Regular Transducer Expressions for Regular Transformations

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Joint work with Paul Gastin (LSV, ENS Paris-Saclay) and S. Krishna (IIT Bombay)

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$$f : \Sigma^* \to \Gamma^*$$

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Running example: $\Sigma = \{\#, a\}$, $\Gamma = \{b, c\}$ and $\mathsf{dom}(f) = (\#a^+)^+\#$

$$f(\#a^{m_1}\#a^{m_2} \cdots \#a^{m_{k-1}}\#a^{m_k}\#) = c^{m_2}b^{m_1}c^{m_3}b^{m_2} \cdots c^{m_k}b^{m_{k-1}}$$

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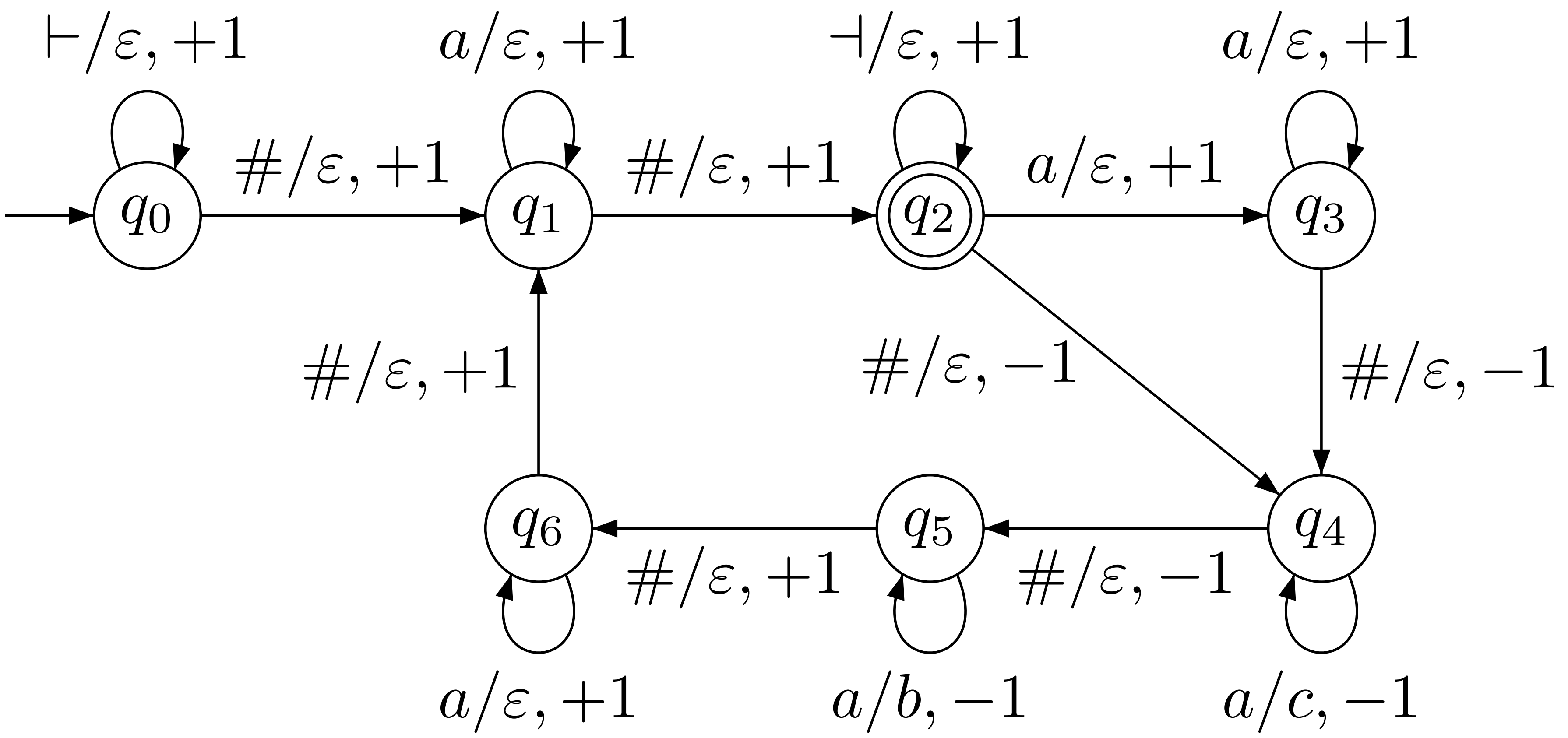

