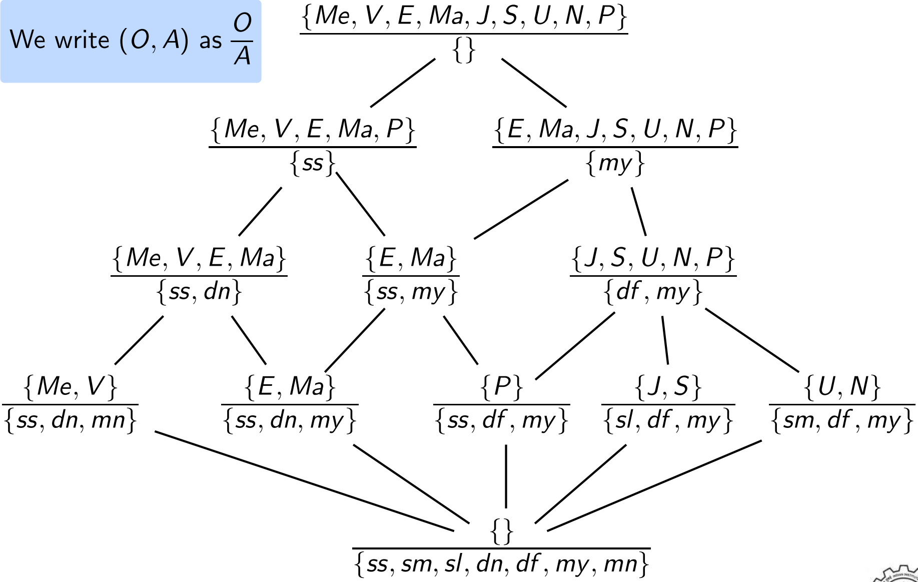

Variants of Products

In each case $L \subseteq L_1 \times L_2$

- *Cartesian Product*

  $$(x_1, x_2) \sqsubseteq (y_1, y_2) \text{ iff } x_1 \sqsubseteq_1 y_1 \wedge x_2 \sqsubseteq_2 y_2$$

- *Interval Product*

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Part 5

Data Flow Values: Details

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Meet semilattices satisfying the descending chain condition

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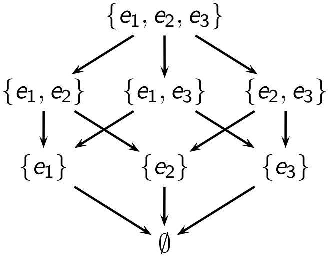
The Set of Data Flow Values For Available Expressions Analysis

- The powerset of the universal set of expressions
- Partial order is the subset relation



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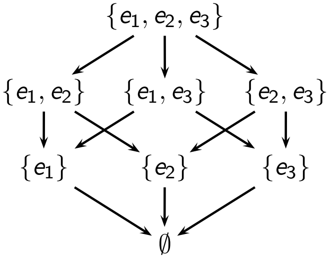

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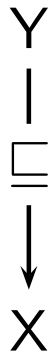

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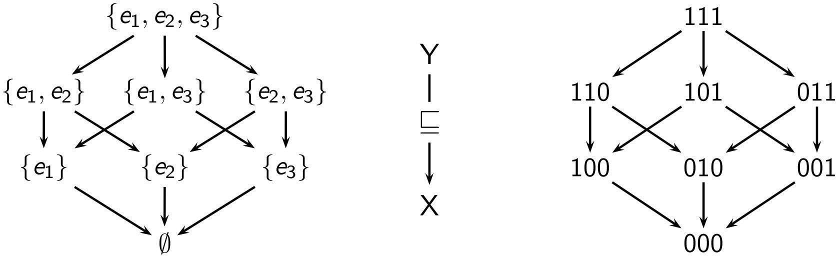

Set View of the Lattice   Bit Vector View

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  - Approximating Time

    Travel time: 2 hours required can be safely approximated by 3 hours
    Study time: 3 available days can be safely assumed to be only 2 days

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- The discovered data flow information should be
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- The intended use of data flow information ($\equiv$ context) determines validity of approximations

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May prohibit correct optimization

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Partial Order Captures Approximation

- $\sqsubseteq$ captures valid approximations for safety

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We want most exhaustive information which is also safe

Most Approximate Values in a Complete Lattice

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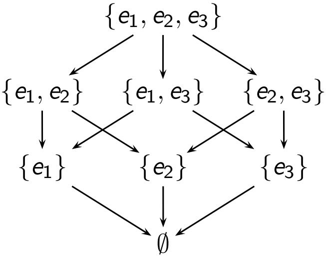
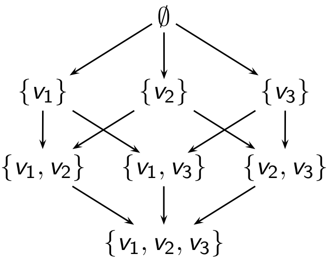
Appropriate orientation chosen by design

Setting Up Lattices

Available Expressions Analysis	Live Variables Analysis
	
$\sqsubseteq$ is $\subseteq$	$\sqsubseteq$ is $\supseteq$
$\sqcap$ is $\cap$	$\sqcap$ is $\cup$

Partial Order Relation

Reflexive	$x \sqsubseteq x$
Transitive	$x \sqsubseteq y, y \sqsubseteq z$ $\Rightarrow x \sqsubseteq z$
Antisymmetric	$x \sqsubseteq y, y \sqsubseteq x$ $\Leftrightarrow x = y$

Partial Order Relation

Reflexive	$x \sqsubseteq x$	x can be safely used in place of x
Transitive	$x \sqsubseteq y, y \sqsubseteq z$ $\Rightarrow x \sqsubseteq z$	If x can be safely used in place of y and y can be safely used in place of z, then x can be safely used in place of z
Antisymmetric	$x \sqsubseteq y, y \sqsubseteq x$ $\Leftrightarrow x = y$	If x can be safely used in place of y and y can be safely used in place of x, then x must be same as y

Merging Information

- $x \sqcap y$ computes the *greatest lower bound* of x and y i.e.
  largest z such that $z \sqsubseteq x$ and $z \sqsubseteq y$

  The largest safe approximation of combining data flow information x and y

Merging Information

- $x \sqcap y$ computes the *greatest lower bound* of x and y i.e. largest z such that $z \sqsubseteq x$ and $z \sqsubseteq y$

  The largest safe approximation of combining data flow information x and y

- Commutative $x \sqcap y = y \sqcap x$

  Associative $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$

  Idempotent $x \sqcap x = x$

Merging Information

- $x \sqcap y$ computes the *greatest lower bound* of x and y i.e. largest z such that $z \sqsubseteq x$ and $z \sqsubseteq y$

  The largest safe approximation of combining data flow information x and y

Commutative	$x \sqcap y = y \sqcap x$	The order in which the data flow information is merged, does not matter
Associative	$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	Allow n-ary merging without any restriction on the order
Idempotent	$x \sqcap x = x$	No loss of information if x is merged with itself

Merging Information

- $x \sqcap y$ computes the *greatest lower bound* of x and y i.e.

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Commutative	$x \sqcap y = y \sqcap x$	The order in which the data flow information is merged, does not matter
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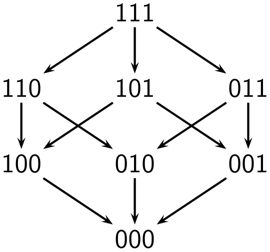
Idempotent	$x \sqcap x = x$	No loss of information if x is merged with itself

- $\top$ is the identity of $\sqcap$
  - Presence of loops $\Rightarrow$ self dependence of data flow information
  - Using $\top$ as the initial value ensure exhaustiveness

More on Lattices in Data Flow Analysis

L = Lattice for all expressions	$\widehat{L}$ = Lattice for a single expression
	(Expression e is available) 1 or $\{e\}$ $\downarrow$ 0 or $\emptyset$ (Expressions e is not available)

More on Lattices in Data Flow Analysis

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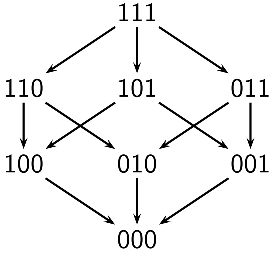
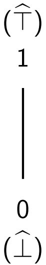
	(Expression e is available) 1 or $\{e\}$ ↓ 0 or $\emptyset$ (Expressions e is not available)

Cartesian products if sets are used, vectors (or tuples) if bit are used

- $L = \widehat{L} \times \widehat{L} \times \widehat{L}$ and $x = \langle \widehat{x}_1, \widehat{x}_2, \widehat{x}_3 \rangle \in L$ where $\widehat{x}_i \in \widehat{L}$
- $\sqsubseteq = \widehat{\sqsubseteq} \times \widehat{\sqsubseteq} \times \widehat{\sqsubseteq}$ and $\sqcap = \widehat{\sqcap} \times \widehat{\sqcap} \times \widehat{\sqcap}$
- $\top = \widehat{\top} \times \widehat{\top} \times \widehat{\top}$ and $\bot = \widehat{\bot} \times \widehat{\bot} \times \widehat{\bot}$

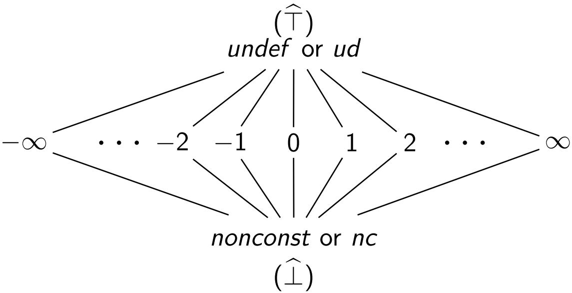
Component Lattice for Data Flow Information Represented By Bit Vectors



$\sqcap$ is $\cap$ or Boolean AND

$\sqcap$ is $\cup$ or Boolean OR

Component Lattice for Integer Constant Propagation



- Overall lattice L is the set of mappings from variables to $\widehat{L}$
- $\sqcap$ and $\widehat{\sqcap}$ get defined by $\sqsubseteq$ and $\widehat{\sqsubseteq}$

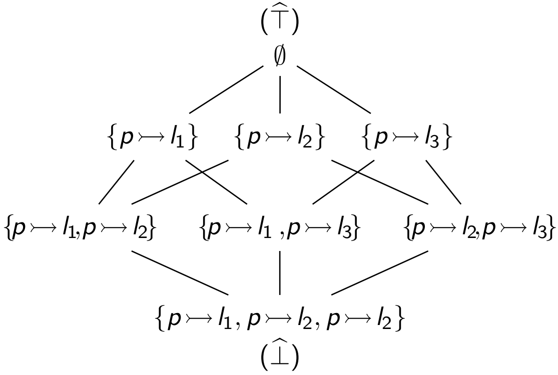
$\widehat{\sqcap}$	$\langle a, ud\rangle$	$\langle a, nc\rangle$	$\langle a, c_1\rangle$
$\langle a, ud\rangle$	$\langle a, ud\rangle$	$\langle a, nc\rangle$	$\langle a, c_1\rangle$
$\langle a, nc\rangle$	$\langle a, nc\rangle$	$\langle a, nc\rangle$	$\langle a, nc\rangle$
$\langle a, c_2\rangle$	$\langle a, c_2\rangle$	$\langle a, nc\rangle$	If $c_1 = c_2$ then $\langle a, c_1\rangle$ else $\langle a, nc\rangle$

Component Lattice for May Points-To Analysis

- Relation between pointer variables and locations in the memory

Component Lattice for May Points-To Analysis

- Relation between pointer variables and locations in the memory
- Assuming three locations l_1, l_2, and l_3, the component lattice for pointer p is



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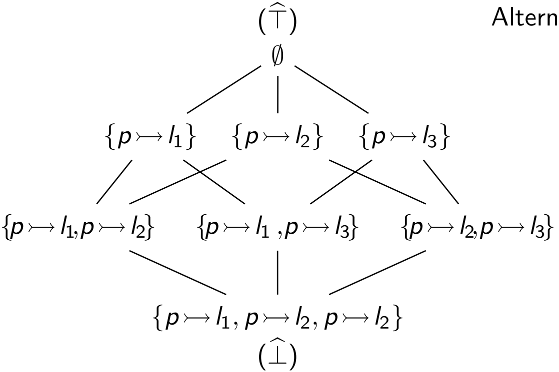
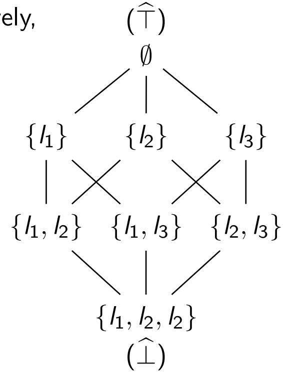
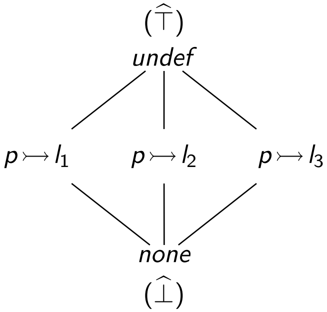


Alternatively,



Component Lattice for Must Points-To Analysis

- A pointer can point to at most one location



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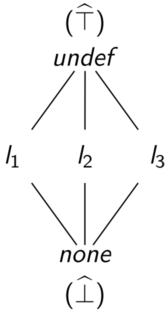
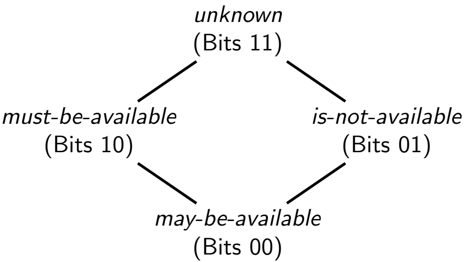


Combined Total and Partial Availability Analysis

- Two bits per expression rather than one. Can be implemented using AND (as below) or using OR (reversed lattice)

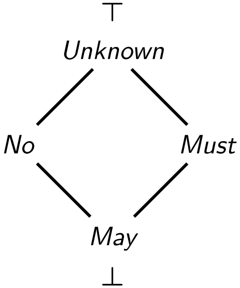


  Can also be implemented as a product of 1-0 and 0-1 lattice with AND for the first bit and OR for the second bit

- What approximation of safety does this lattice capture?

  Uncertain information (= no optimization) is guaranteed to be safe

General Lattice for May-Must Analysis



Interpreting data flow values

- *Unknown.* Nothing is known as yet
- *No.* Information does not hold along any path
- *Must.* Information must hold along all paths
- *May.* Information may hold along some path

Possible Applications

- Pointer Analysis : No need of separate of *May* and *Must* analyses
  eg. $(p \rightarrowtail l, \mathit{May})$, $(p \rightarrowtail l, \mathit{Must})$, $(p \rightarrowtail l, \mathit{No})$, or $(p \rightarrowtail l, \mathit{Unknown})$
- Type Inferencing for Dynamically Checked Languages

Part 6

Flow Functions

Flow Functions: An Outline of Our Discussion

- Defining flow functions
- Properties of flow functions
  (Some properties discussed in the context of solutions of data flow analysis)

The Set of Flow Functions

- F is the set of functions $f : L \rightarrow L$ such that
  - F contains an identity function

    To model "empty" statements, i.e. statements which do not influence the data flow information
  - F is closed under composition

    Cumulative effect of statements should generate data flow information from the same set
  - For every $x \in L$, there must be a finite set of flow functions $\{f_1, f_2, \ldots f_m\} \subseteq F$ such that

$$x = \prod_{1 \leq i \leq m} f_i(BI)$$

- Properties of f
  - Monotonicity and Distributivity
  - Loop Closure Boundedness and Separability

Flow Functions in Bit Vector Data Flow Frameworks

- Bit Vector Frameworks: Available Expressions Analysis, Reaching Definitions Analysis Live variable Analysis, Anticipable Expressions Analysis, Partial Redundancy Elimination etc
  - All functions can be defined in terms of constant Gen and Kill

$$f(x) = \text{Gen} \cup (x - \text{Kill})$$

  - Lattices are powersets with partial orders as $\subseteq$ or $\supseteq$ relations
  - Information is merged using $\cap$ or $\cup$

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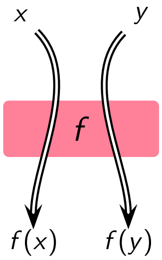
$$f(x) = \text{Gen} \cup (x - \text{Kill})$$

  - Lattices are powersets with partial orders as $\subseteq$ or $\supseteq$ relations
  - Information is merged using $\cap$ or $\cup$
- Flow functions in Strong Liveness Analysis, Pointer Analyses, Constant Propagation, Possibly Uninitialized Variables cannot be expressed using constant Gen and Kill

  Local context alone is not sufficient to describe the effect of statements fully

Monotonicity of Flow Functions

- Partial order is preserved: If x can be safely used in place of y then $f(x)$ can be safely used in place of $f(y)$



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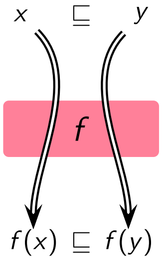

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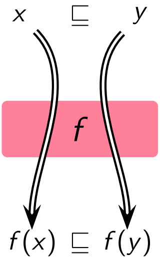
$$\forall x, y \in L, x \sqsubseteq y \Rightarrow f(x) \sqsubseteq f(y)$$



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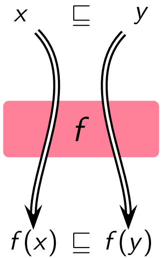

- Alternative definition

$$\forall x, y \in L, f(x \sqcap y) \sqsubseteq f(x) \sqcap f(y)$$

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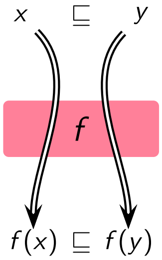


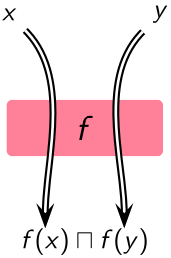
- Alternative definition

$$\forall x, y \in L, f(x \sqcap y) \sqsubseteq f(x) \sqcap f(y)$$

- Merging at intermediate points in shared segments of paths is safe (However, it may lead to imprecision)

Distributivity of Flow Functions

- Merging distributes over function application



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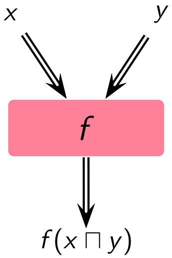

Distributivity of Flow Functions

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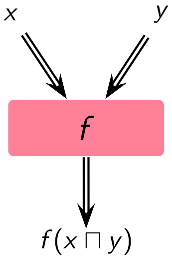
$\forall x, y \in L, f(x \sqcap y) = f(x) \sqcap f(y)$



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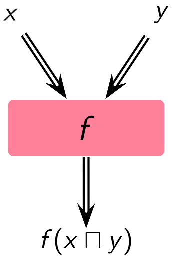
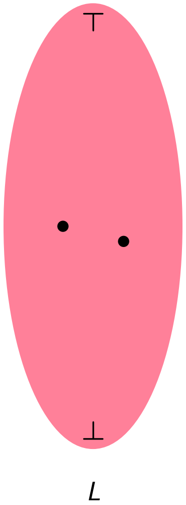


- Merging at intermediate points in shared segments of paths does not lead to imprecision

Monotonicity and Distributivity



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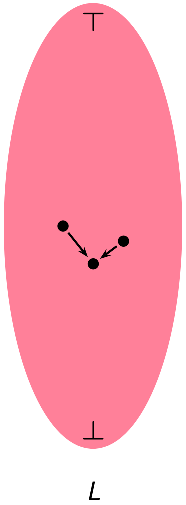

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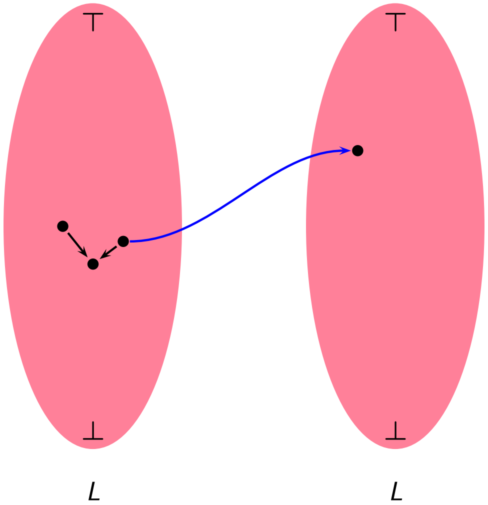

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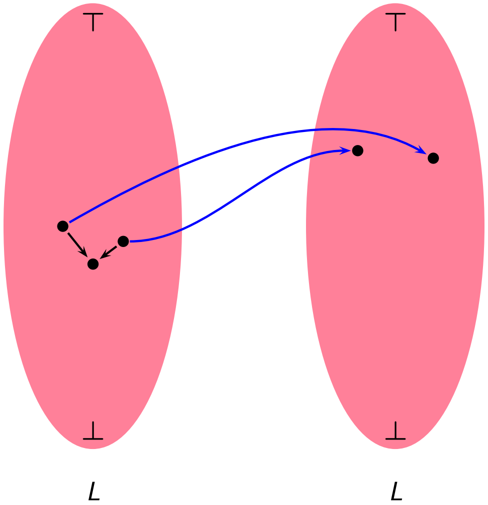

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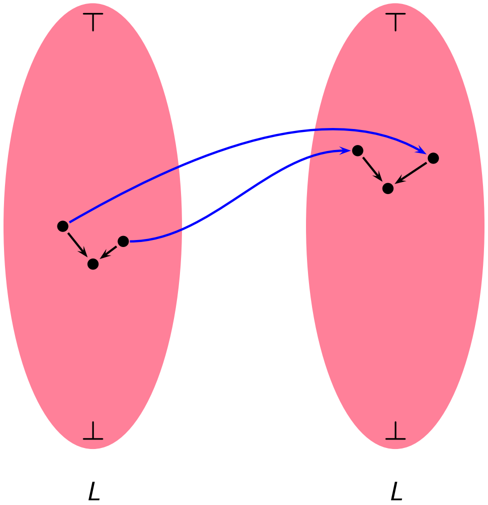

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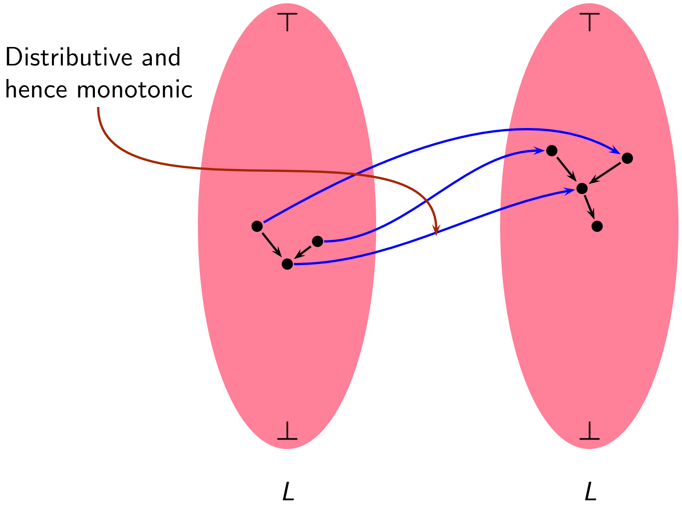

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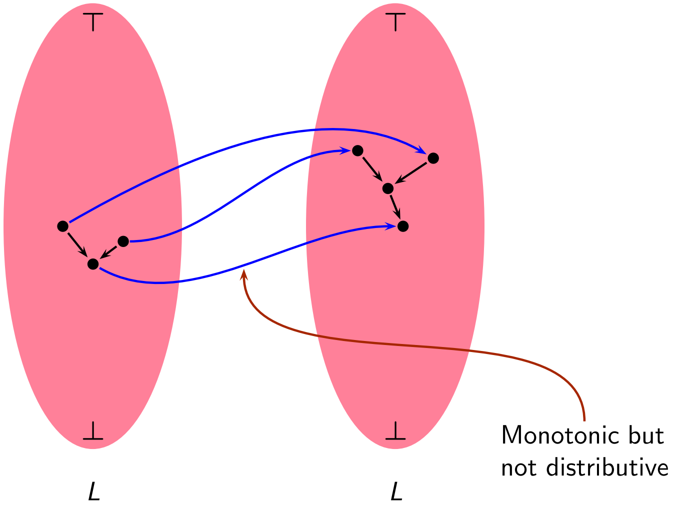


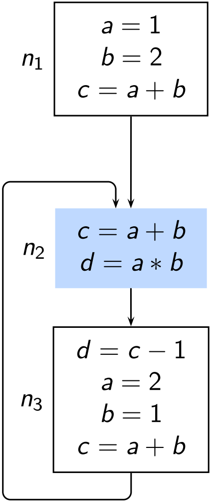
Distributivity of Bit Vector Frameworks

$$
\begin{aligned}
f(x) &= \text{Gen} \cup (x - \text{Kill}) \\
f(y) &= \text{Gen} \cup (y - \text{Kill})
\end{aligned}
$$

$$
\begin{aligned}
f(x \cup y) &= \text{Gen} \cup ((x \cup y) - \text{Kill}) \\
&= \text{Gen} \cup ((x - \text{Kill}) \cup (y - \text{Kill})) \\
&= (\text{Gen} \cup (x - \text{Kill}) \cup \text{Gen} \cup (y - \text{Kill})) \\
&= f(x) \cup f(y)
\end{aligned}
$$

$$
\begin{aligned}
f(x \cap y) &= \text{Gen} \cup ((x \cap y) - \text{Kill}) \\
&= \text{Gen} \cup ((x - \text{Kill}) \cap (y - \text{Kill})) \\
&= (\text{Gen} \cup (x - \text{Kill}) \cap \text{Gen} \cup (y - \text{Kill})) \\
&= f(x) \cap f(y)
\end{aligned}
$$

Non-Distributivity of Constant Propagation



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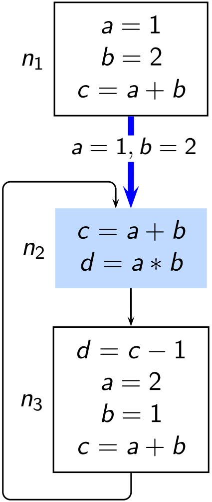
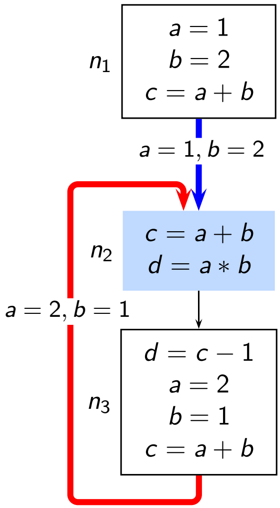


- $x = \langle 1, 2, 3, ud \rangle$ (Along $Out_{n_1} \rightarrow In_{n_2}$)

Non-Distributivity of Constant Propagation



- $x = \langle 1, 2, 3, ud \rangle$ (Along $Out_{n_1} \rightarrow In_{n_2}$)
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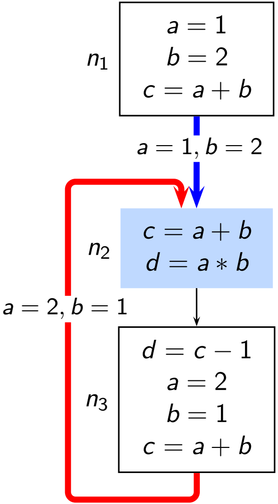
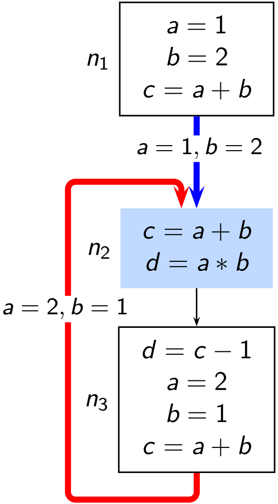
Non-Distributivity of Constant Propagation



- $x = \langle 1, 2, 3, ud \rangle$ (Along $Out_{n_1} \rightarrow In_{n_2}$)
- $y = \langle 2, 1, 3, 2 \rangle$ (Along $Out_{n_3} \rightarrow In_{n_2}$)
- Function application for block n_2 before merging

$$\begin{aligned} f(x) \sqcap f(y) &= f(\langle 1, 2, 3, ud \rangle) \sqcap f(\langle 2, 1, 3, 2 \rangle) \\ &= \langle 1, 2, 3, 2 \rangle \sqcap \langle 2, 1, 3, 2 \rangle \\ &= \langle \widehat{\bot}, \widehat{\bot}, 3, 2 \rangle \end{aligned}$$

Non-Distributivity of Constant Propagation



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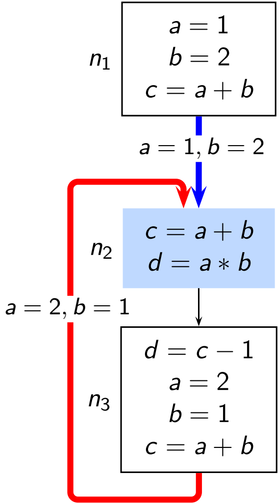
- Function application for block n_2 after merging

$$\begin{aligned} f(x \sqcap y) &= f(\langle 1, 2, 3, ud \rangle \sqcap \langle 2, 1, 3, 2 \rangle) \\ &= f(\langle \widehat{\perp}, \widehat{\perp}, 3, 2 \rangle) \\ &= \langle \widehat{\perp}, \widehat{\perp}, \widehat{\perp}, \widehat{\perp} \rangle \end{aligned}$$

Non-Distributivity of Constant Propagation



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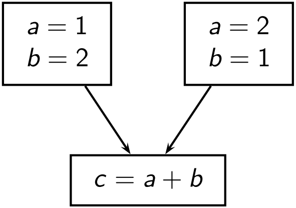
- Function application for block n_2 after merging

$$\begin{aligned} f(x \sqcap y) &= f(\langle 1, 2, 3, ud \rangle \sqcap \langle 2, 1, 3, 2 \rangle) \\ &= f(\langle \widehat{\bot}, \widehat{\bot}, 3, 2 \rangle) \\ &= \langle \widehat{\bot}, \widehat{\bot}, \widehat{\bot}, \widehat{\bot} \rangle \end{aligned}$$

- $f(x \sqcap y) \sqsubset f(x) \sqcap f(y)$

Why is Constant Propagation Non-Distributive?



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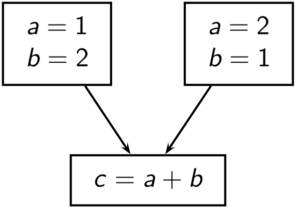
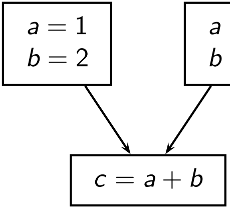


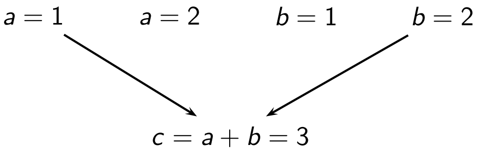
Possible combinations due to merging

$a = 1$ $a = 2$ $b = 1$ $b = 2$

Why is Constant Propagation Non-Distributive?

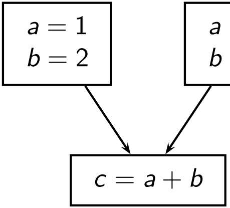


Possible combinations due to merging



- Correct combination

Why is Constant Propagation Non-Distributive?



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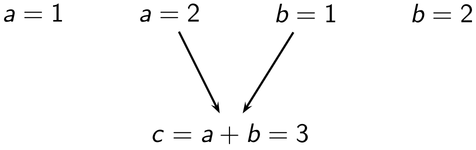

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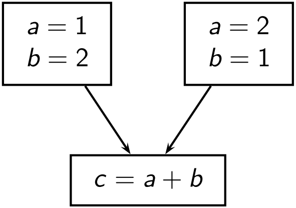

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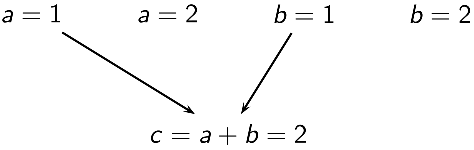

- Wrong combination
- Mutually exclusive information
- No execution path along which this information holds

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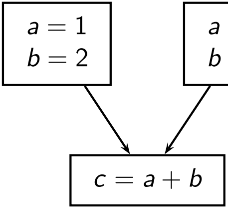

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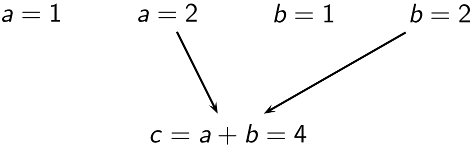

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Part 7

Solutions of Data Flow Analysis

Solutions of Data Flow Analysis: An Outline of Our Discussion

- MoP and MFP assignments and their relationship
- Existence of MoP assignment
  - Boundedness of flow functions
- Existence and Computability of MFP assignment
  - Flow functions Vs. function computed by data flow equations
- Safety of MFP solution

Solutions of Data Flow Analysis

- An assignment A associates data flow values with program points
  $A \sqsubseteq B$ if for all program points p, $A(p) \sqsubseteq B(p)$
- Performing data flow analysis

  Given

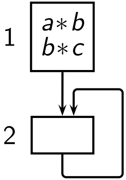
  - A set of flow functions, a lattice, and merge operation
  - A program flow graph with a mapping from nodes to flow functions

  Find out

  - An assignment A which is as exhaustive as possible and is safe

An Example For Available Expressions Analysis

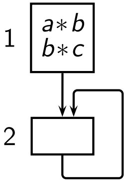
Program



Some Assignments							
	A_0	A_1	A_2	A_3	A_4	A_5	A_6
In_1	11	00	00	00	00	00	00
Out_1	11	11	00	11	11	11	11
In_2	11	11	00	00	10	01	01
Out_2	11	11	00	00	10	01	10

An Example For Available Expressions Analysis

Program



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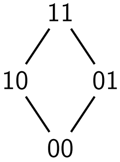
	A_0	A_1	A_2	A_3	A_4	A_5	A_6
In_1	11	00	00	00	00	00	00
Out_1	11	11	00	11	11	11	11
In_2	11	11	00	00	10	01	01
Out_2	11	11	00	00	10	01	10

Lattice L of data flow values at a node



An Example For Available Expressions Analysis

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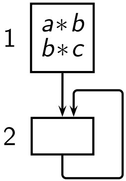

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Out_2	11	11	00	00	10	01	10

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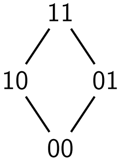
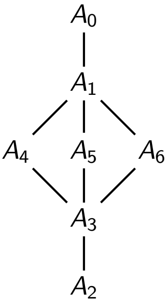
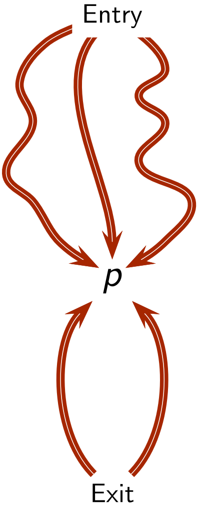


Lattice $L \times L \times L \times L$ for data flow values at all nodes



Meet Over Paths (MoP) Assignment



- The largest safe approximation of the information reaching a program point along all information flow paths

$$MoP(p) \quad = \bigsqcap_{\rho \,\in\, Paths(p)} f_\rho(BI)$$

  - f_ρ represents the compositions of flow functions along ρ
  - BI refers to the relevant information from the calling context
  - All execution paths are considered potentially executable by ignoring the results of conditionals

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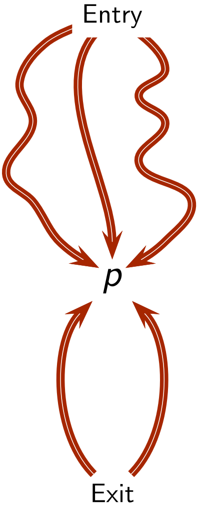

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- Any $Info(p) \sqsubseteq MoP(p)$ is safe

Maximum Fixed Point (MFP) Assignment

- Difficulties in computing MoP assignment

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    - In the presence of cycles there are infinite paths

      If all paths need to be traversed $\Rightarrow$ Undecidability

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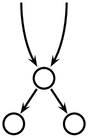
    If all paths need to be traversed ⇒ Intractability



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Path based specification

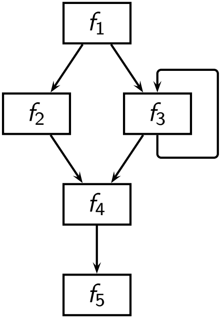
Edge based specifications

Computing MFP Vs. Computing MoP

Expression Tree for MFP

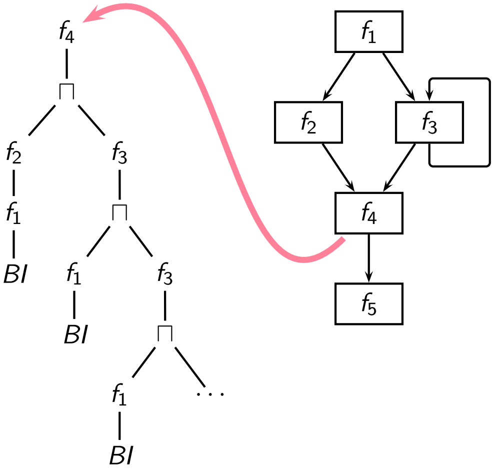
Program

Expression Tree for MoP

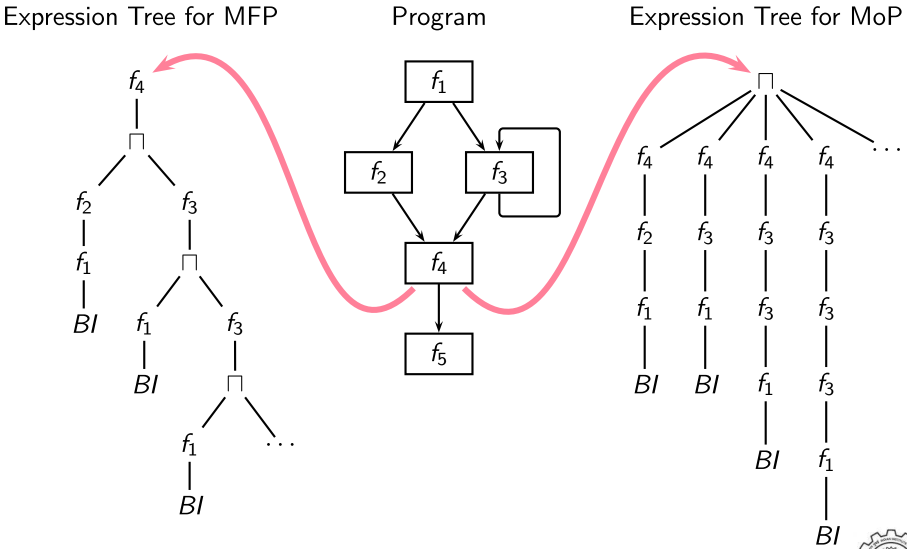


Computing MFP Vs. Computing MoP

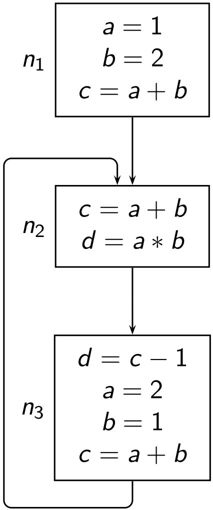
Expression Tree for MFP | Program | Expression Tree for MoP



Computing MFP Vs. Computing MoP



Assignments for Constant Propagation Example



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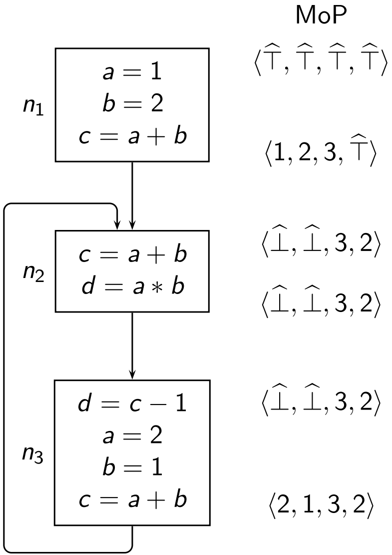
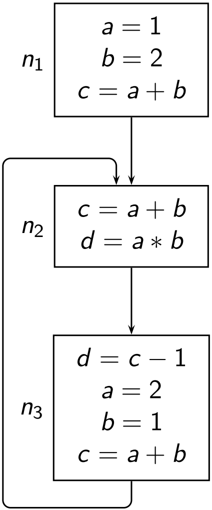
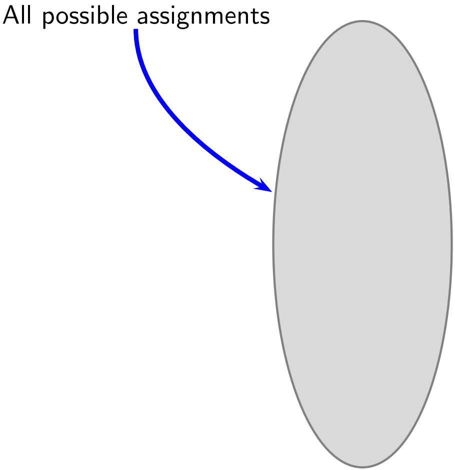


Assignments for Constant Propagation Example



MoP	MFP
$\langle \widehat{\top}, \widehat{\top}, \widehat{\top}, \widehat{\top} \rangle$	$\langle \widehat{\top}, \widehat{\top}, \widehat{\top}, \widehat{\top} \rangle$
$\langle 1, 2, 3, \widehat{\top} \rangle$	$\langle 1, 2, 3, \widehat{\top} \rangle$
$\langle \widehat{\bot}, \widehat{\bot}, 3, 2 \rangle$	$\langle \widehat{\bot}, \widehat{\bot}, 3, \widehat{\bot} \rangle$
$\langle \widehat{\bot}, \widehat{\bot}, 3, 2 \rangle$	$\langle \widehat{\bot}, \widehat{\bot}, \widehat{\bot}, \widehat{\bot} \rangle$
$\langle \widehat{\bot}, \widehat{\bot}, 3, 2 \rangle$	$\langle \widehat{\bot}, \widehat{\bot}, \widehat{\bot}, \widehat{\bot} \rangle$
$\langle 2, 1, 3, 2 \rangle$	$\langle 2, 1, 3, \widehat{\bot} \rangle$

Possible Assignments as Solutions of Data Flow Analyses



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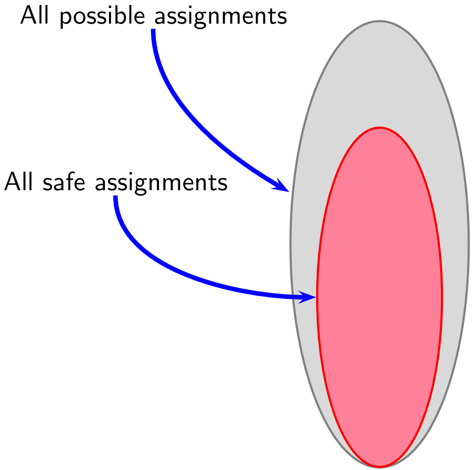

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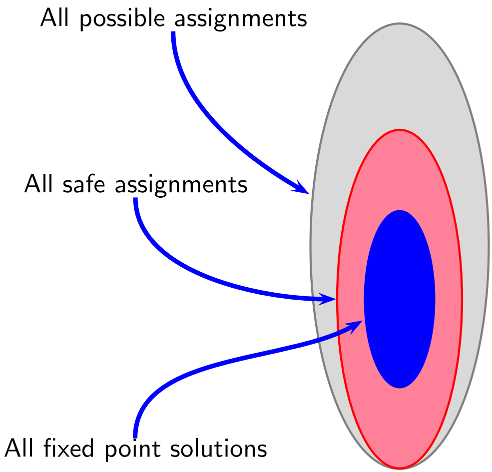

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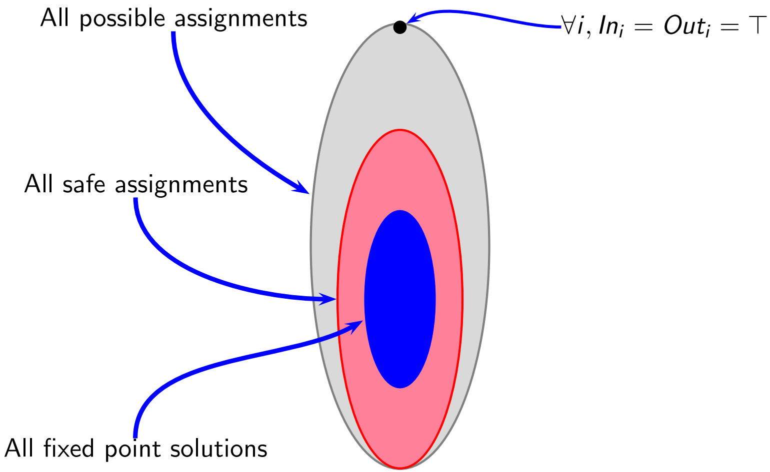

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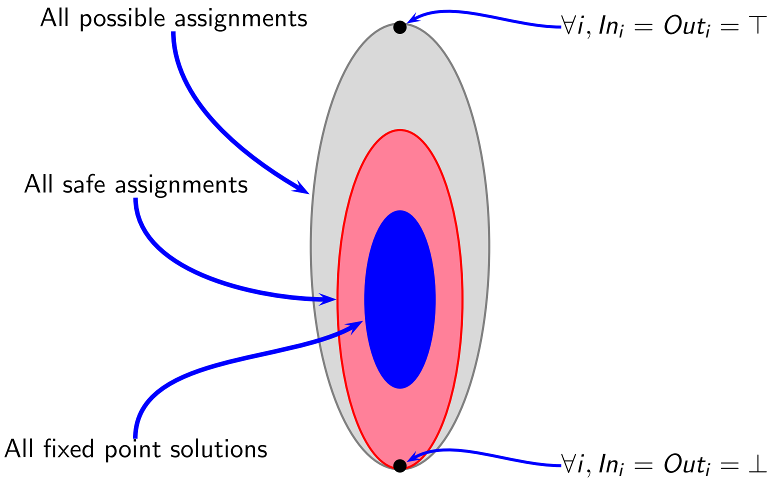