Language and Transformation

Regular Language $\equiv$ Regular Expressions

Atomic: $\epsilon \mid \emptyset \mid a \in \Sigma$

Operators: $\cup \mid . \mid \text{Kleene}-^{*}$

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What is known

- The above problem is solved for SST over finite words

  [Alur, Freilich, and Raghothaman 2014]

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What is known

- The above problem is solved for SST over infinite words

  [Alur, Freilich, and Raghothaman 2014]

Our contribution:
Regular Transducer Expressions (RTE)
New proof technique
Works directly with 2DFT
Based on Algebra
Extension to infinite words

Summary

- Regular Transducer Expressions (RTE)
- Transition Monoid
- Good Rational Expressions
- From 2DFT to RTE
- Extension to Infinite words
- Conclusion

Regular Transducer Expressions

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$$[\![C]\!] \colon \Sigma^* \to \Gamma^* \text{ partial function}$$

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$K \subseteq \Sigma^*$ regular

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$[\![{C_a}^{\boxplus}]\!](baabbbab) = a^3$

Unambiguous Kleene-plus

$$f^{\boxplus}(w) = f(u_1) f(u_2) \cdots f(u_n)$$

If $w = u_1 u_2 \cdots u_n$ with $n \geq 1$ and $u_1, u_2, \ldots, u_n \in \mathsf{dom}(f)$

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$$C ::= d \mid K\,?\,C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

$[\![C]\!] \colon \Sigma^* \to \Gamma^*$ partial function

Unambiguous Kleene-plus

$$f^{\boxplus}(w) = f(u_1) f(u_2) \cdots f(u_n)$$

If $w = u_1 u_2 \cdots u_n$ with $n \geq 1$ and $u_1, u_2, \ldots, u_n \in \mathsf{dom}(f)$

Regular Transducer Expressions

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$$f(\#a^{m_1}\#a^{m_2}\cdots\#a^{m_{k-1}}\#a^{m_k}\#) = c^{m_2}b^{m_1}c^{m_3}b^{m_2}\cdots c^{m_k}b^{m_{k-1}}$$

$$C_b = (a\,?\,b : \bot)^{\boxplus} \qquad C_c = (a\,?\,c : \bot)^{\boxplus}$$

$$C = [\#a^+, ((\#\,?\,\varepsilon : \bot) \boxdot C_b) \overleftarrow{\boxdot} ((\#\,?\,\varepsilon : \bot) \boxdot C_c)]^{2\boxplus} \boxdot (\#\,?\,\varepsilon : \bot)$$