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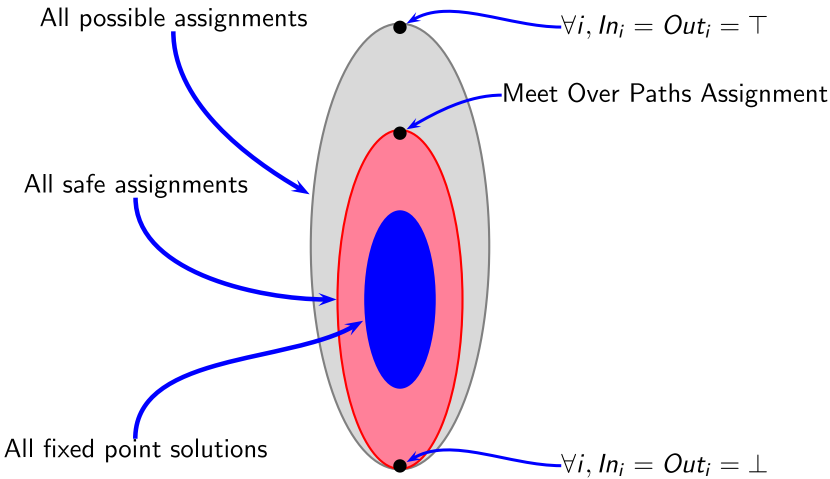

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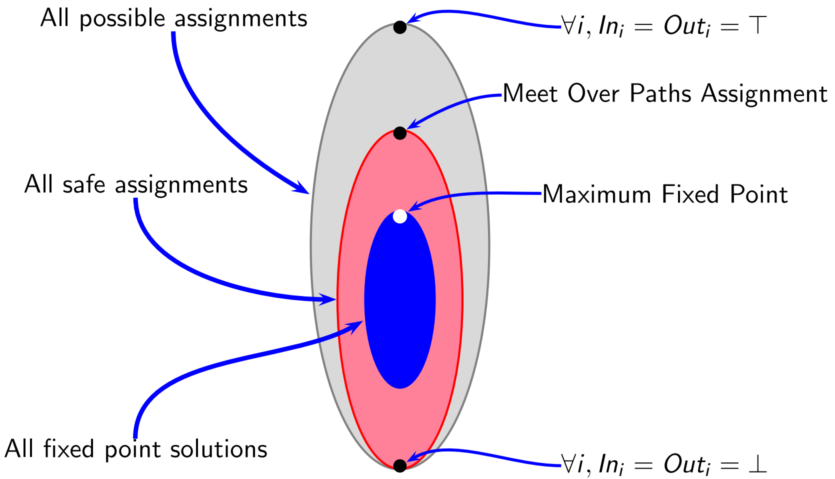

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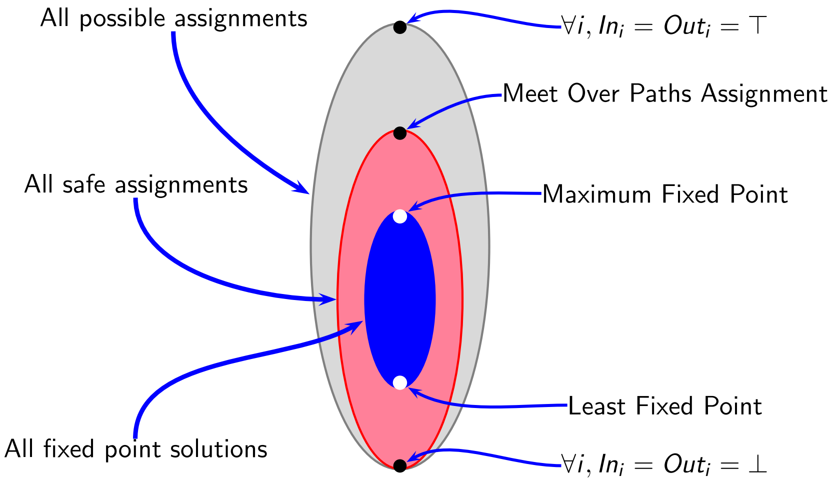
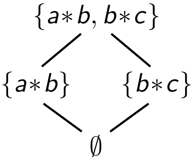


An Instance of Available Expressions Analysis

Lattice



Constant Functions		Dependent Functions	
f	$f(x)$	f	$f(x)$
$f_\top$	$\{a*b, b*c\}$	f_{id}	x
$f_\bot$	$\emptyset$	f_c	$x \cup \{a*b\}$
f_a	$\{a*b\}$	f_d	$x \cup \{b*c\}$
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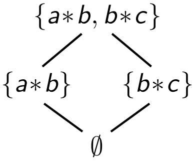

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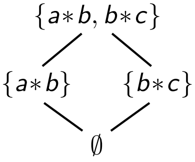
- Is the lattice a meet semilattice?

An Instance of Available Expressions Analysis

Lattice



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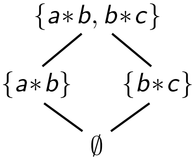
- Is the lattice a meet semilattice?
- What is the meet operation that computes glb?

An Instance of Available Expressions Analysis

Lattice



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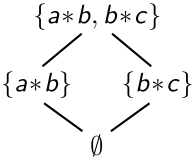
- Is the lattice a meet semilattice?
- What is the meet operation that computes glb?
- Are all strictly descending chains finite?

An Instance of Available Expressions Analysis

Lattice



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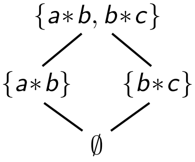
- Is the lattice a meet semilattice?
- What is the meet operation that computes glb?
- Are all strictly descending chains finite?
- Does the function space have an identity function?

An Instance of Available Expressions Analysis

Lattice



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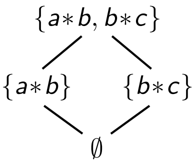
- Is the lattice a meet semilattice?
- What is the meet operation that computes glb?
- Are all strictly descending chains finite?
- Does the function space have an identity function?
- Are all values in the lattice computable from a finite merge of flow functions?

An Instance of Available Expressions Analysis

Lattice



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- What is the meet operation that computes glb?
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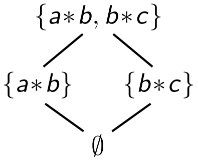
- Are all values in the lattice computable from a finite merge of flow functions?
- Is the function space closed under composition?

An Instance of Available Expressions Analysis

Lattice



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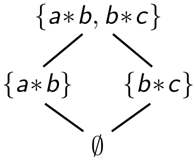

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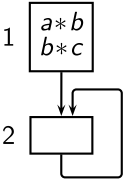
Program



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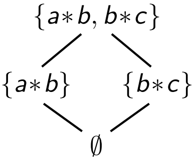
Lattice



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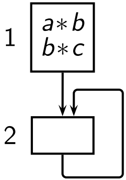
		f_f	$x - \{b*c\}$

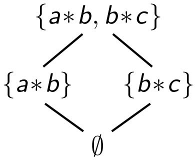
Program



Flow Functions	
Node	Flow Function
1	$f_\top$
2	f_{id}

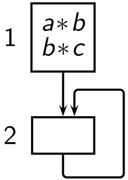
An Instance of Available Expressions Analysis

Lattice



Constant Functions		Dependent Functions	
f	$f(x)$	f	$f(x)$
$f_\top$	$\{a*b, b*c\}$	f_{id}	x
$f_\bot$	$\emptyset$	f_c	$x \cup \{a*b\}$
f_a	$\{a*b\}$	f_d	$x \cup \{b*c\}$
f_b	$\{b*c\}$	f_e	$x - \{a*b\}$
		f_f	$x - \{b*c\}$

Program

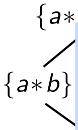


Flow Functions	
Node	Flow Function
1	$f_\top$
2	f_{id}

Some Possible Assignments						
	A_1	A_2	A_3	A_4	A_5	A_6
In_1	00	00	00	00	00	00
Out_1	11	00	11	11	11	11
In_2	11	00	00	10	01	01
Out_2	11	00	00	10	01	10

An Instance of Available Expressions Analysis

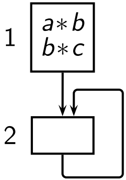
Lattice



Constant Functions		Dependent Functions	
f	$f(x)$	f	$f(x)$
$f_\top$	$\{a*b, b*c\}$	f_{id}	x
			$x \cup \{a*b\}$
			$x \cup \{b*c\}$
			$x - \{a*b\}$
			$x - \{b*c\}$

- Maximum fixed point assignment
- Initialization for round robin iterative method: 11
- Safe assignment

Program

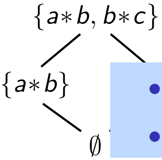


Flow Functions	
Node	Flow Function
1	$f_\top$
2	f_{id}

Some Possible Assignments						
	A_1	A_2	A_3	A_4	A_5	A_6
In_1	00	00	00	00	00	00
Out_1	11	00	11	11	11	11
In_2	11	00	00	10	01	01
Out_2	11	00	00	10	01	10

An Instance of Available Expressions Analysis

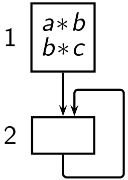
Lattice



Constant Functions		Dependent Functions	
f	$f(x)$	f	$f(x)$
$f_\top$	$\{a*b, b*c\}$	f_{id}	x
$f_\bot$	$\emptyset$	f_c	$x \cup \{a*b\}$
		f_d	$x \cup \{b*c\}$
		f_e	$x - \{a*b\}$
		f_f	$x - \{b*c\}$

- Not a fixed point assignment
- Safe assignment

Program

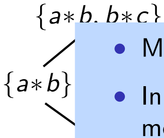


Flow Functions	
Node	Flow Function
1	$f_\top$
2	f_{id}

Some Possible Assignments						
	A_1	A_2	A_3	A_4	A_5	A_6
In_1	00	00	00	00	00	00
Out_1	11	00	11	11	11	11
In_2	11	00	00	10	01	01
Out_2	11	00	00	10	01	10

An Instance of Available Expressions Analysis

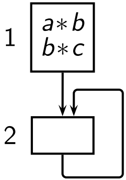
Lattice



Constant Functions		Dependent Functions	
f	$f(x)$	f	$f(x)$
$f_\top$	$\{a*b, b*c\}$	f_{id}	x
			$x \cup \{a*b\}$
			$x \cup \{b*c\}$
			$x - \{a*b\}$
			$x - \{b*c\}$

- Minimum fixed point assignment
- Initialization for round robin iterative method: 00
- Safe assignment

Program

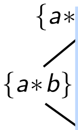


Flow Functions	
Node	Flow Function
1	$f_\top$
2	f_{id}

Some Possible Assignments						
	A_1	A_2	A_3	A_4	A_5	A_6
In_1	00	00	00	00	00	00
Out_1	11	00	11	11	11	11
In_2	11	00	00	10	01	01
Out_2	11	00	00	10	01	10

An Instance of Available Expressions Analysis

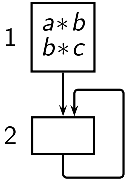
Lattice



Constant Functions		Dependent Functions	
f	$f(x)$	f	$f(x)$
			x
			$x \cup \{a*b\}$
			$x \cup \{b*c\}$
			$x - \{a*b\}$
			$x - \{b*c\}$

- Fixed point assignment which is neither maximum nor minimum
- Initialization for round robin iterative method: 10
- Safe assignment

Program



Flow Functions	
Node	Flow Function
1	$f_{\top}$
2	f_{id}

Some Possible Assignments						
	A_1	A_2	A_3	A_4	A_5	A_6
In_1	00	00	00	00	00	00
Out_1	11	00	11	11	11	11
In_2	11	00	00	10	01	01
Out_2	11	00	00	10	01	10

An Instance of Available Expressions Analysis

Lattice

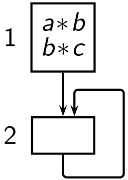
$\{a*$ ...

$\{a*b\}$

Constant Functions		Dependent Functions	
f	$f(x)$	f	$f(x)$
			x
			$x \cup \{a*b\}$
			$x \cup \{b*c\}$
			$x - \{a*b\}$
			$x - \{b*c\}$

- Fixed point assignment which is neither maximum nor minimum
- Initialization for round robin iterative method: 01
- Safe assignment

Program

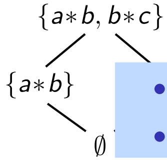


Flow Functions	
Node	Flow Function
1	$f_{\top}$
2	f_{id}

Some Possible Assignments						
	A_1	A_2	A_3	A_4	A_5	A_6
In_1	00	00	00	00	00	00
Out_1	11	00	11	11	11	11
In_2	11	00	00	10	01	01
Out_2	11	00	00	10	01	10

An Instance of Available Expressions Analysis

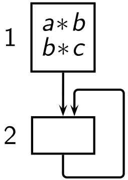
Lattice



Constant Functions		Dependent Functions	
f	$f(x)$	f	$f(x)$
$f_\top$	$\{a*b, b*c\}$	f_{id}	x
$f_\bot$	$\emptyset$	f_c	$x \cup \{a*b\}$
		f_d	$x \cup \{b*c\}$
		f_e	$x - \{a*b\}$
		f_f	$x - \{b*c\}$

- Not a fixed point assignment
- Safe assignment

Program



Flow Functions	
Node	Flow Function
1	$f_\top$
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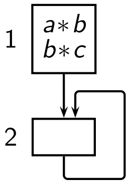
Some Possible Assignments						
	A_1	A_2	A_3	A_4	A_5	A_6
In_1	00	00	00	00	00	00
Out_1	11	00	11	11	11	11
In_2	11	00	00	10	01	01
Out_2	11	00	00	10	01	10

Lattice of Assignments for Available Expressions Analysis

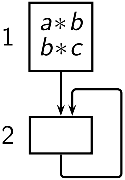
Program



Some Assignments							
	A_0	A_1	A_2	A_3	A_4	A_5	A_6
In_1	11	00	00	00	00	00	00
Out_1	11	11	00	11	11	11	11
In_2	11	11	00	00	10	01	01
Out_2	11	11	00	00	10	01	10

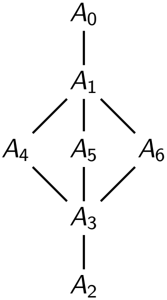
Lattice of Assignments for Available Expressions Analysis

Program



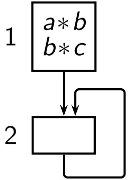
Some Assignments							
	A_0	A_1	A_2	A_3	A_4	A_5	A_6
In_1	11	00	00	00	00	00	00
Out_1	11	11	00	11	11	11	11
In_2	11	11	00	00	10	01	01
Out_2	11	11	00	00	10	01	10

Lattice $L \times L \times L \times L$ for all assignments (many assignments omitted, e.g. node 1 could have data flow values 10 and 01)



Lattice of Assignments for Available Expressions Analysis

Program



Some Assignments							
	A_0	A_1	A_2	A_3	A_4	A_5	A_6
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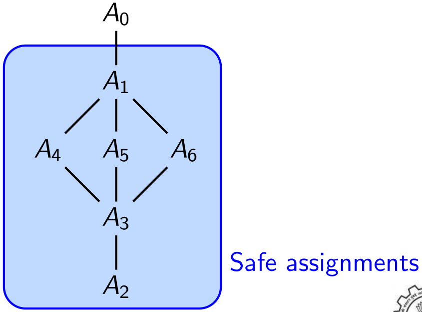
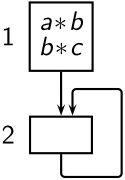


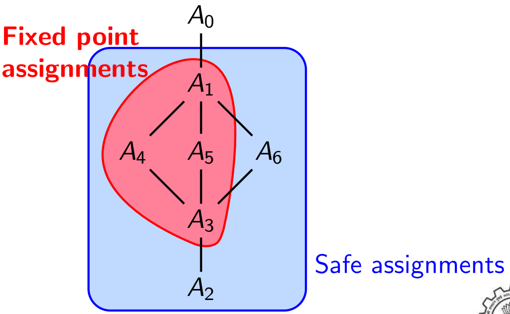
Lattice of Assignments for Available Expressions Analysis

Program



Some Assignments							
	A_0	A_1	A_2	A_3	A_4	A_5	A_6
In_1	11	00	00	00	00	00	00
Out_1	11	11	00	11	11	11	11
In_2	11	11	00	00	10	01	01
Out_2	11	11	00	00	10	01	10

Lattice $L \times L \times L \times L$ for all assignments (many assignments omitted, e.g. node 1 could have data flow values 10 and 01)



Existence of an MoP Assignment (1)

$$MoP(p) \quad = \bigsqcap_{\rho \in Paths(p)} f_\rho(BI)$$

- If a finite number of paths reach p, then existence of solution trivially follows
  - Function space is closed under composition
  - glb exists for all non-empty finite subsets of the lattice (Assuming that the data flow values form a meet semilattice)

Existence of an MoP Assignment (2)

$$MoP(p) \quad = \bigsqcap_{\rho \in Paths(p)} f_\rho(BI)$$

- If an infinite number of paths reach p then,

$$MoP(p) = f_{\rho_1}(BI) \sqcap f_{\rho_2}(BI) \sqcap f_{\rho_3}(BI) \sqcap \ldots$$

Existence of an MoP Assignment (2)

$$MoP(p) \quad = \prod_{\rho \,\in\, Paths(p)} f_\rho(BI)$$

- If an infinite number of paths reach p then,

$$MoP(p) = \underbrace{f_{\rho_1}(BI)}_{X_1} \sqcap f_{\rho_2}(BI) \sqcap f_{\rho_3}(BI) \sqcap \ldots$$

Existence of an MoP Assignment (2)

$$MoP(p) \quad = \bigsqcap_{\rho \in Paths(p)} f_\rho(BI)$$

- If an infinite number of paths reach p then,

$$MoP(p) = \underbrace{\underbrace{f_{\rho_1}(BI)}_{X_1} \sqcap f_{\rho_2}(BI)}_{X_2} \sqcap f_{\rho_3}(BI) \sqcap \ldots$$

- Every meet results in a weaker value

Existence of an MoP Assignment (2)

$$MoP(p) \quad = \bigsqcap_{\rho \in Paths(p)} f_\rho(BI)$$

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$$MoP(p) = \underbrace{\underbrace{\underbrace{f_{\rho_1}(BI)}_{X_1} \sqcap f_{\rho_2}(BI)}_{X_2} \sqcap f_{\rho_3}(BI)}_{X_3} \sqcap \ldots$$

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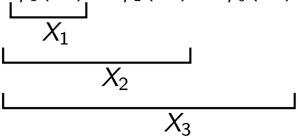
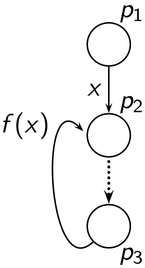


- Every meet results in a weaker value
- The sequence $X_1, X_2, X_3, \ldots$ follows a descending chain
- Since all strictly descending chains are finite, MoP exists
  (Assuming that our meet semilattice satisfies DCC)

Computability of MoP

Does existence of MoP imply it is computable?



Paths reaching the entry of p_2	Data Flow Value
p_1, p_2	x
p_1, p_2, p_3, p_2	$f(x)$
$p_1, p_2, p_3, p_2, p_3, p_2$	$f(f(x)) = f^2(x)$
$p_1, p_2, p_3, p_2, p_3, p_2, p_3, p_2$	$f(f(f(x))) = f^3(x)$
...	...

$$MoP(p_2) = x \sqcap f(x) \sqcap f^2(x) \sqcap f^3(x) \sqcap f^4(x) \sqcap \ldots$$

MoP Computation is Undecidable

There does not exist any algorithm that can compute MoP assignment for every possible instance of every possible monotone data flow framework

- Reducing MPCP (Modified Post's Correspondence Problem) to constant propagation
- MPCP is known to be undecidable
- If an algorithm exists for detecting all constants

  $\Rightarrow$ MPCP would be decidable
- Since MPCP is undecidable

  $\Rightarrow$ There does not exist an algorithm for detecting all constants

  $\Rightarrow$ Static analysis is undecidable

Post's Correspondence Problem (PCP)

- Given strings $u_i, v_i \in \Sigma^+$ for some alphabet Σ, and two k-tuples,

$$\begin{aligned} U &= (u_1, u_2, \ldots, u_k) \\ V &= (v_1, v_2, \ldots, v_k) \end{aligned}$$

Is there a sequence $i_1, i_2, \ldots, i_m$ of one or more integers such that

$$u_{i_1} u_{i_2} \ldots u_{i_m} = v_{i_1} v_{i_2} \ldots v_{i_m}$$

Post's Correspondence Problem (PCP)

- Given strings $u_i, v_i \in \Sigma^+$ for some alphabet Σ, and two k-tuples,

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  Is there a sequence $i_1, i_2, \ldots, i_m$ of one or more integers such that

$$u_{i_1} u_{i_2} \ldots u_{i_m} = v_{i_1} v_{i_2} \ldots v_{i_m}$$

- For $U = (101, 11, 100)$ and $V = (01, 1, 11001)$ the solution is $2, 3, 2$

$$\begin{aligned} u_2 u_3 u_2 &= 1110011 \\ v_2 v_3 v_2 &= 1110011 \end{aligned}$$

Post's Correspondence Problem (PCP)

- Given strings $u_i, v_i \in \Sigma^+$ for some alphabet Σ, and two k-tuples,

$$\begin{aligned} U &= (u_1, u_2, \ldots, u_k) \\ V &= (v_1, v_2, \ldots, v_k) \end{aligned}$$

  Is there a sequence $i_1, i_2, \ldots, i_m$ of one or more integers such that

$$u_{i_1} u_{i_2} \ldots u_{i_m} = v_{i_1} v_{i_2} \ldots v_{i_m}$$

- For $U = (101, 11, 100)$ and $V = (01, 1, 11001)$ the solution is $2, 3, 2$

$$\begin{aligned} u_2 u_3 u_2 &= 1110011 \\ v_2 v_3 v_2 &= 1110011 \end{aligned}$$

- For $U = (1, 10111, 10)$, $V = (111, 10, 0)$, the solution is $2, 1, 1, 3$

Post's Correspondence Problem (PCP)

- Given strings $u_i, v_i \in \Sigma^+$ for some alphabet Σ, and two k-tuples,

$$\begin{aligned} U &= (u_1, u_2, \ldots, u_k) \\ V &= (v_1, v_2, \ldots, v_k) \end{aligned}$$

  Is there a sequence $i_1, i_2, \ldots, i_m$ of one or more integers such that

$$u_{i_1} u_{i_2} \ldots u_{i_m} = v_{i_1} v_{i_2} \ldots v_{i_m}$$

- For $U = (101, 11, 100)$ and $V = (01, 1, 11001)$ the solution is $2, 3, 2$

$$\begin{aligned} u_2 u_3 u_2 &= 1110011 \\ v_2 v_3 v_2 &= 1110011 \end{aligned}$$

- For $U = (1, 10111, 10)$, $V = (111, 10, 0)$, the solution is $2, 1, 1, 3$
- For $U = (01, 110)$, $V = (00, 11)$, there is no solution

Post's Correspondence Problem (PCP)

- Given strings $u_i, v_i \in \Sigma^+$ for some alphabet Σ, and two k-tuples,

$$\begin{aligned} U &= (u_1, u_2, \ldots, u_k) \\ V &= (v_1, v_2, \ldots, v_k) \end{aligned}$$

Is there a sequence $i_1, i_2, \ldots, i_m$ of one or more integers such that

$$u_{i_1} u_{i_2} \ldots u_{i_m} = v_{i_1} v_{i_2} \ldots v_{i_m}$$

- Sets U and V are finite and contain the same number of strings
- The strings in U and V are finite and are of varying lengths
- For constructing the new strings using the strings in U and V
  - The strings at the same the index of must be used
  - There is no limit on the length of the new string