Unambiguous 2-chained Kleene-plus

$$[K, f]^{2\boxplus}(w) = f(u_1u_2)f(u_2u_3)\cdots f(u_{n-1}u_n)$$

If $w = u_1u_2\cdots u_n$ with $n \geq 1$ and $u_1, u_2, \ldots, u_n \in K$

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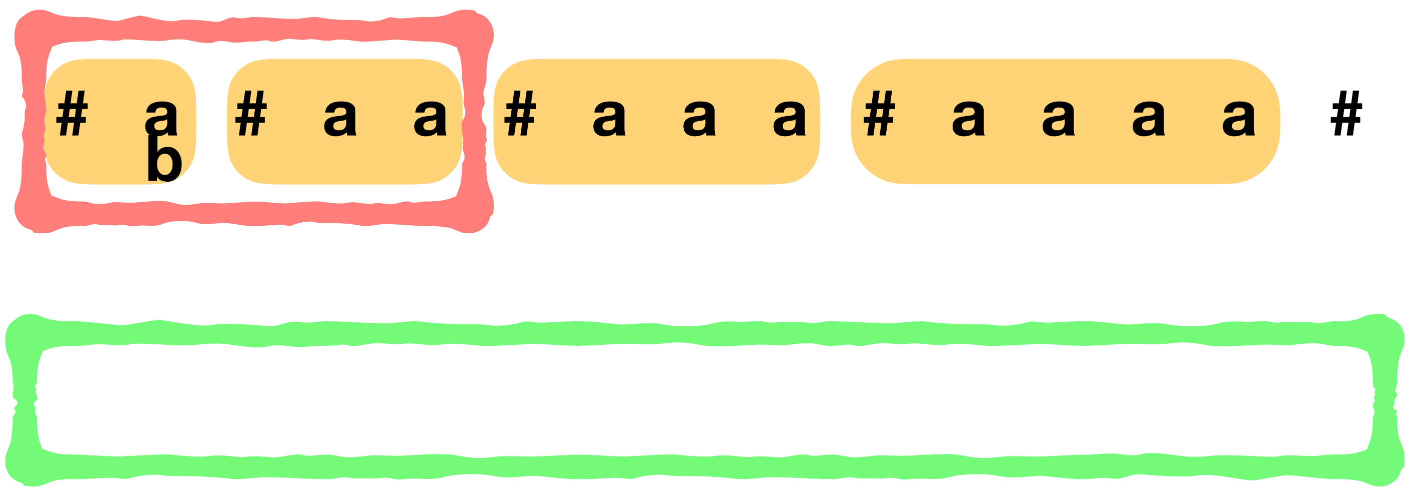

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Unambiguous reversed 2-chained Kleene-plus

$$[K, f]^{\overleftarrow{2\boxplus}}(w) = f(u_{n-1}u_n)f(u_{n-2}u_{n-1}) \cdots f(u_1u_2)$$

Unambiguous 2-chained Kleene-plus

$$[K, f]^{2\boxplus}(w) = f(u_1u_2)f(u_2u_3) \cdots f(u_{n-1}u_n)$$

If $w = u_1u_2 \cdots u_n$ with $n \geq 1$ and $u_1, u_2, \ldots, u_n \in K$

RTE and 2DFT

Main Theorem:

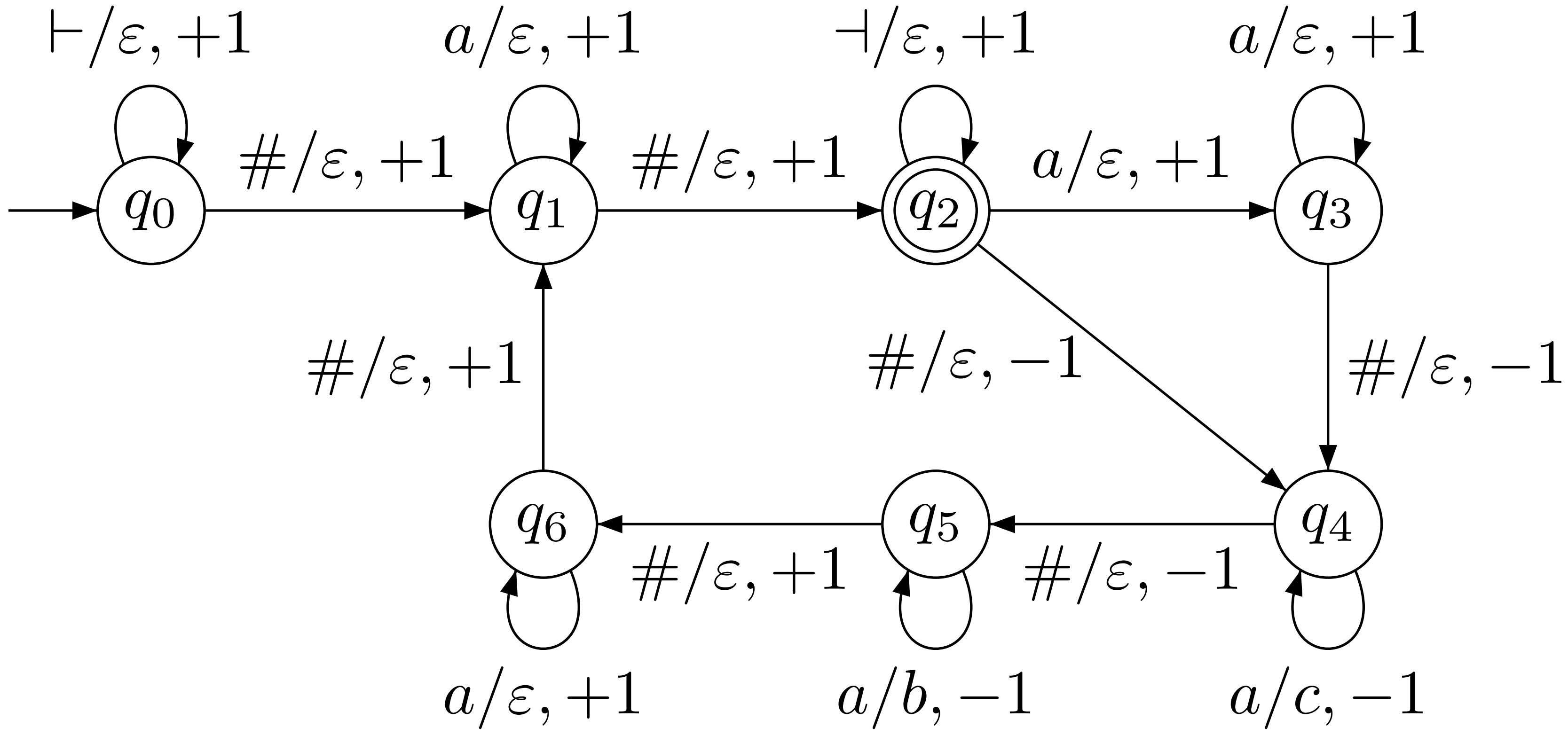
2DFTs and RTEs define the same class of functions. More precisely,