Indices could repeat without any bound

Modified Post's Correspondence Problem (MPCP)

- The first string in the correspondence relation should be the first string from the *k*-tuple

$$u_1 u_{i_1} u_{i_2} \ldots u_{i_m} = v_1 v_{i_1} v_{i_2} \ldots v_{i_m}$$

Modified Post's Correspondence Problem (MPCP)

- The first string in the correspondence relation should be the first string from the k-tuple

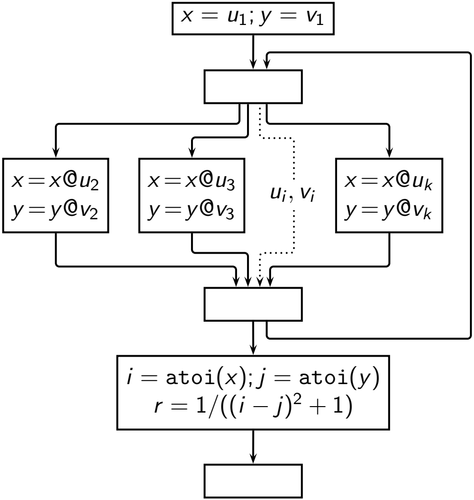
$$u_1 u_{i_1} u_{i_2} \ldots u_{i_m} = v_1 v_{i_1} v_{i_2} \ldots v_{i_m}$$

- For $U = (11, 1, 0111, 10)$, $V = (1, 111, 10, 0)$, the solution is $3, 2, 2, 4$

$$\begin{aligned} u_1 u_3 u_2 u_2 u_4 &= 1101111110 \\ v_1 v_3 v_2 v_2 v_4 &= 1101111110 \end{aligned}$$

Hecht's Reduction of MPCP to Constant Propagation

Given: An instance of MPCP with $\Sigma = \{0, 1\}$



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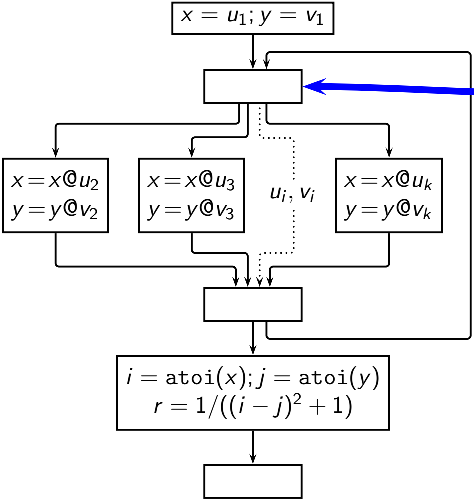

Each block in the loop corresponds to a particular index

Random branching for random selection of index

Hecht's Reduction of MPCP to Constant Propagation

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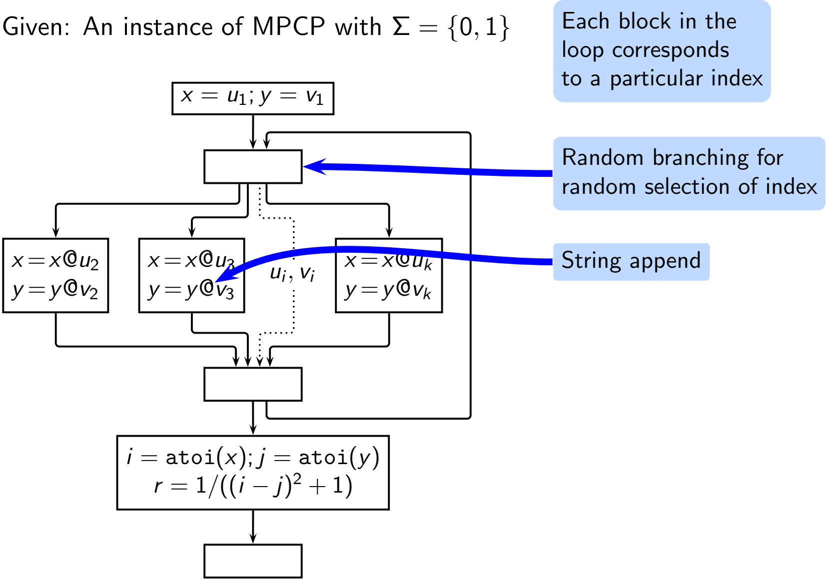
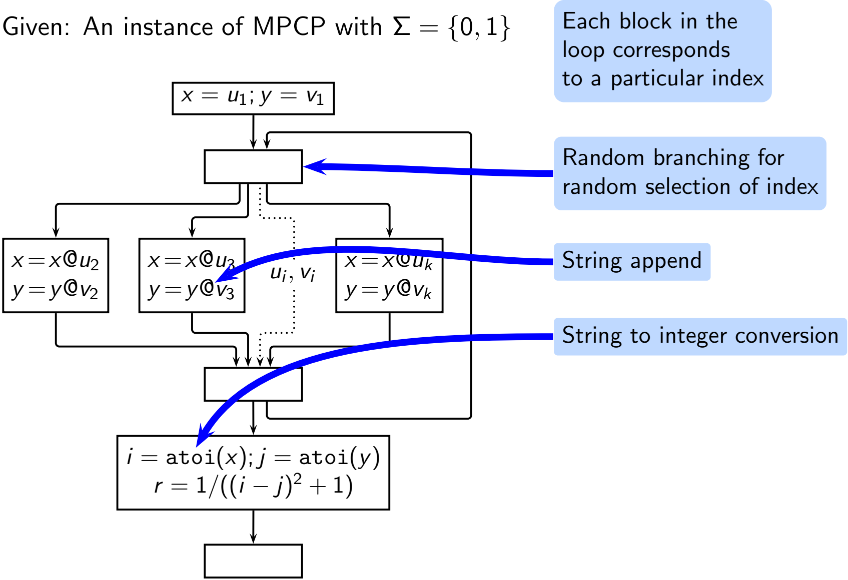


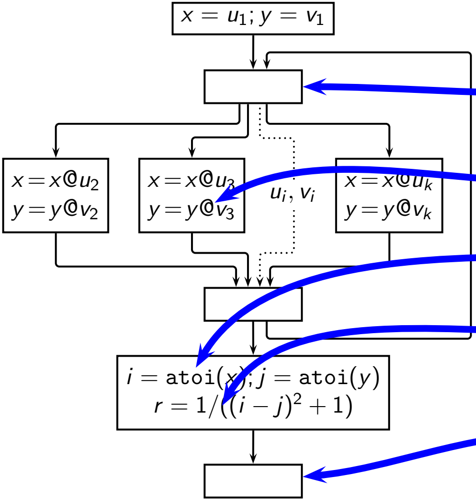
Hecht's Reduction of MPCP to Constant Propagation

Given: An instance of MPCP with $\Sigma = \{0, 1\}$



Hecht's Reduction of MPCP to Constant Propagation

Given: An instance of MPCP with $\Sigma = \{0, 1\}$



Each block in the loop corresponds to a particular index

Random branching for random selection of index

String append

String to integer conversion

Integer division

MoP computation. No merge at intermediate points. Merge only at the point of interest

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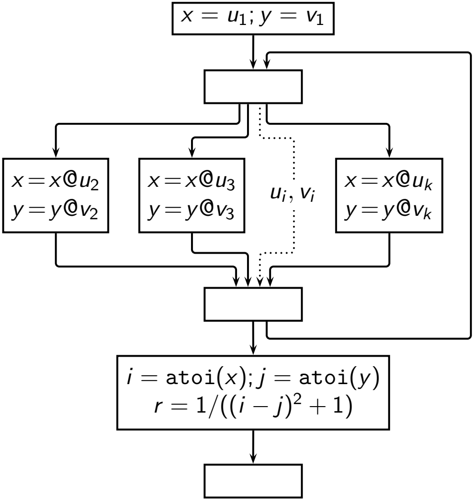

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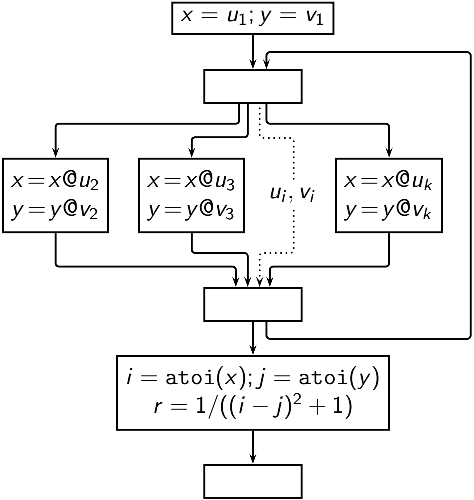

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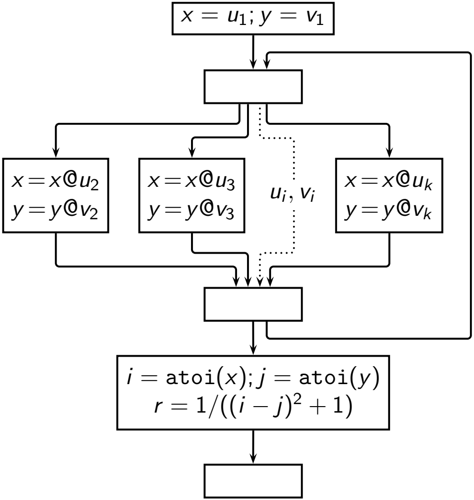

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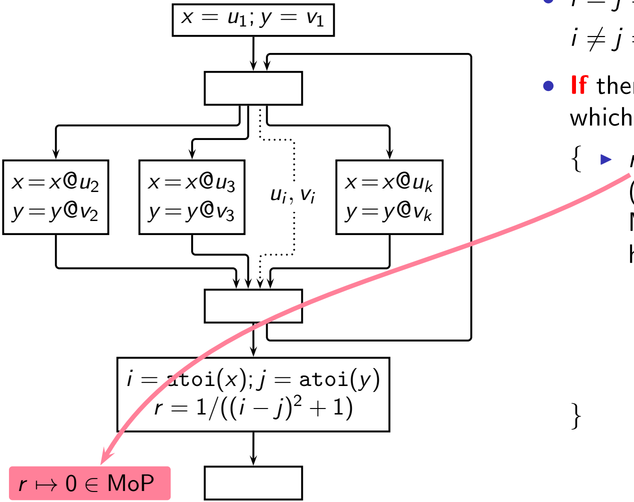

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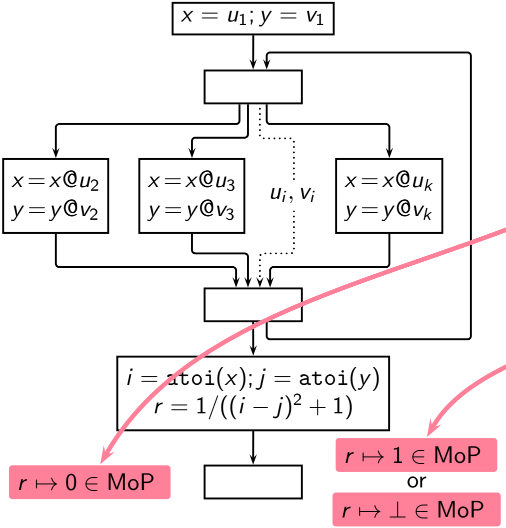

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Then MPCP is decidable

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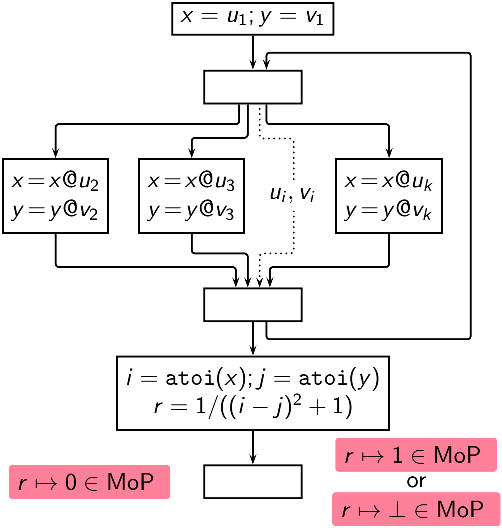

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Hecht's Reduction of MPCP to Constant Propagation

Given: An instance of MPCP with $\Sigma = \{0, 1\}$

- Asserting that no x is equal to y requires us to examine infinitely many (x, y) pairs
- If we keep finding x and y that are unequal, how long do we wait to decide that there is no x that is equal to y?
- In a lucky case we may find an x that is equal to y, but there is no guarantee

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MPCP is not decidable
⇒ Constant Propagation is not decidable

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MPCP is not decidable
$\Rightarrow$ *Constant Propagation is not decidable*

- Descending chains consist of sets of pairs (x, y) with $\top$ as $\emptyset$

  Since there is no bound on the length of x and y, the number of these sets is infinite

  $\Rightarrow$ DCC is violated

The tricky part!!

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Then MPCP is decidable

Is MFP Always Computable?

MFP assignment may not be computable

- if the flow functions are non-monotonic, or
- if some strictly descending chain is not finite

Computability of MFP

- If f is not monotonic, the computation may not converge

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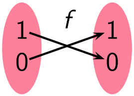

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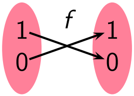

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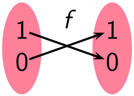

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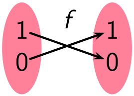

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- Computing MFP iteratively

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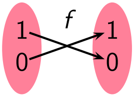

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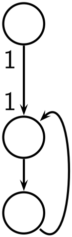

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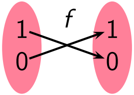

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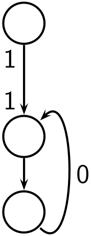

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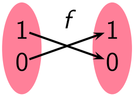

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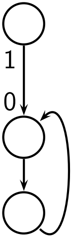

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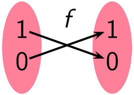

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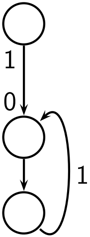

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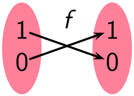

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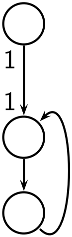

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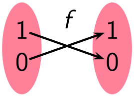

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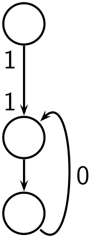

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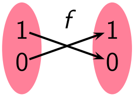

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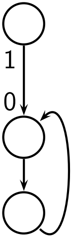

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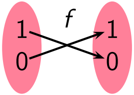

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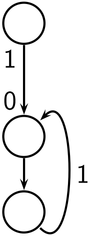

MFP does not exist and is not computable

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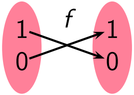

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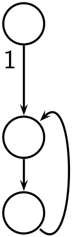

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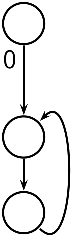

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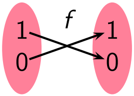

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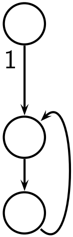

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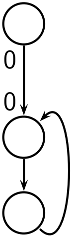

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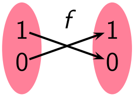

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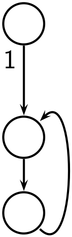

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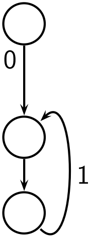

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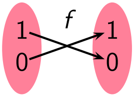

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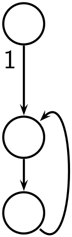

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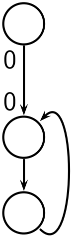

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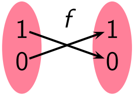

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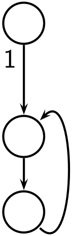

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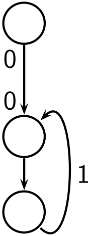

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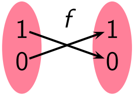

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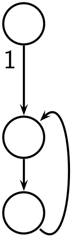
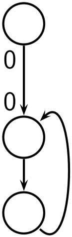
$$MoP = x \sqcap f(x) \sqcap f^2(x) \sqcap f^3(x) \sqcap \ldots \ = \ 0$$

- Computing MFP iteratively

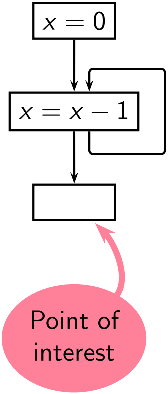


MFP does not exist and is not computable



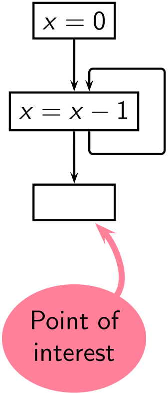
MFP exist and is computable

Computability of MFP



$\sqsubseteq$	$\leq$	$\leq$
$\sqcap$	min	min
Hasse diagram	0 -1 -2 -3 . . .	0 -1 -2 -3 . . . $-\infty$
MFP exists?		
MFP computable?		
MoP exists?		

Computability of MFP



$\sqsubseteq$	$\leq$	$\leq$
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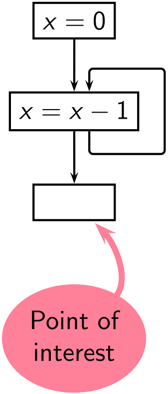
Hasse diagram	0 | -1 | -2 | -3 | . . .	0 | -1 | -2 | -3 | . . . | $-\infty$
MFP exists?	No	
MFP computable?	No	
MoP exists?	No	

Computability of MFP



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MFP exists?	No	Yes
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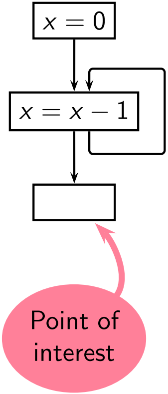
Computability of MFP



$\sqsubseteq$	$\leq$	$\leq$
$\sqcap$	min	min
	0 | -1 ...	0 | -1 ... | $-\infty$
MFP exists?	No	Yes
MFP computable?	No	No
MoP exists?	No	Yes

- Flow functions are monotonic
- Strictly descending chains are not finite

Existence and Computation of the Maximum Fixed Point

If L is a meet semilattice satisfying DCC, $f : L \rightarrow L$ is monotonic, then
$MFP(f) = f^{k+1}(\top) = f^{k}(\top)$ such that $f^{j+1}(\top) \neq f^{j}(\top)$, $j < k$

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Claims being made:

- $\exists k \ s.t. \ f^{k+1}(\top) = f^{k}(\top)$
- Since k is finite, $f^{k}(\top)$ exists and is computable
- $f^{k}(\top)$ is a fixed point
- $f^{k}(\top)$ is a the *maximum* fixed point

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The proof depends on:

- The existence of glb for every pair of values in L
- Finiteness of strictly descending chains
- Monotonicity of f

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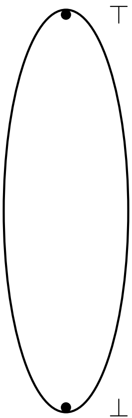

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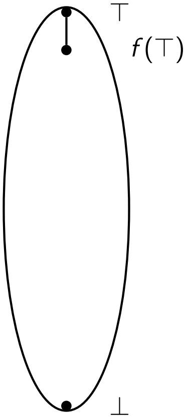

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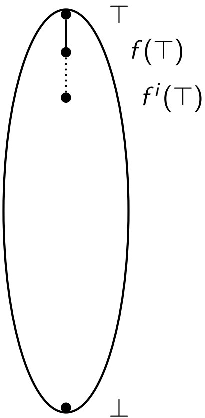

- $\top \sqsupseteq f(\top) \sqsupseteq f^2(\top) \sqsupseteq f^3(\top) \sqsupseteq f^4(\top) \sqsupseteq \ldots$

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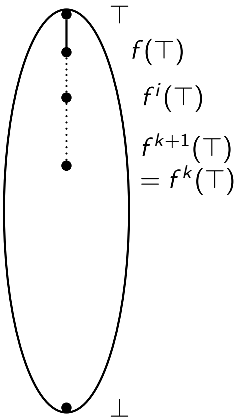

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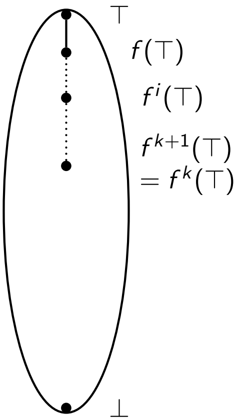

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  Proof strategy: Induction on i for $f^i(\top)$

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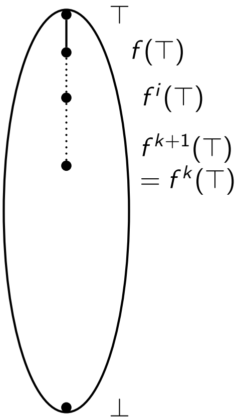

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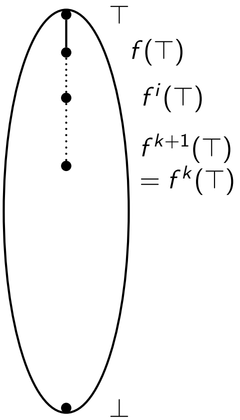

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  - Proof: $f(p) \sqsubseteq f(f^i(\top))$ (f is monotonic)

    $\Rightarrow\ p \sqsubseteq f(f^i(\top))$ ($f(p) = p$)

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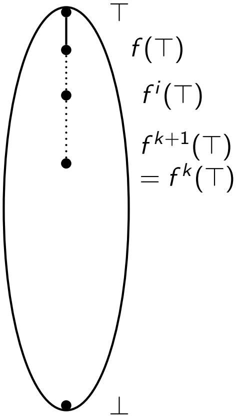

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    $\Rightarrow \quad p \sqsubseteq f(f^i(\top))$ ($f(p) = p$)
    $\Rightarrow \quad p \sqsubseteq f^{i+1}(\top)$
- Since this holds for every p that is a fixed point, $f^{k+1}(\top)$ must be the Maximum Fixed Point

Fixed Points Computation: Flow Functions Vs. Equations

- Recall that

$$MFP(f) = f^{k+1}(\top) = f^{k}(\top) \text{ such that } f^{j+1}(\top) \neq f^{j}(\top),\ j < k.$$

Fixed Points Computation: Flow Functions Vs. Equations

- Recall that

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  - What is f in the above?

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  - Flow function of a block? Which block?

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  - What is f in the above?
  - Flow function of a block? Which block?
- Our method computes the maximum fixed point of data flow equations!
- What is the relation between the maximum fixed point of data flow equations and the MFP defined above?