1. given an RTE C, we can construct a 2DFT $\mathcal{A}$ such that $[\![\mathcal{A}]\!] = [\![C]\!]$,
2. given a 2DFT $\mathcal{A}$, we can construct an RTE C such that $[\![\mathcal{A}]\!] = [\![C]\!]$.

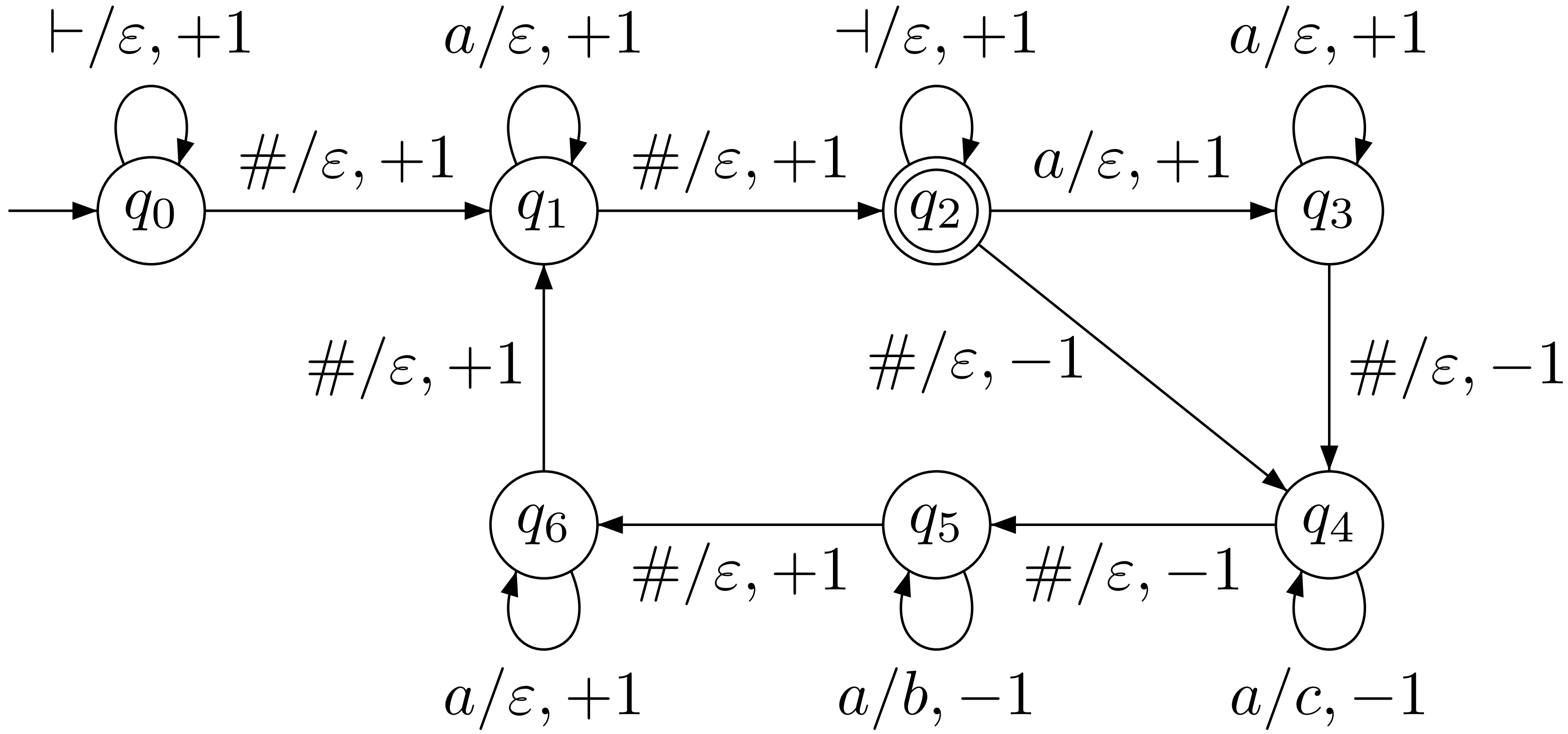
Summary

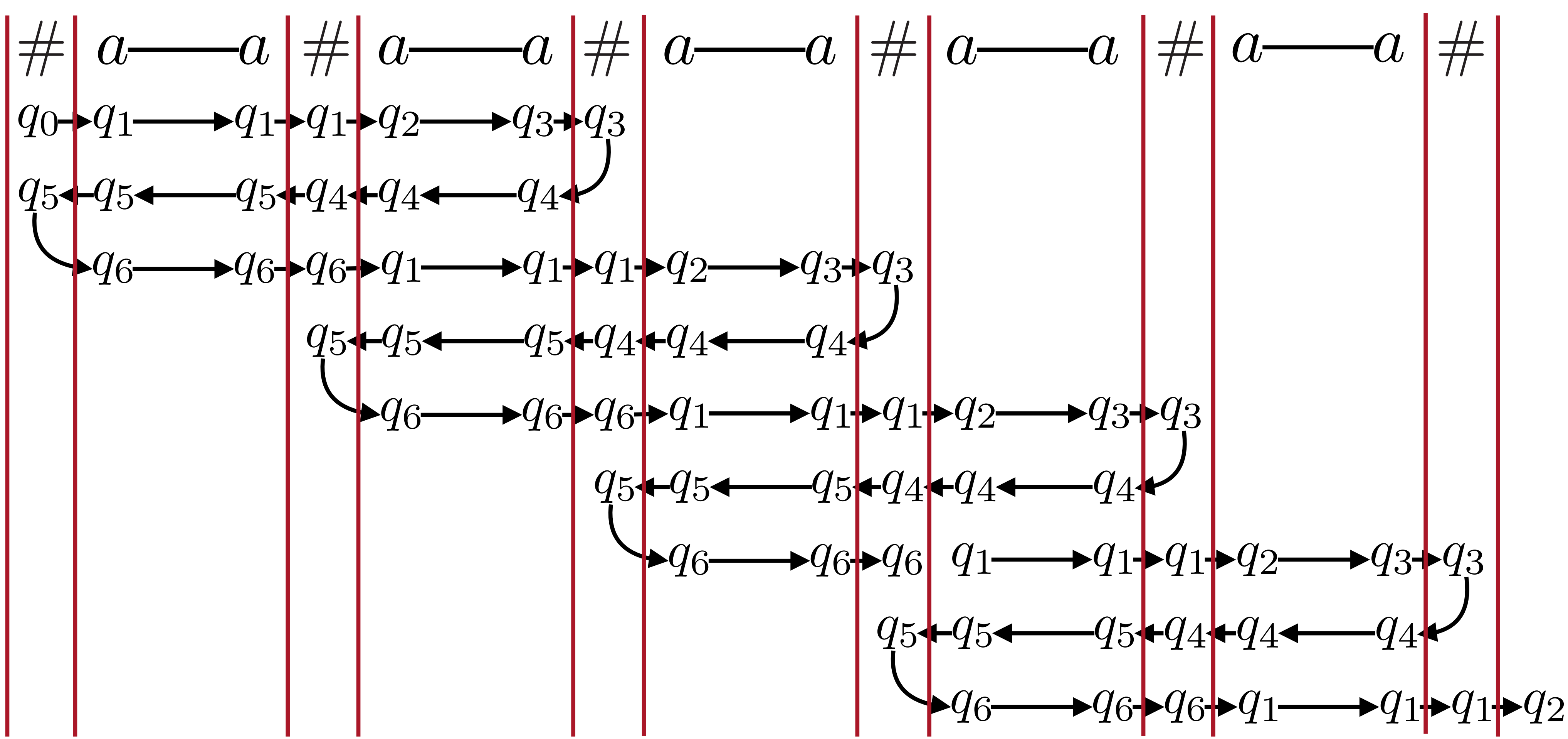
- Regular Transducer Expressions (RTE)
- Transition Monoid
- Good Rational Expressions
- From 2DFT to RTE
- Extension to Infinite words
- Conclusion