Fixed Points Computation: Flow Functions Vs. Equations

- Data flow equations for a CFG with N nodes can be written as

$$
\begin{aligned}
In_1 &= BI \\
Out_1 &= f_1(In_1) \\
In_2 &= Out_1 \sqcap \ldots \\
Out_2 &= f_2(In_2) \\
&\ldots \\
In_N &= Out_{N-1} \sqcap \ldots \\
Out_N &= f_N(In_N)
\end{aligned}
$$

Fixed Points Computation: Flow Functions Vs. Equations

- Data flow equations for a CFG with N nodes can be written as

$$
\begin{aligned}
In_1 &= f_{In_1}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle) \\
Out_1 &= f_{Out_1}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle) \\
In_2 &= f_{In_2}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle) \\
Out_2 &= f_{Out_2}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle) \\
& \ldots \\
In_N &= f_{In_N}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle) \\
Out_N &= f_{Out_N}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle)
\end{aligned}
$$

where each flow function is of the form $L \times L \times \ldots \times L \rightarrow L$

Fixed Points Computation: Flow Functions Vs. Equations

- Data flow equations for a CFG with N nodes can be written as

$$
\begin{array}{lll}
\langle In_1, Out_1, \ldots, In_N, Out_N \rangle & = & \langle \quad f_{In_1}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle), \\
 & & \quad\;\; f_{Out_1}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle), \\
 & & \quad\;\; \ldots \\
 & & \quad\;\; f_{In_N}(\langle In_1, Out_1, \ldots, In_N, Out_N \rangle), \\
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\end{array}
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Fixed Points Computation: Flow Functions Vs. Equations

- Data flow equations for a CFG with N nodes can be written as

$$
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& & \qquad f_{Out_1}(\mathcal{X}), \\
& & \qquad \ldots \\
& & \qquad f_{In_N}(\mathcal{X}), \\
& & \qquad f_{Out_N}(\mathcal{X}), \\
& & \rangle
\end{array}
$$

where $\mathcal{X} = \langle In_1, Out_1, \ldots, In_N, Out_N \rangle$

Fixed Points Computation: Flow Functions Vs. Equations

- Data flow equations for a CFG with N nodes can be written as

$$\mathcal{X} = \mathcal{F}(\mathcal{X})$$

where

$$\begin{aligned} \mathcal{X} &= \langle In_1, Out_1, \ldots, In_N, Out_N \rangle \\ \mathcal{F}(\mathcal{X}) &= \langle f_{In_1}(\mathcal{X}), f_{Out_1}(\mathcal{X}), \ldots, f_{In_N}(\mathcal{X}), f_{Out_N}(\mathcal{X}) \rangle \end{aligned}$$

Fixed Points Computation: Flow Functions Vs. Equations

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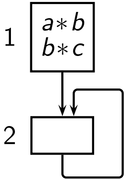
We compute the fixed points of function $\mathcal{F}$ defined above

An Instance of Available Expressions Analysis

Program



- Conventional data flow equations

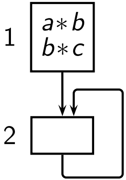
$$In_1 = 00$$
$$Out_1 = 11$$

An Instance of Available Expressions Analysis

- Conventional data flow equations

$$In_1 = 00 \qquad In_2 = Out_1 \cap Out_2$$
$$Out_1 = 11 \qquad Out_2 = In_2$$

Program



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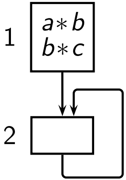

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- Data Flow Equation $\mathcal{X} = \mathcal{F}(\mathcal{X})$ is

$$\mathcal{F}(\langle In_1, Out_1, In_2, Out_2 \rangle) = \langle 00, 11, Out_1 \cap Out_2, In_2 \rangle$$

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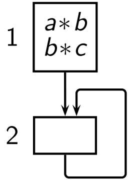

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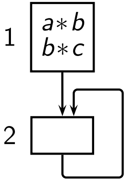
- The maximum fixed point assignment is

$$\mathcal{F}(\langle 11, 11, 11, 11 \rangle) = \langle 00, 11, 11, 11 \rangle$$

An Instance of Available Expressions Analysis

Program



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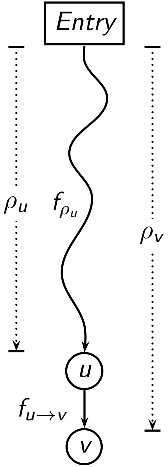
$$\mathcal{F}(\langle 11, 11, 11, 11 \rangle) = \langle 00, 11, 11, 11 \rangle$$

- The minimum fixed point assignment is

$$\mathcal{F}(\langle 00, 00, 00, 00 \rangle) = \langle 00, 11, 00, 00 \rangle$$

Safety of FP Assignment: FP $\sqsubseteq$ MoP



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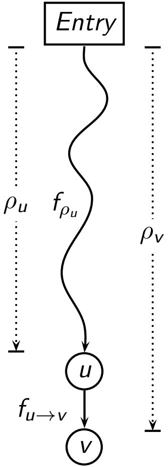
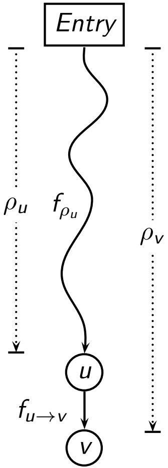


- $MoP(v) \quad = \bigsqcap_{\rho \in Paths(v)} f_\rho(BI)$

Safety of FP Assignment: FP $\sqsubseteq$ MoP



- $MoP(v) = \bigsqcap_{\rho \in Paths(v)} f_\rho(BI)$
- Proof Obligation: $\forall \rho_v \; FP(v) \sqsubseteq f_{\rho_v}(BI)$

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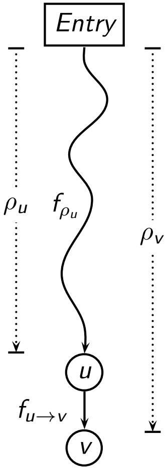

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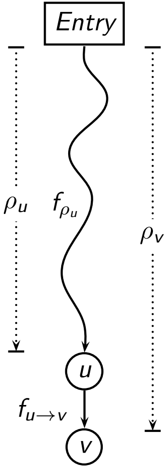

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- Proof Outline: Induction on path length

  Base case: Path of length 0

  $FP(Entry) = MoP(Entry) = BI$

  Inductive hypothesis: Assume it holds for paths consisting of k edges (say at u)

$$\begin{array}{llr} & FP(u) \sqsubseteq f_{\rho_u}(BI) & \text{(Inductive hypothesis)} \\ & FP(v) \sqsubseteq f_{u\to v}(FP(u)) & \text{(Claim 1)} \\ \Rightarrow & FP(v) \sqsubseteq f_{u\to v}(f_{\rho_u}(BI)) & \\ \Rightarrow & FP(v) \sqsubseteq f_{\rho_v}(BI) & \end{array}$$

This holds for every *FP* an hence for *MFP* also

Part 8

Theoretical Abstractions: A Summary

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Necessary and sufficient conditions for designing a data flow framework

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- A meet semilattice satisfying dcc

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- A meet semilattice satisfying dcc

- A function space
  - Monotonic functions

Theoretical Abstractions: A Summary

Necessary and sufficient conditions for designing a data flow framework

- A meet semilattice satisfying dcc
  - Meet: commutative, associative, and idempotent
  - Partial order: reflexive, transitive, and antisymmetric
  - Existence of $\bot$
- A function space
  - Monotonic functions

Theoretical Abstractions: A Summary

Necessary and sufficient conditions for designing a data flow framework

- A meet semilattice satisfying dcc
  - Meet: commutative, associative, and idempotent
  - Partial order: reflexive, transitive, and antisymmetric
  - Existence of $\bot$
- A function space
  - Existence of the identity function
  - Closure under composition
  - Monotonic functions

Part 9

Performing Data Flow Analysis

Performing Data Flow Analysis

- Algorithms for computing MFP solution
- Complexity of data flow analysis
- Factor affecting the complexity of data flow analysis

Iterative Methods of Performing Data Flow Analysis

Successive recomputation after conservative initialization ($\top$)

- *Round Robin*. Repeated traversals over nodes in a fixed order

  Termination : After values stabilise

  + Simplest to understand and implement
  − May perform unnecessary computations

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Our examples use this method

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- *Round Robin*. Repeated traversals over nodes in a fixed order

  Termination : After values stabilise

  + Simplest to understand and implement
  − May perform unnecessary computations

Our examples use this method

- *Work List*. Dynamic list of nodes which need recomputation

  Termination : When the list becomes empty

  + Demand driven. Avoid unnecessary computations
  − Overheads of maintaining work list

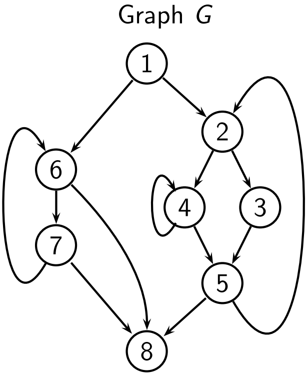
Elimination Methods of Performing Data Flow Analysis

Delayed computations of dependent data flow values of dependent nodes

Find suitable single-entry regions

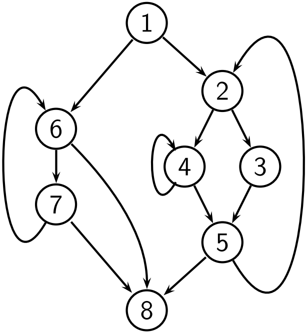
- *Interval Based Analysis*. Uses graph partitioning
- T_1, T_2 *Based Analysis*. Uses graph parsing

Classification of Edges in a Graph

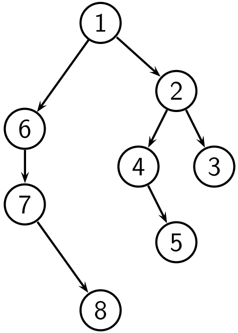


Classification of Edges in a Graph

Graph G



A depth first spanning tree of G



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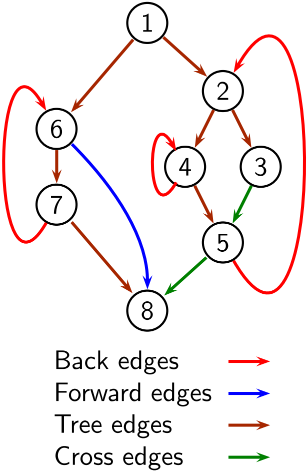

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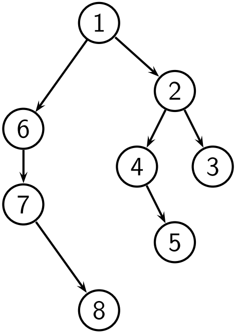

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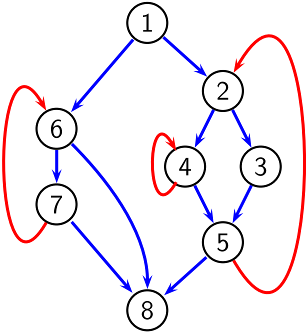
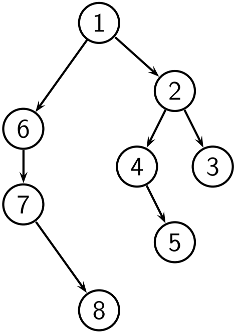


Back edges →

Forward edges →

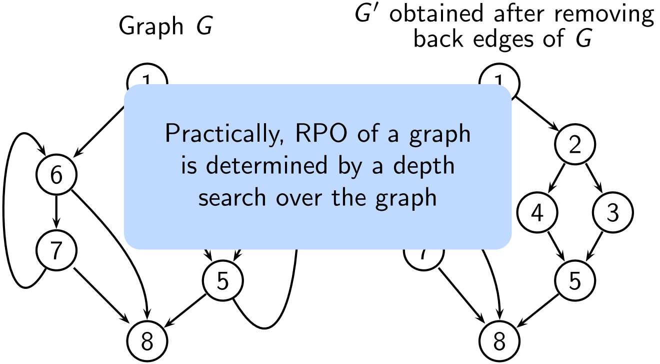
A depth first spanning tree of G



For data flow analysis, we club *tree*, *forward*, and *cross* edges into *forward* edges. Thus we have just forward or back edges in a control flow graph

Reverse Post Order Traversal

- A reverse post order (rpo) is a topological sort of the graph obtained after removing back edges



- Some possible RPOs for G are: $(1, 2, 3, 4, 5, 6, 7, 8)$, $(1, 6, 7, 2, 3, 4, 5, 8)$, $(1, 6, 2, 7, 4, 3, 5, 8)$, and $(1, 2, 6, 7, 3, 4, 5, 8)$

Round Robin Iterative Algorithm

$In_0 = BI$
for all $j \neq 0$ **do**
    $In_j = \top$
$change = true$
while $change$ **do**
{ $change = false$
    for $j = 1$ to $N - 1$ **do**
    { $temp = \prod_{p \in pred(j)} f_p(In_p)$
        if $temp \neq In_j$ **then**
        { $In_j = temp$
            $change = true$
        }
    }
}

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  Works fine for unidirectional frameworks
- $\top$ is the identity of $\sqcap$ (line 3)
- Reverse postorder (rpo) traversal for efficiency (line 7)
- rpo traversal AND no loops

  $\Rightarrow$ no need of initialization

Complexity of Round Robin Iterative Algorithm

- Unidirectional bit vector frameworks
  - Construct a spanning tree T of G to identify postorder traversal
  - Traverse G in reverse postorder for forward problems and Traverse G in postorder for backward problems
  - Depth $d(G, T)$: Maximum number of back edges in any acyclic path

Task	Number of iterations
First computation of *In* and *Out*	1
Convergence (until *change* remains true)	$d(G, T)$
Verifying convergence (*change* becomes false)	1

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Verifying convergence (*change* becomes false)	1

- What about bidirectional bit vector frameworks?
- What about other frameworks?

Example C Program with d(G,T) = 2

```
void fun(int m, int n)
{
   int i,j,a,b,c;
   c=a+b;
   i=0;
   while(i<m)
   {
        j=0;
        while(j<n)
        {
           a=i+j;
           j=j+1;
        }
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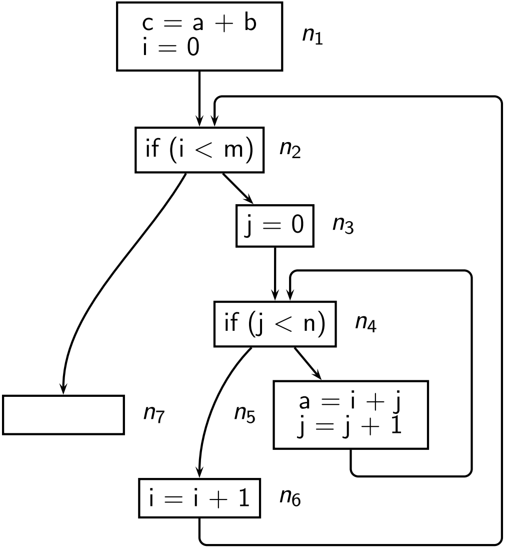

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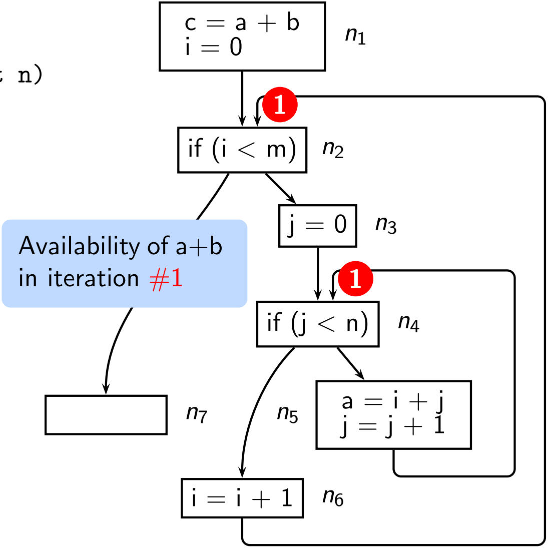

Example C Program with d(G,T) = 2

```c
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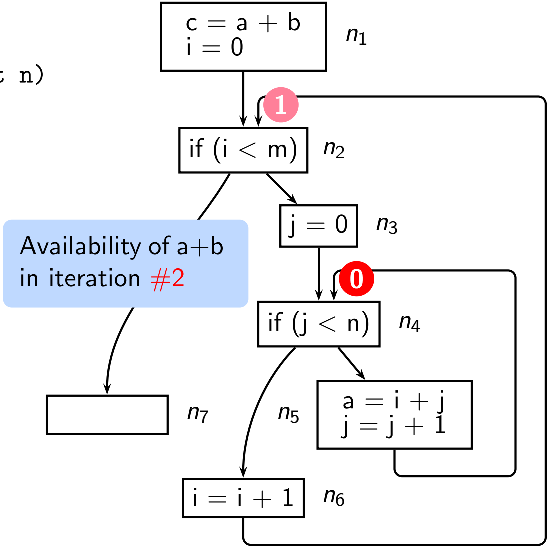

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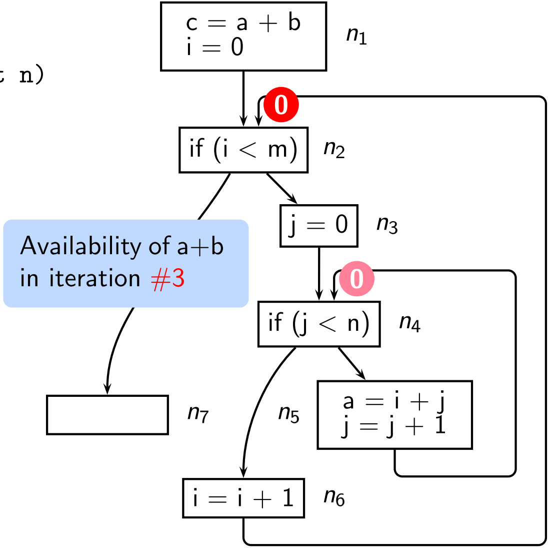

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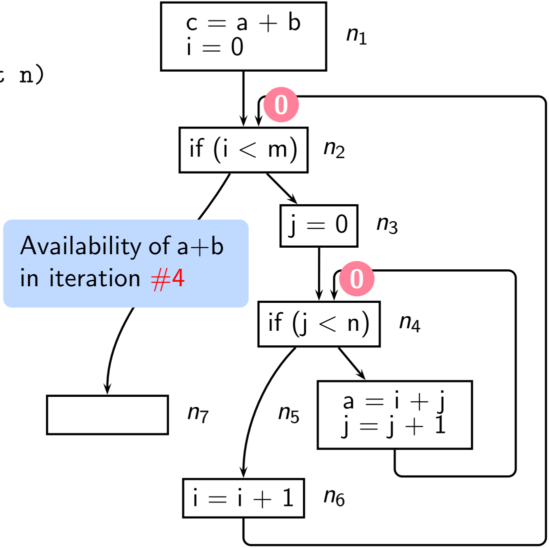

3 + 1 iterations for available expressions analysis

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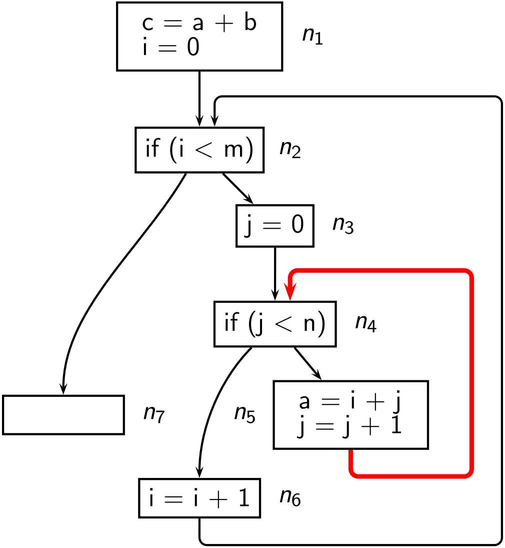

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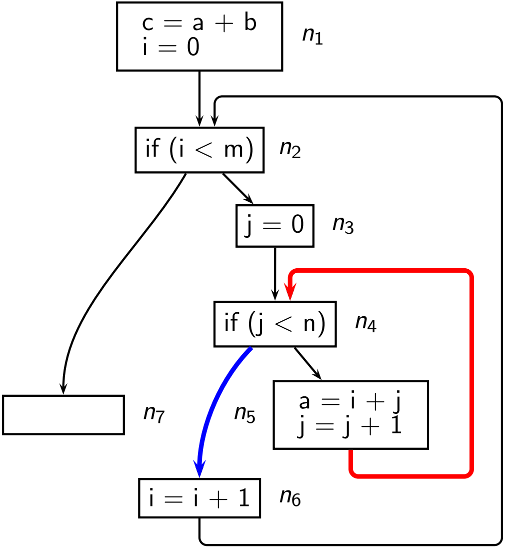

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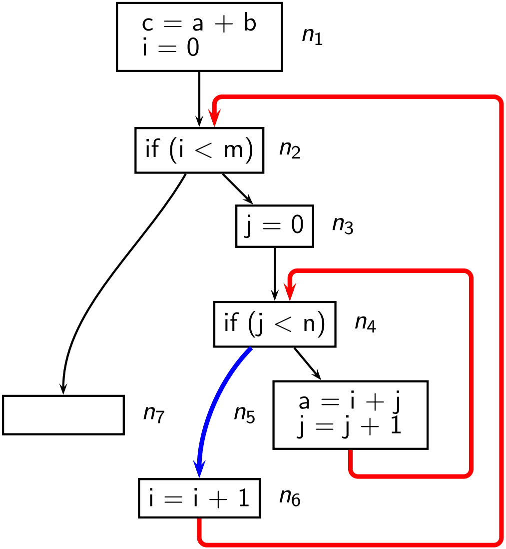
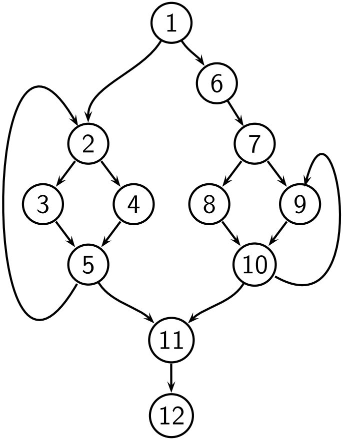


Complexity of Bidirectional Bit Vector Frameworks

Example: Consider the following CFG for PRE



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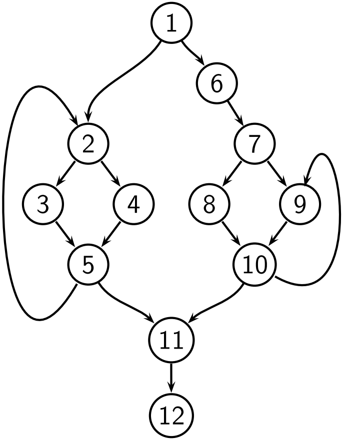

- Node numbers are in reverse post order

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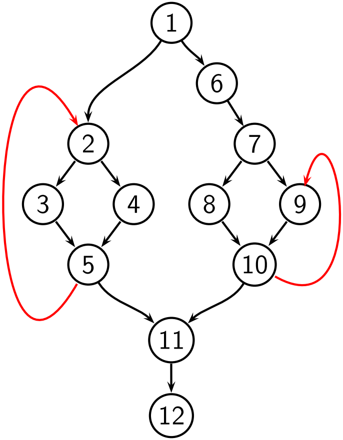

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- Back edges in the graph are $n_5 \rightarrow n_2$ and $n_{10} \rightarrow n_9$

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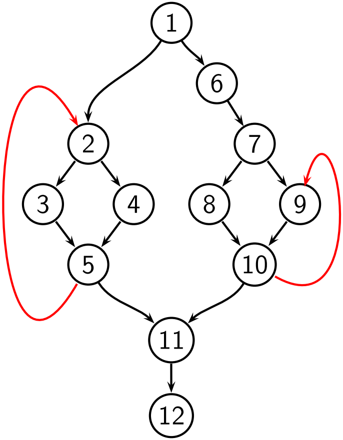

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- Back edges in the graph are $n_5 \rightarrow n_2$ and $n_{10} \rightarrow n_9$
- $d(G, T) = 1$

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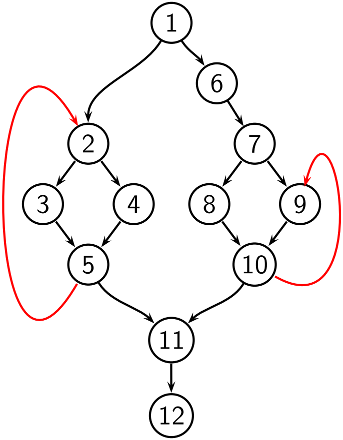
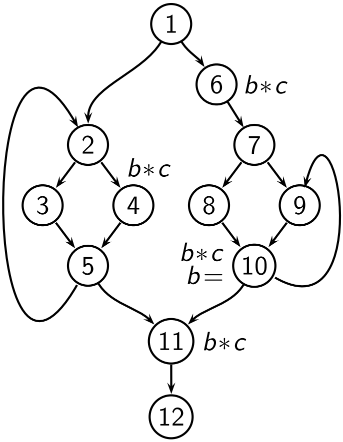


- Node numbers are in reverse post order
- Back edges in the graph are $n_5 \rightarrow n_2$ and $n_{10} \rightarrow n_9$
- $d(G, T) = 1$
- Actual iterations : 5

Complexity of Bidirectional Bit Vector Frameworks



	Pairs of *Out*,*In* Values							
	Initia-lization	Changes in Iterations					Final values & transformation	
		#1	#2	#3	#4	#5		
	O,I	O,I	O,I	O,I	O,I	O,I	O,I	
12	0,1							
11	1,1							
10	1,1							
9	1,1							
8	1,1							
7	1,1							
6	1,1							
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4	1,1							
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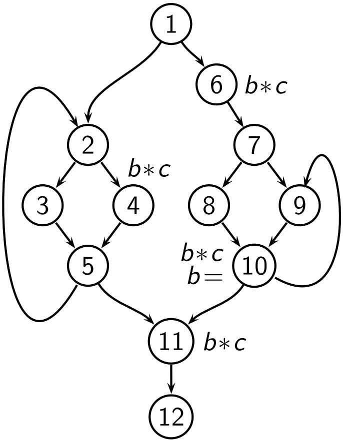

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12	0,1	0,0						
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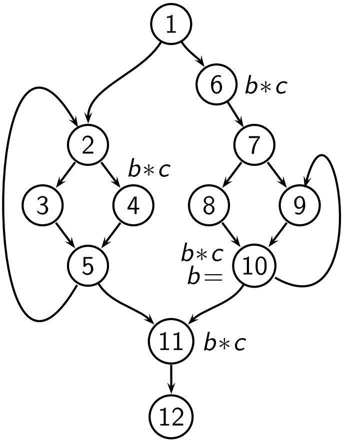

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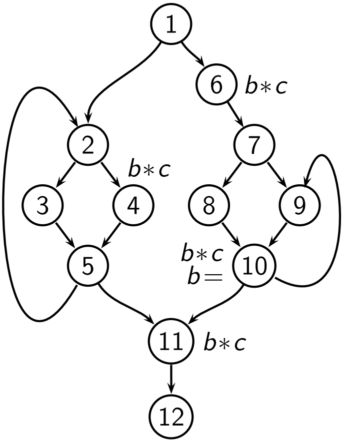

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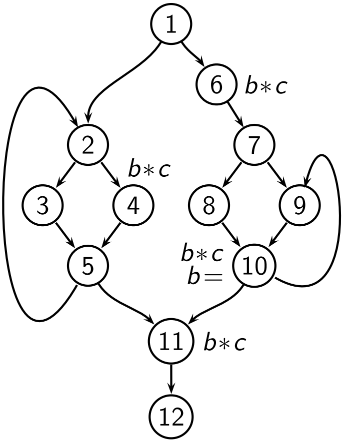

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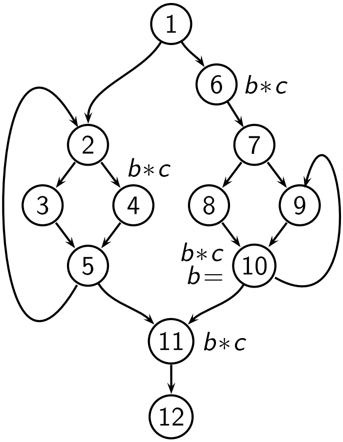

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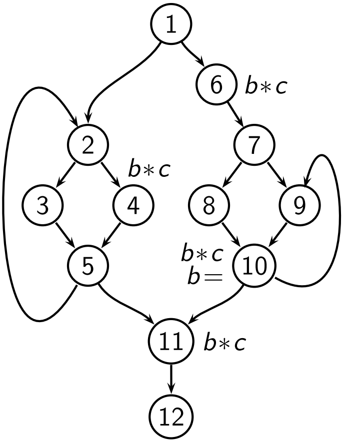

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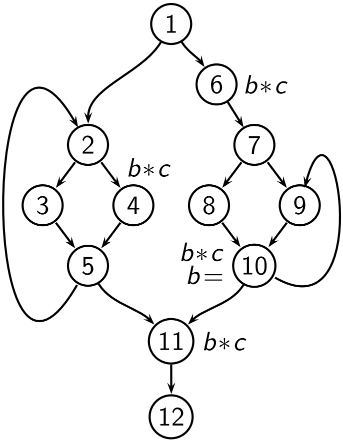
	O,I	O,I	O,I	O,I	O,I	O,I	O,I	
12	0,1	0,0					0,0	
11	1,1	0,1			0,0		0,0	
10	1,1				0,1		0,1	
9	1,1				1,0		1,0	
8	1,1					1,0	1,0	
7	1,1				0,0		0,0	
6	1,1	1,0			0,0		0,0	
5	1,1			0,0			0,0	
4	1,1			0,1	0,0		0,0	
3	1,1			0,0			0,0	
2	1,1		1,0	0,0			0,0	
1	1,1	0,0					0,0	

Complexity of Bidirectional Bit Vector Frameworks



	Pairs of *Out*,*In* Values							
	Initia-lization	Changes in Iterations					Final values & transformation	
		#1	#2	#3	#4	#5		
	O,I	O,I	O,I	O,I	O,I	O,I	O,I	
12	0,1	0,0					0,0	
11	1,1	0,1			0,0		0,0	
10	1,1				0,1		0,1	Delete
9	1,1				1,0		1,0	Insert
8	1,1					1,0	1,0	Insert
7	1,1				0,0		0,0	
6	1,1	1,0			0,0		0,0	
5	1,1			0,0			0,0	
4	1,1			0,1	0,0		0,0	
3	1,1			0,0			0,0	
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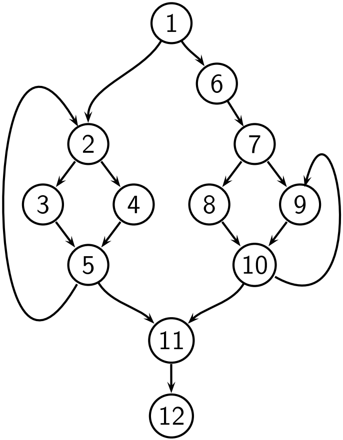
An Example of Information Flow in Our PRE Analysis



- $PavIn_6$ becomes 0 in the first iteration
- This cause many all other values to become 0
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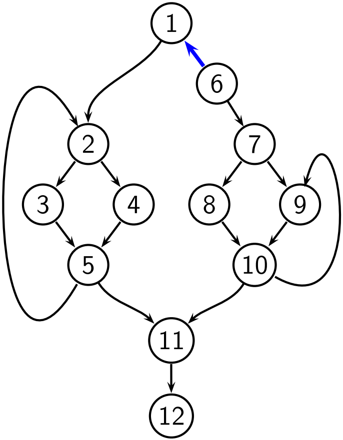

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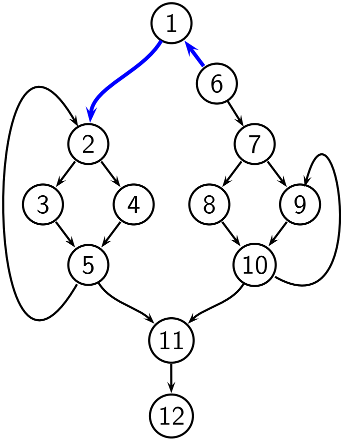

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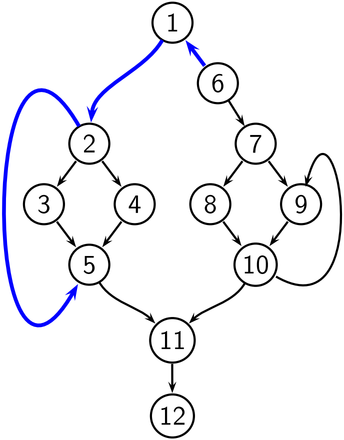

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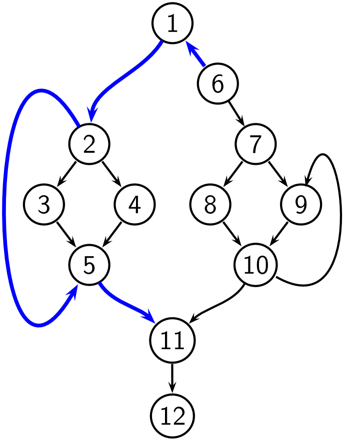

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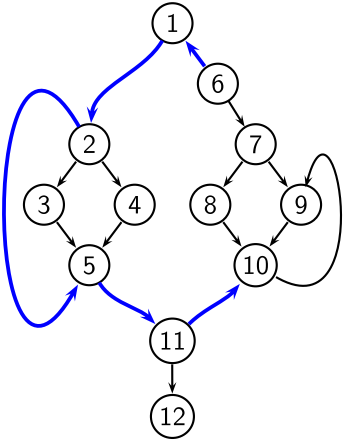

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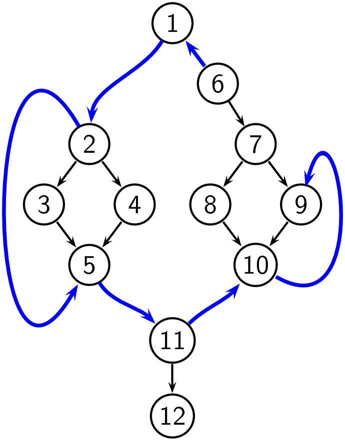

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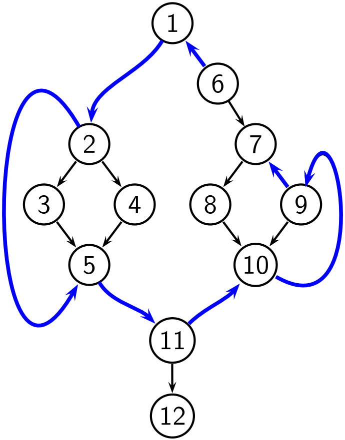

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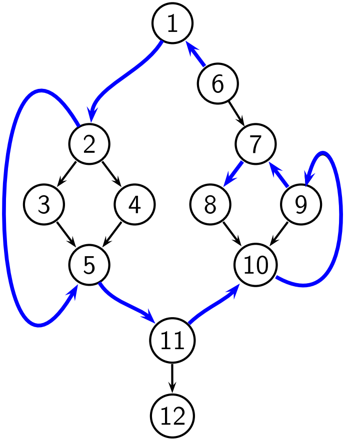

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Information Flow and Information Flow Paths

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- *Information flow path*

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    Change from $\top$ to a non-$\top$ due to local effect (i.e. $f(\top) \neq \top$)
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    Change from x to y such that $y \sqsubseteq x$ due to global effect

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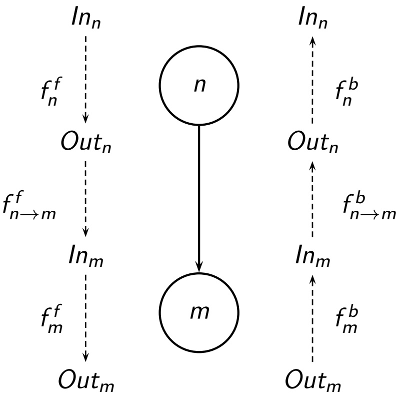
    Change from x to y such that $y \sqsubseteq x$ due to global effect
- Information flow path (ifp) need not be a graph theoretic path

Edge and Node Flow Functions



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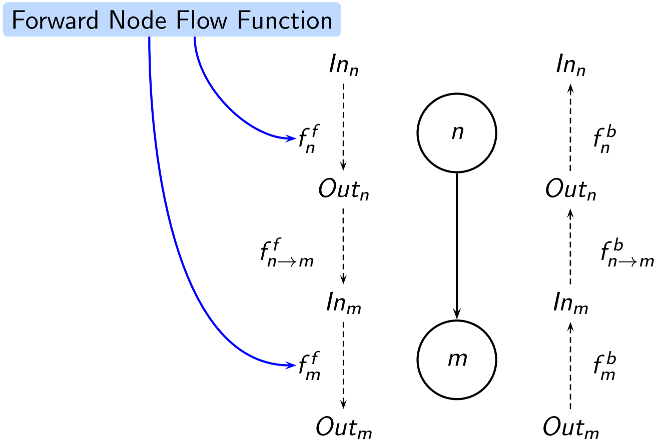

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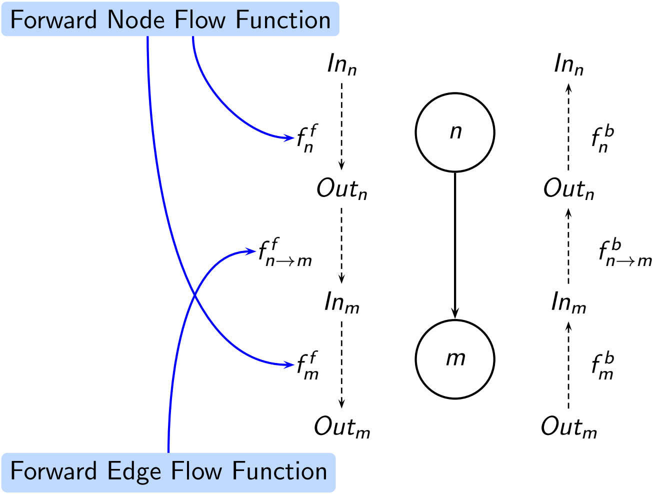

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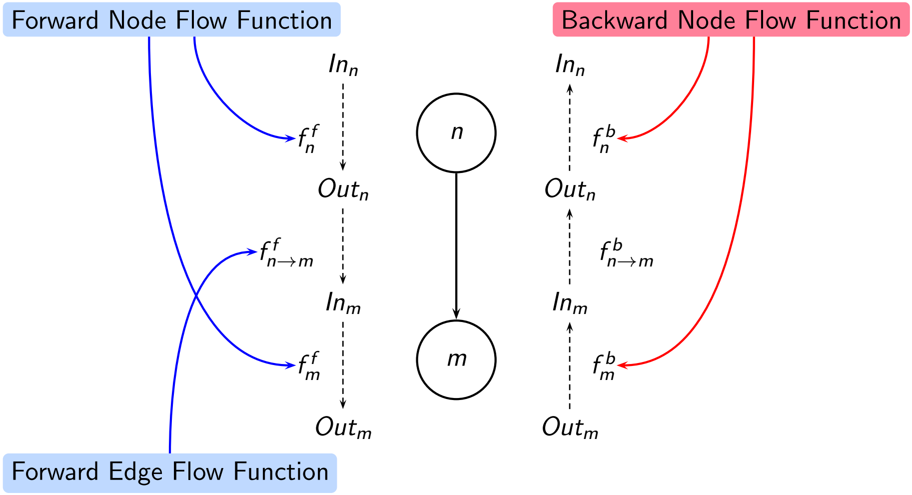

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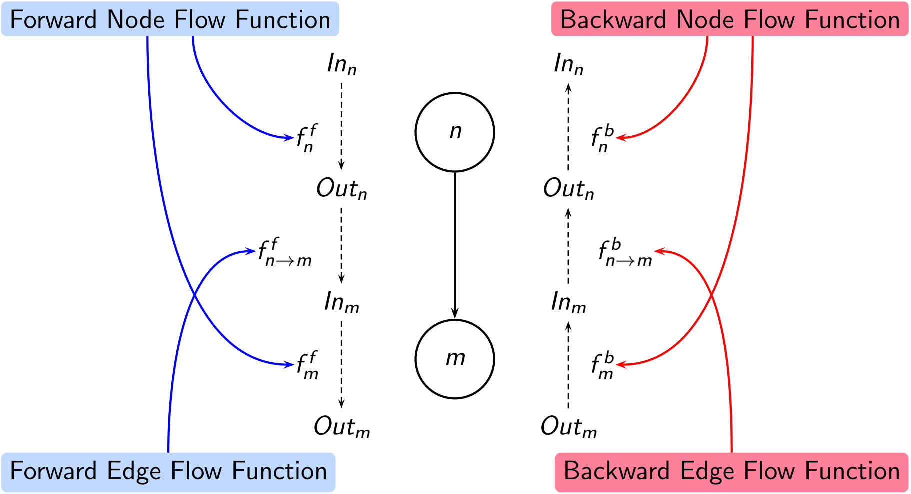


General Data Flow Equations

$$In_n = \begin{cases} BI_{Start} \sqcap f_n^b(Out_n) & n = Start \\ \left(\bigsqcap_{m \in pred(n)} f_{m\rightarrow n}^f(Out_m) \right) \sqcap f_n^b(Out_n) & \text{otherwise} \end{cases}$$

$$Out_n = \begin{cases} BI_{End} \sqcap f_n^f(In_n) & n = End \\ \left(\bigsqcap_{m \in succ(n)} f_{m\rightarrow n}^b(In_m) \right) \sqcap f_n^f(In_n) & \text{otherwise} \end{cases}$$

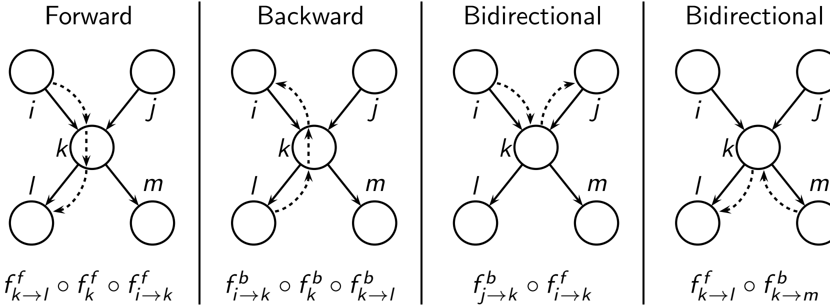
- Edge flow functions are typically identity

$$\forall x \in L,\ f(x) = x$$

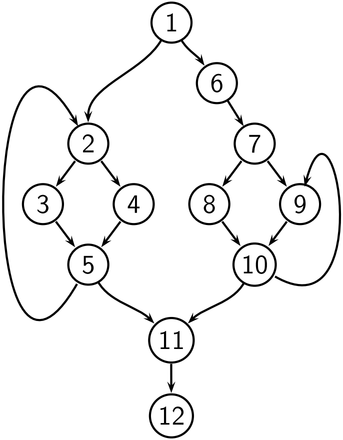
- If particular flows are absent, the corresponding flow functions are

$$\forall x \in L,\ f(x) = \top$$

Modelling Information Flows Using Edge and Node Flow Functions



Information Flow Paths in PRE



- Information could flow along arbitrary paths

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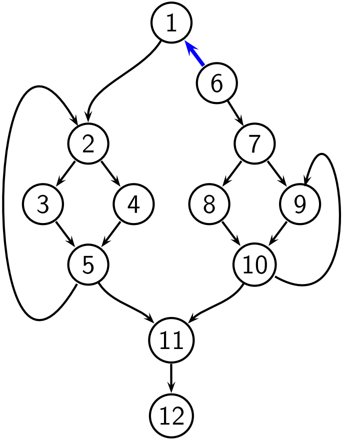

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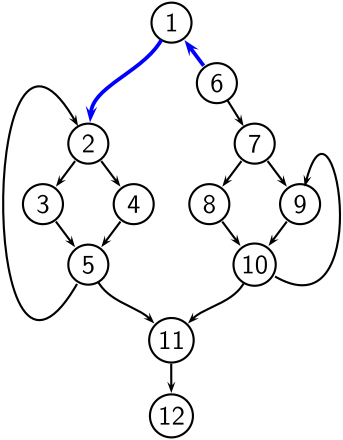

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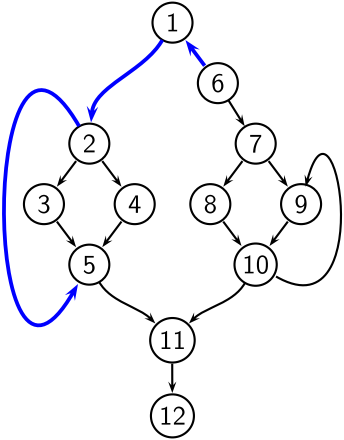

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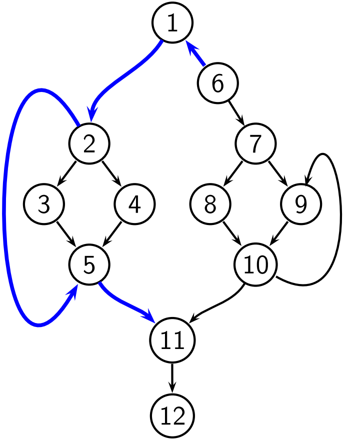

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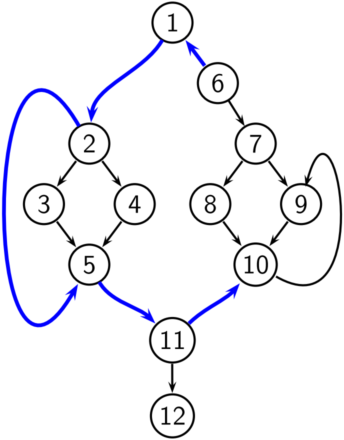

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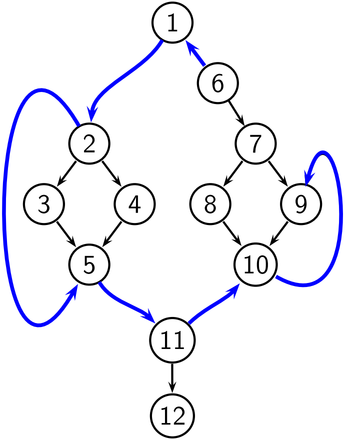

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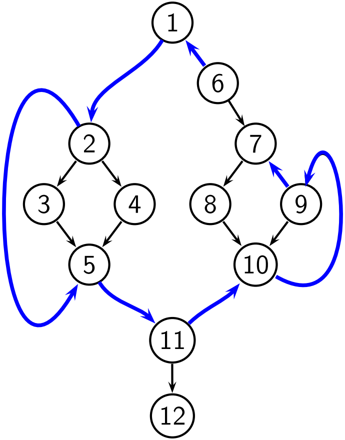

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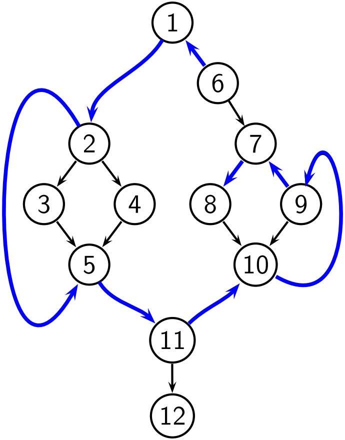

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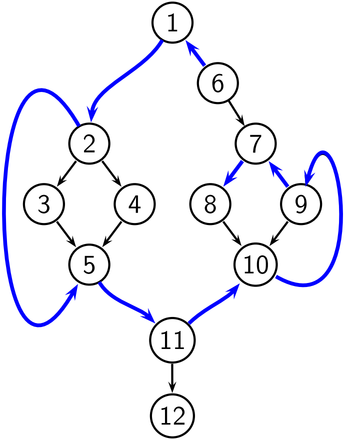

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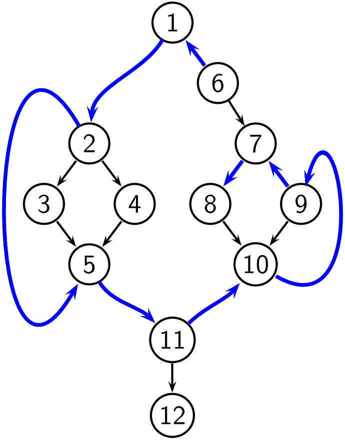

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- Not related to depth (1)

Complexity of Worklist Algorithms for Bit Vector Frameworks

- Assume n nodes and r entities
- Total number of data flow values $= 2 \cdot n \cdot r$
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- *Must be same for both unidirectional and bidirectional frameworks*
  (Number of data flow values does not change!)

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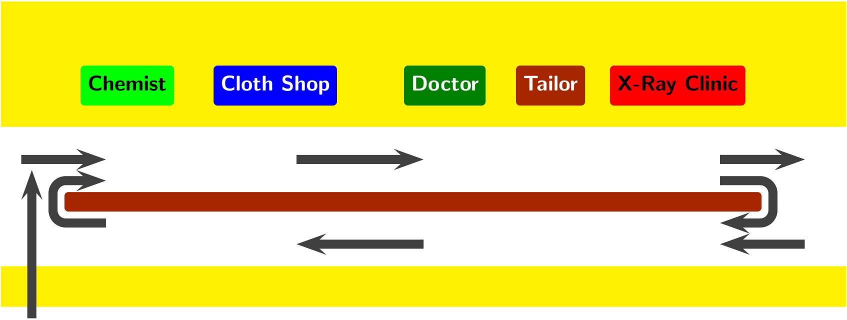
Complexity of Round Robin Iterative Method



- Buy OTC (Over-The-Counter) medicine   No U-Turn   1 Trip

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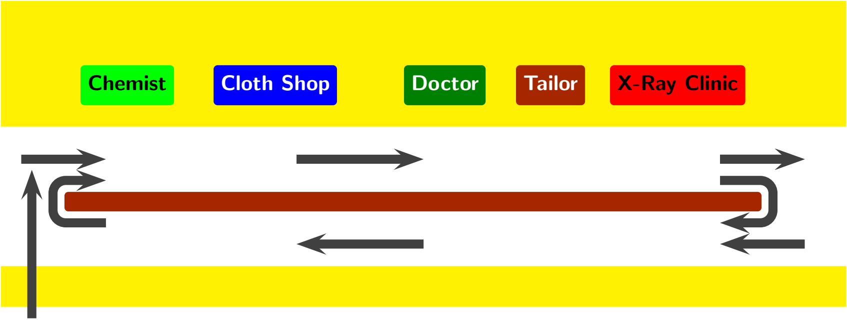

• Buy OTC (Over-The-Counter) medicine	No U-Turn	1 Trip
• Buy cloth. Give it to the tailor for stitching	No U-Turn	1 Trip

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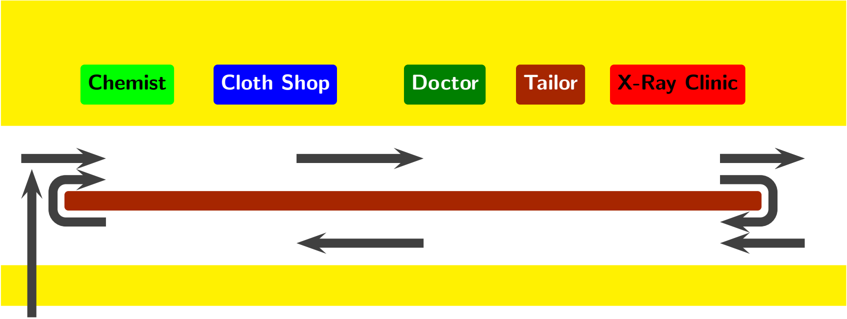

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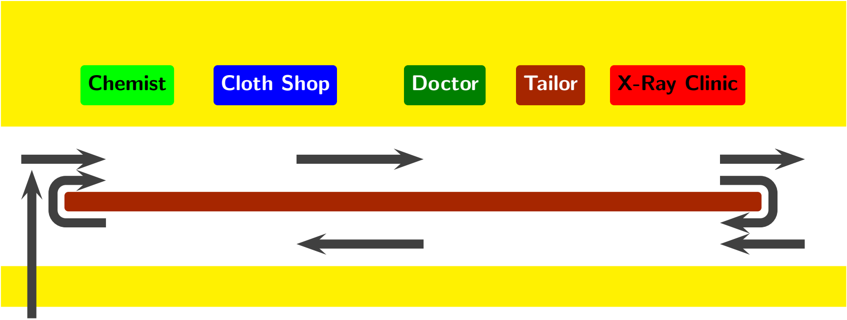

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• Buy medicine with doctor's prescription The diagnosis requires X-Ray	2 U-Turns	3 Trips

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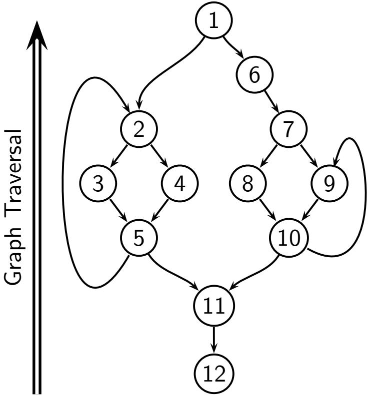
- Width + 1 iterations are sufficient to converge on MFP solution

  (1 additional iteration may be required for verifying convergence)

Complexity of Bidirectional Bit Vector Frameworks



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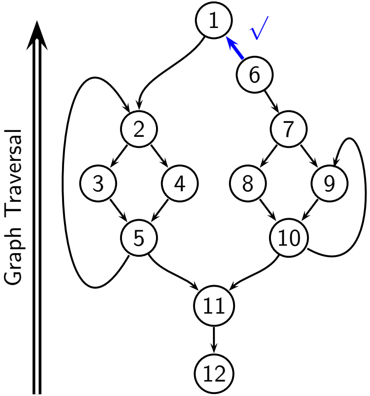

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- Max. Incompatible edge traversals
  = *Width* of the graph = **0?**
- Maximum number of traversals =
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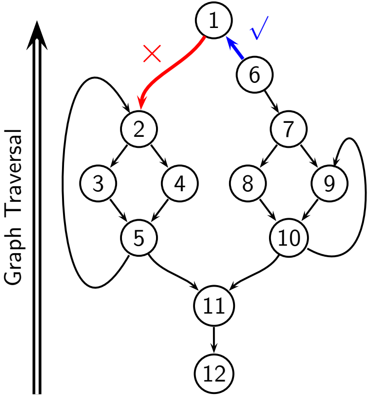

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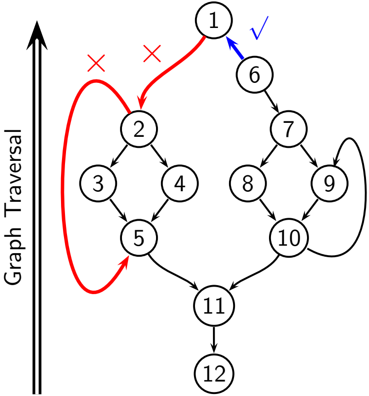

- **Every "incompatible" edge traversal**
  $\Rightarrow$ One additional graph traversal
- Max. Incompatible edge traversals
  $=$ *Width* of the graph $=$ **2?**
- Maximum number of traversals $=$
  $1 +$ Max. incompatible edge traversals

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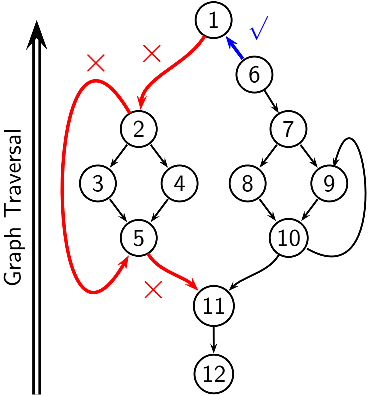
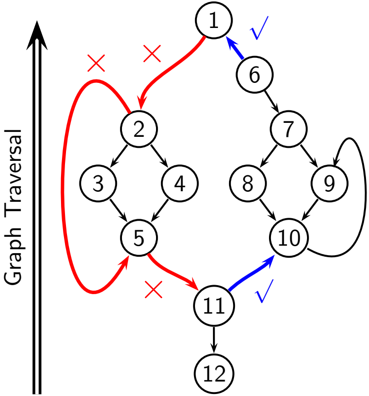


- **Every "incompatible" edge traversal ⇒ One additional graph traversal**
- Max. Incompatible edge traversals = *Width* of the graph = **3?**
- Maximum number of traversals = 1 + Max. incompatible edge traversals

Complexity of Bidirectional Bit Vector Frameworks



- **Every "incompatible" edge traversal $\Rightarrow$ One additional graph traversal**
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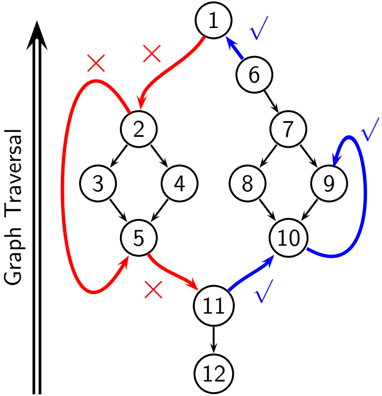
- Maximum number of traversals $=$ $1 +$ Max. incompatible edge traversals

Complexity of Bidirectional Bit Vector Frameworks



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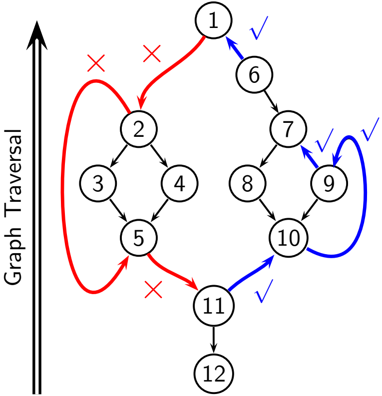
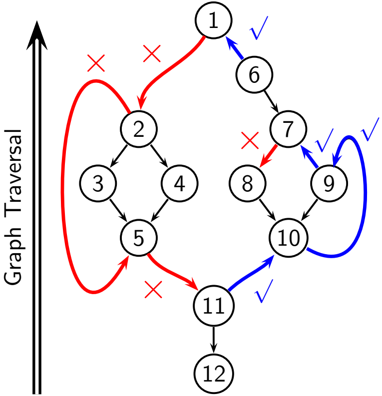
  1 + Max. incompatible edge traversals

Complexity of Bidirectional Bit Vector Frameworks



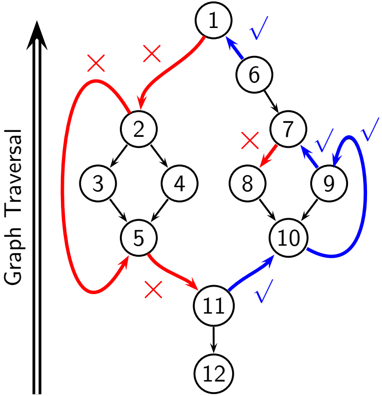
- Every "incompatible" edge traversal
  $\Rightarrow$ **One additional graph traversal**
- Max. Incompatible edge traversals
  $=$ *Width* of the graph $=$ **3?**
- Maximum number of traversals $=$
  $1 +$ Max. incompatible edge traversals

Complexity of Bidirectional Bit Vector Frameworks



- Every "incompatible" edge traversal
  $\Rightarrow$ **One additional graph traversal**
- Max. Incompatible edge traversals
  = *Width* of the graph = **4**
- Maximum number of traversals =
  1 + Max. incompatible edge traversals

Complexity of Bidirectional Bit Vector Frameworks



- Every "incompatible" edge traversal
  ⇒ **One additional graph traversal**
- Max. Incompatible edge traversals
  = *Width* of the graph = **4**
- Maximum number of traversals =
  1 + 4 = 5

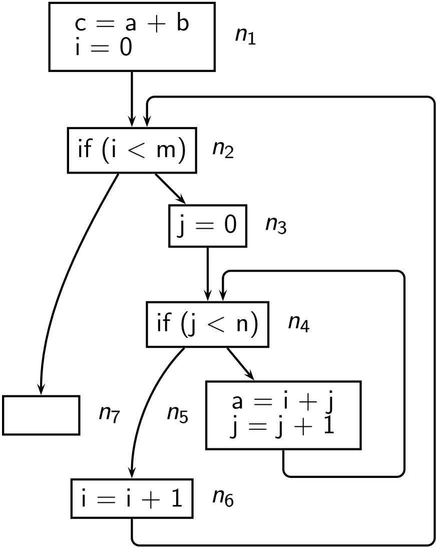
Width Subsumes Depth

- Depth is applicable only to unidirectional data flow frameworks
- Width is applicable to both unidirectional and bidirectional frameworks
- For a given graph for a unidirectional bit vector framework,
  Width $\leq$ Depth
  Width provides a tighter bound

Comparison Between Width and Depth

- Depth is purely a graph theoretic property whereas width depends on control flow graph as well as the data framework
- Comparison between width and depth is meaningful only
  - For unidirectional frameworks
  - When the direction of traversal for computing width is the natural direction of traversal
- Since width excludes bypassed path segments, width can be smaller than depth

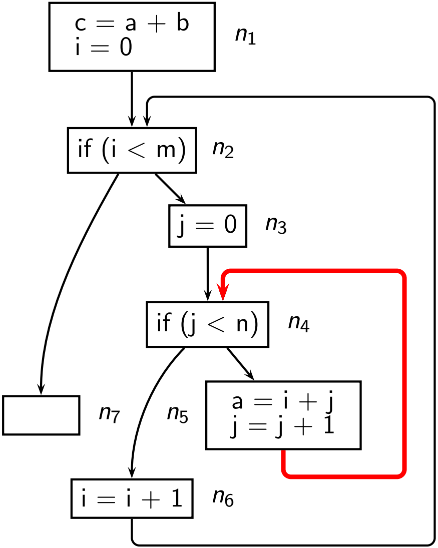
Width and Depth



Assuming reverse postorder traversal for available expressions analysis

- Depth = 2

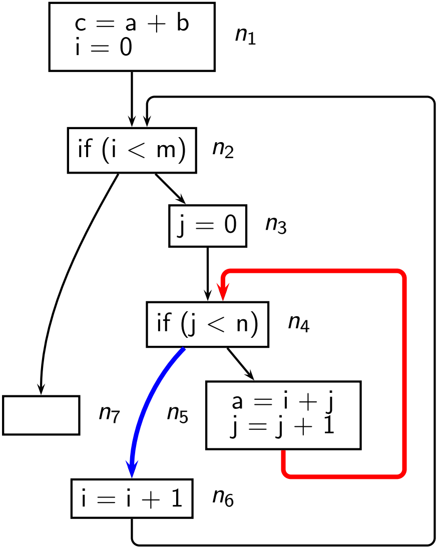
Width and Depth



Assuming reverse postorder traversal for available expressions analysis

- Depth = 2
- Information generation point n_5 kills expression "a + b"

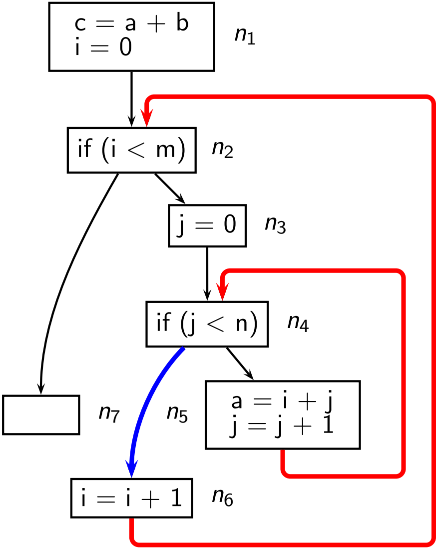
Width and Depth



Assuming reverse postorder traversal for available expressions analysis

- Depth = 2
- Information generation point
  n_5 kills expression "a + b"
- Information propagation path
  $n_5 \rightarrow n_4 \rightarrow n_6 \rightarrow n_2$
  No Gen or Kill for "a + b" along this path

Width and Depth



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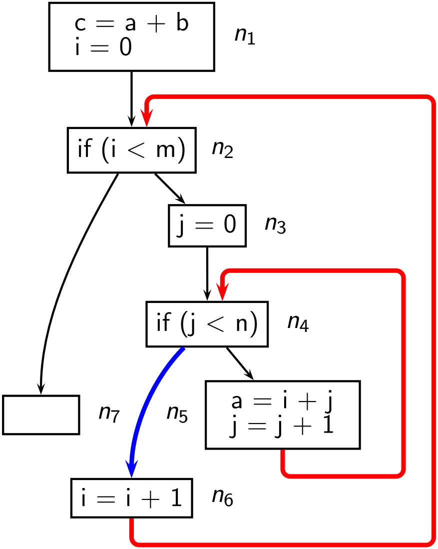

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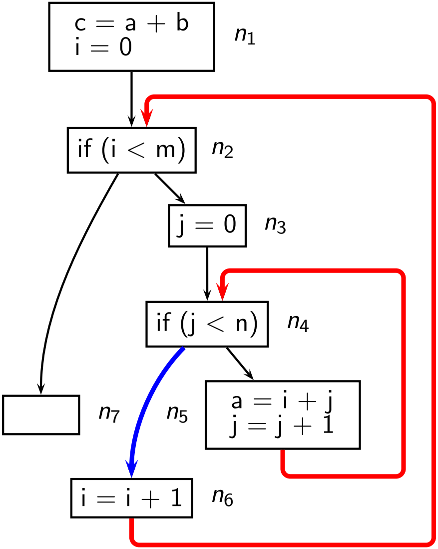
  No Gen or Kill for "a + b" along this path
- Width = 2
- What about "j + 1"?

Width and Depth



Assuming reverse postorder traversal for available expressions analysis

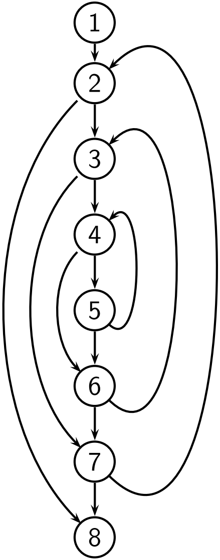
- Depth = 2
- Information generation point

  n_5 kills expression "a + b"
- Information propagation path

  $n_5 \rightarrow n_4 \rightarrow n_6 \rightarrow n_2$

  No Gen or Kill for "a + b" along this path
- Width = 2
- What about "j + 1"?
- Not available on entry to the loop

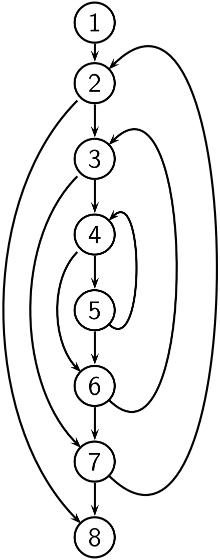
Width and Depth



Structures resulting from repeat-until loops with premature exits

- Depth = 3

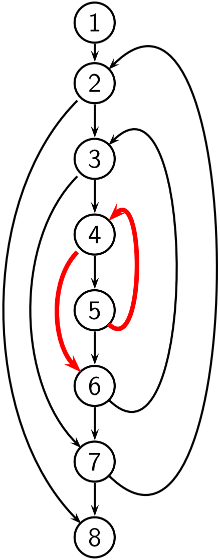
Width and Depth



Structures resulting from repeat-until loops with premature exits

- Depth = 3
- However, any unidirectional bit vector analysis is guaranteed to converge in 2 + 1 iterations

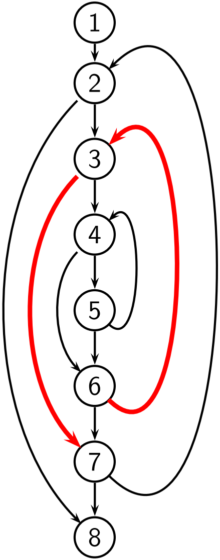
Width and Depth



Structures resulting from repeat-until loops with premature exits

- Depth = 3
- However, any unidirectional bit vector analysis is guaranteed to converge in 2 + 1 iterations
- ifp $5 \rightarrow 4 \rightarrow 6$ is bypassed by the edge $5 \rightarrow 6$

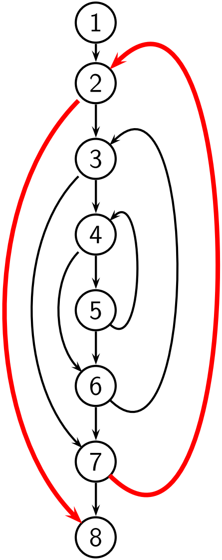
Width and Depth



Structures resulting from repeat-until loops with premature exits

- Depth = 3
- However, any unidirectional bit vector analysis is guaranteed to converge in 2 + 1 iterations
- ifp $5 \rightarrow 4 \rightarrow 6$ is bypassed by the edge $5 \rightarrow 6$
- ifp $6 \rightarrow 3 \rightarrow 7$ is bypassed by the edge $6 \rightarrow 7$

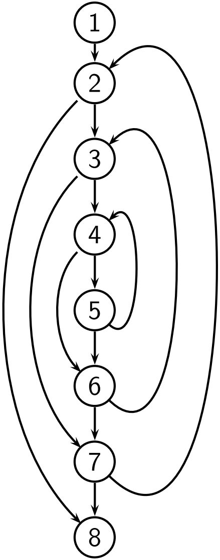
Width and Depth



Structures resulting from repeat-until loops with premature exits

- Depth = 3
- However, any unidirectional bit vector analysis is guaranteed to converge in 2 + 1 iterations
- ifp $5 \rightarrow 4 \rightarrow 6$ is bypassed by the edge $5 \rightarrow 6$
- ifp $6 \rightarrow 3 \rightarrow 7$ is bypassed by the edge $6 \rightarrow 7$
- ifp $7 \rightarrow 2 \rightarrow 8$ is bypassed by the edge $7 \rightarrow 8$

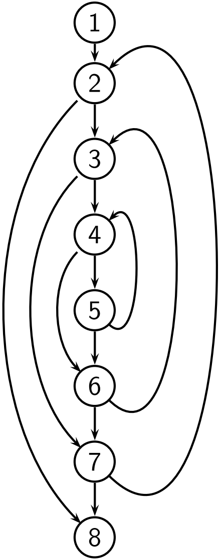
Width and Depth



Structures resulting from repeat-until loops with premature exits

- Depth = 3
- However, any unidirectional bit vector analysis is guaranteed to converge in 2 + 1 iterations
- ifp 5 → 4 → 6 is bypassed by the edge 5 → 6
- ifp 6 → 3 → 7 is bypassed by the edge 6 → 7
- ifp 7 → 2 → 8 is bypassed by the edge 7 → 8
- For forward unidirectional frameworks, width is 1

Width and Depth



Structures resulting from repeat-until loops with premature exits

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- However, any unidirectional bit vector analysis is guaranteed to converge in 2 + 1 iterations
- ifp 5 → 4 → 6 is bypassed by the edge 5 → 6
- ifp 6 → 3 → 7 is bypassed by the edge 6 → 7
- ifp 7 → 2 → 8 is bypassed by the edge 7 → 8
- For forward unidirectional frameworks, width is 1
- Splitting the bypassing edges and inserting nodes along those edges increases the width

Work List Based Iterative Algorithm

Directly traverses information flow paths

```
In_0 = BI
for all j ≠ 0 do
{   In_j = ⊤
    Add j to LIST
}
while LIST is not empty do
{   Let j be the first node in LIST. Remove it from LIST
    temp = ⊓_{p ∈ pred(j)} f_p(In_p)
    if temp ≠ In_j then
    {   In_j = temp
        Add all successors of j to LIST
    }
}
```

$$temp = \bigsqcap_{p \in pred(j)} f_p(In_p)$$

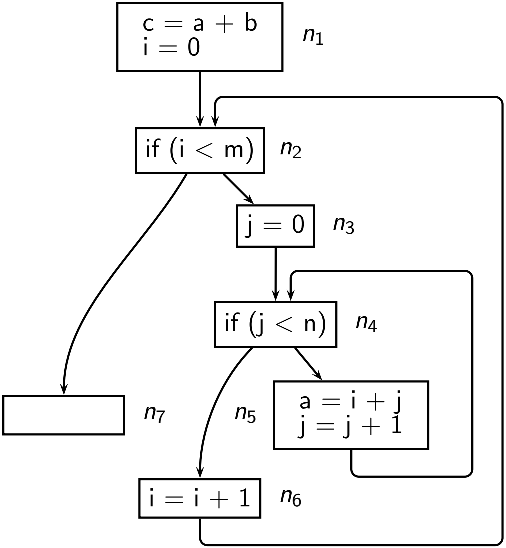
Tutorial Problem

Perform work list based iterative analysis for earlier examples. Assume that the work list follows FIFO (First in First Out) policy

Show the trace of the analysis in the following format:

Step	Node	Remaining work list	*Out* DFV	Change?	Node Added	Resulting work list

Tutorial Problem for Work List Based Analysis



For available expressions analysis

- Round robin method needs 3+1 iterations

  Total number of nodes processed $= 7 \times 4 = 28$
- We illustrate work list method for expression $a + b$

  (other expressions are unavailable in the first iteration because of BI)

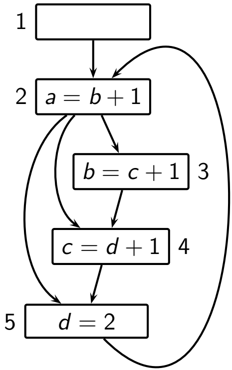
Tutorial Problem for Work List Based Analysis

Step	Node	Remaining work list	*Out* DFV	Change?	Node Added	Resulting work list
1	n_1	$n_2, n_3, n_4, n_5, n_6, n_7$	1	No		$n_2, n_3, n_4, n_5, n_6, n_7$
2	n_2	n_3, n_4, n_5, n_6, n_7	1	No		n_3, n_4, n_5, n_6, n_7
3	n_3	n_4, n_5, n_6, n_7	1	No		n_4, n_5, n_6, n_7
4	n_4	n_5, n_6, n_7	1	No		n_5, n_6, n_7
5	n_5	n_6, n_7	0	Yes	n_4	n_6, n_7, n_4
6	n_6	n_7, n_4	1	No		n_7, n_4
7	n_7	n_4	1	No		n_4
8	n_4		0	Yes	n_5, n_6	n_5, n_6
9	n_5	n_6	0	No		n_6
10	n_6		0	Yes	n_2	n_2
11	n_2		0	Yes	n_3, n_7	n_3, n_7
12	n_3	n_7	0	Yes	n_4	n_7, n_4
13	n_7	n_4	0	Yes		n_4
14	n_4		0	No		Empty $\Rightarrow$ End

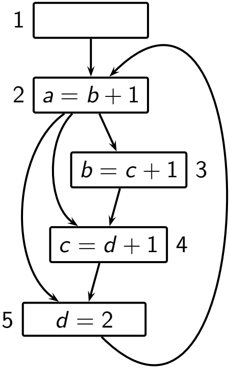
Part 10

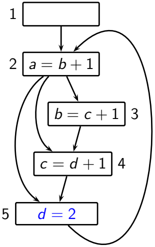
Precise Modelling of General Flows

Complexity of Constant Propagation?



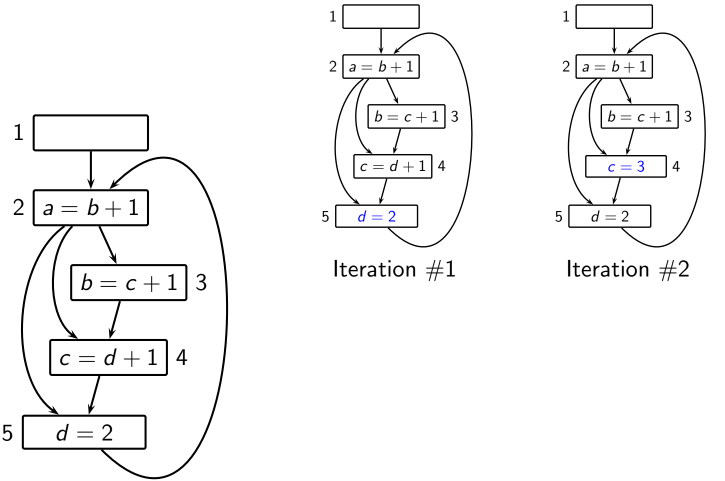
Complexity of Constant Propagation?



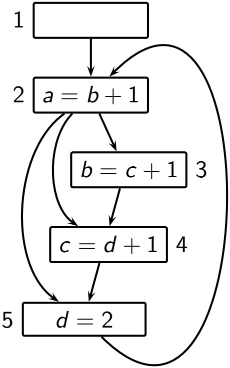


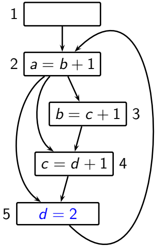
Iteration #1

Complexity of Constant Propagation?

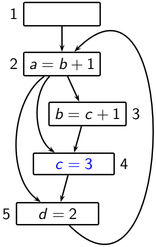


Complexity of Constant Propagation?

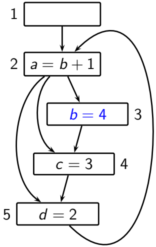




Iteration #1

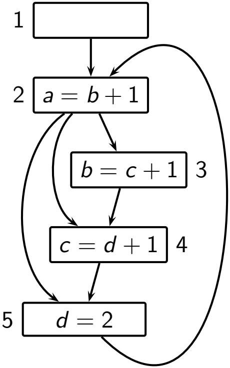


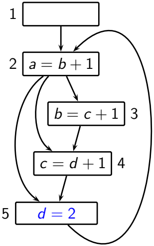
Iteration #2



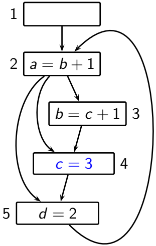
Iteration #3

Complexity of Constant Propagation?

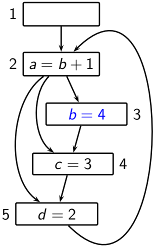




Iteration #1



Iteration #2



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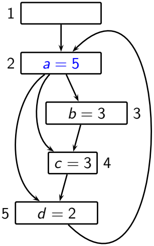

Iteration #4

Part 11

Extra Topics

Tarski's Fixed Point Theorem

Given monotonic $f : L \rightarrow L$ where L is a complete lattice

Define

$$\begin{array}{lrcl} p \text{ is a fixed point of } f: & Fix(f) & = & \{p \mid f(p) = p\} \\ f \text{ is reductive at } p: & Red(f) & = & \{p \mid f(p) \sqsubseteq p\} \\ f \text{ is extensive at } p: & Ext(f) & = & \{p \mid f(p) \sqsupseteq p\} \end{array}$$

Then

$$\begin{array}{rcl} LFP(f) & = & \bigsqcap Red(f) \in Fix(f) \\ MFP(f) & = & \bigsqcup Ext(f) \in Fix(f) \end{array}$$

Tarski's Fixed Point Theorem

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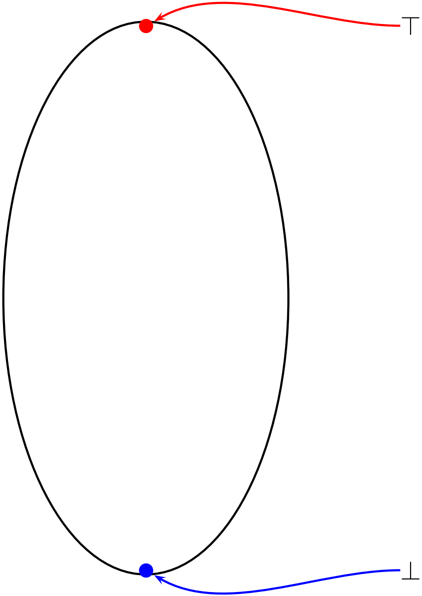
Then

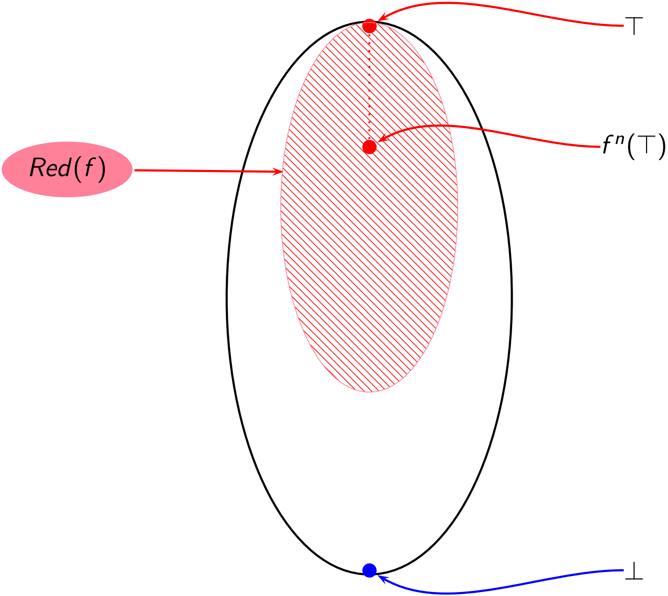
$$\begin{array}{rcl} LFP(f) & = & \bigsqcap Red(f) \in Fix(f) \\ MFP(f) & = & \bigsqcup Ext(f) \in Fix(f) \end{array}$$

Guarantees only existence, not computability of fixed points

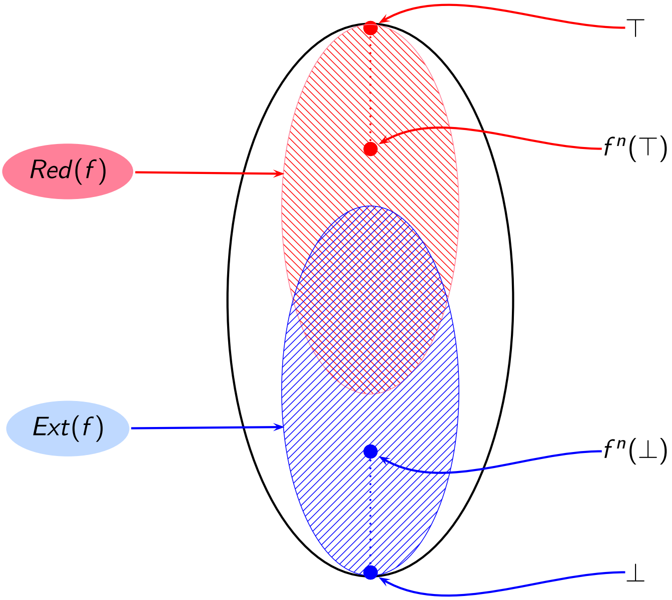
Fixed Points of a Function



Fixed Points of a Function



Fixed Points of a Function



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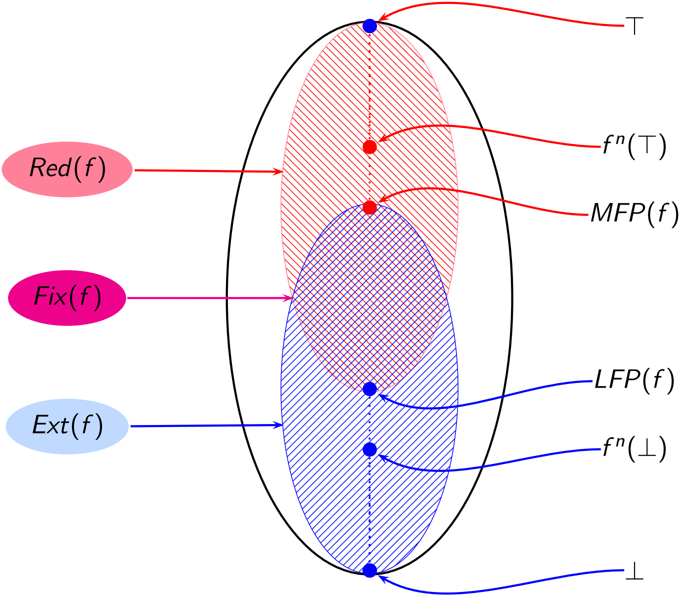
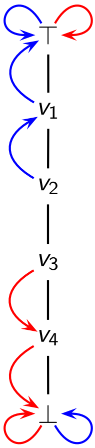


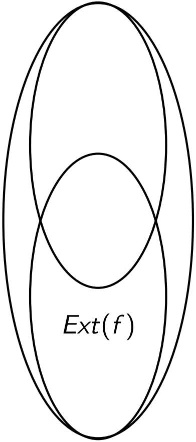
Examples of Reductive and Extensive Sets

Finite L Monotonic $f : L \to L$



$$
\begin{aligned}
Red(f) &= \{\top, v_3, v_4, \bot\} \\
Ext(f) &= \{\top, v_1, v_2, \bot\} \\
Fix(f) &= Red(f) \cap Ext(f) \\
&= \{\top, \bot\} \\
MFP(f) &= lub\,(Ext(f)) \\
&= lub\,(Fix(f)) \\
&= \top \\
LFP(f) &= glb\,(Red(f)) \\
&= glb\,(Fix(f)) \\
&= \bot
\end{aligned}
$$

Existence of MFP: Proof of Tarski's Fixed Point Theorem



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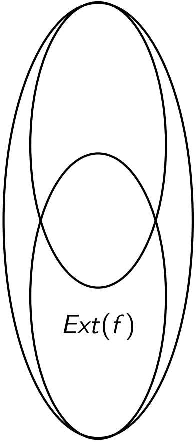
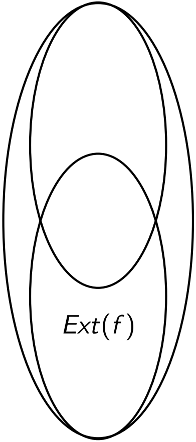


1. Claim 1: Let $X \subseteq L$.
   $\forall x \in X,\ p \sqsupseteq x \Rightarrow p \sqsupseteq \bigsqcup(X)$.

Existence of MFP: Proof of Tarski's Fixed Point Theorem



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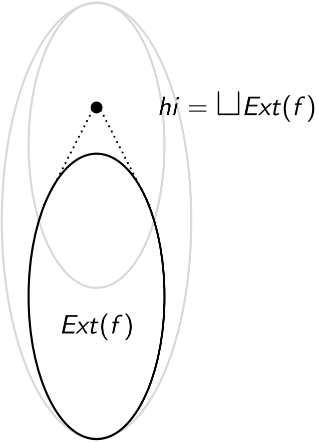
2. In the following we use $Ext(f)$ as X

Existence of MFP: Proof of Tarski's Fixed Point Theorem



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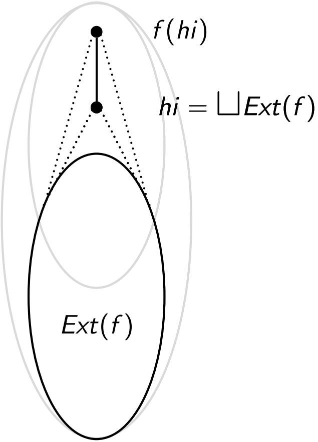

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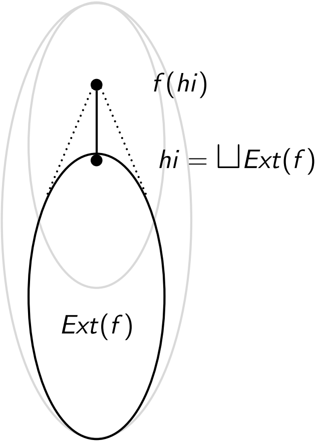
2. In the following we use $Ext(f)$ as X
3. $\forall p \in Ext(f),\ hi \sqsupseteq p$
4. $hi \sqsupseteq p \ \Rightarrow f(hi) \sqsupseteq f(p) \sqsupseteq p$ (monotonicity)
   $\Rightarrow f(hi) \sqsupseteq hi$ (claim 1)

Existence of MFP: Proof of Tarski's Fixed Point Theorem



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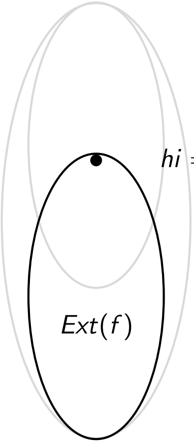
5. f is extensive at hi also: $hi \in Ext(f)$

Existence of MFP: Proof of Tarski's Fixed Point Theorem



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6. $f(hi) \sqsupseteq hi \Rightarrow f^2(hi) \sqsupseteq f(hi)$
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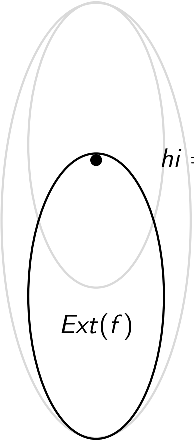
   $\Rightarrow hi \sqsupseteq f(hi)$ (from 3)
   $\Rightarrow hi = f(hi) \Rightarrow hi \in Fix(f)$

Existence of MFP: Proof of Tarski's Fixed Point Theorem



1. Claim 1: Let $X \subseteq L$.
   $\forall x \in X,\ p \sqsupseteq x \Rightarrow p \sqsupseteq \bigsqcup(X)$.
2. In the following we use $Ext(f)$ as X
3. $\forall p \in Ext(f),\ hi \sqsupseteq p$
4. $hi \sqsupseteq p \ \Rightarrow f(hi) \sqsupseteq f(p) \sqsupseteq p$ (monotonicity)
   $\Rightarrow f(hi) \sqsupseteq hi$ (claim 1)
5. f is extensive at hi also: $hi \in Ext(f)$
6. $f(hi) \sqsupseteq hi \Rightarrow f^2(hi) \sqsupseteq f(hi)$
   $\Rightarrow f(hi) \in Ext(f)$
   $\Rightarrow hi \sqsupseteq f(hi)$ (from 3)
   $\Rightarrow hi = f(hi) \Rightarrow hi \in Fix(f)$
7. $Fix(f) \subseteq Ext(f)$ (by definition)
   $\Rightarrow hi \sqsupseteq p,\ \forall p \in Fix(f)$

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  $\Rightarrow$ Completeness of lattice
- Completeness of lattice $\not\Rightarrow$ Finite strictly descending chains
- $\Rightarrow$ Even if MFP exists, it may not be reachable unless all strictly descending chains are finite

Framework Properties Influencing Complexity

Depends on the loop closure properties of the framework

k-Bounded Frameworks

$$f^*(x) = x \sqcap f(x) \sqcap f^2(x) \sqcap \ldots \sqcap f^{k-1}(x)$$

Necessary and sufficient

Framework Properties Influencing Complexity

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k-Bounded Frameworks

Fast Frameworks ($k = 2$)

$f^2(x) \sqsupseteq f(x) \sqcap x$

$f^*(x) = x \sqcap f(x) \sqcap f^2(x) \sqcap \ldots \sqcap f^{k-1}(x)$

Necessary and sufficient

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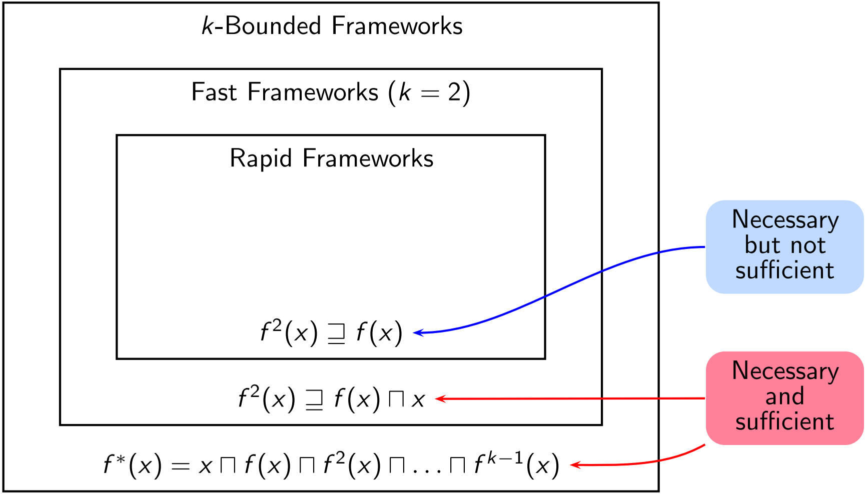

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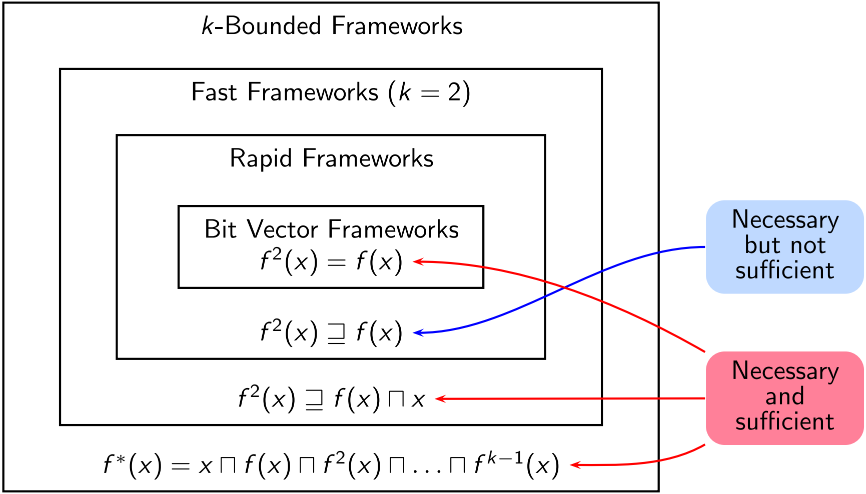

Complexity of Round Robin Iterative Algorithm

- Unidirectional rapid frameworks

Task	Number of iterations	
	Irreducible G	Reducible G
Initialisation	1	1
Convergence (until *change* remains true)	$d(G,T)+1$	$d(G,T)$
Verifying convergence (*change* becomes false)	1	1