Transition Monoid



Transition Monoid





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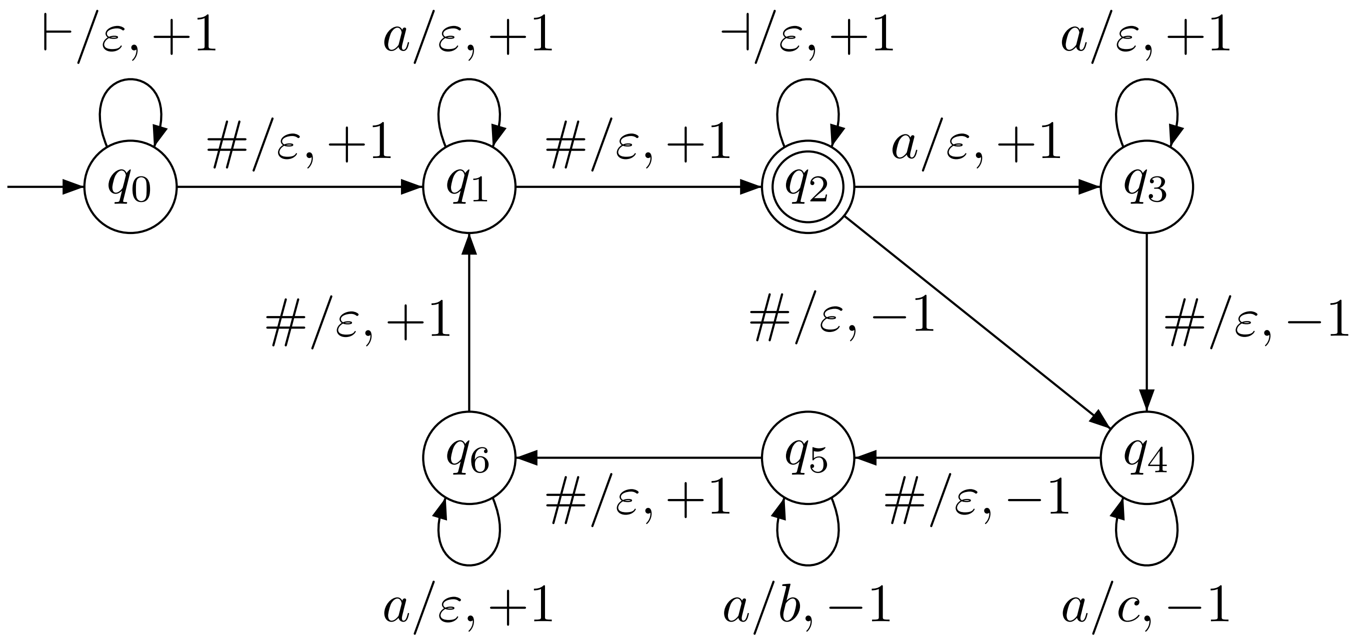
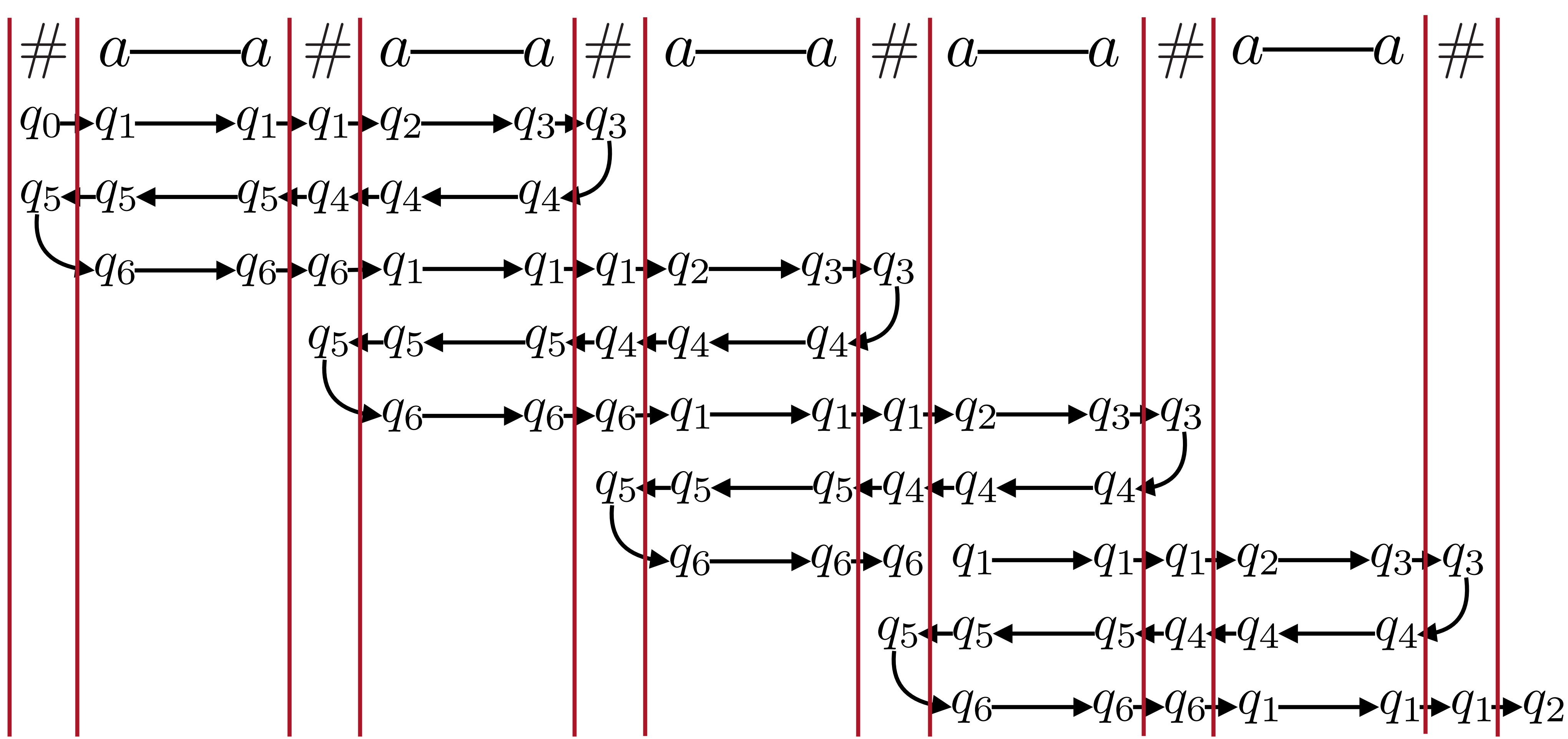
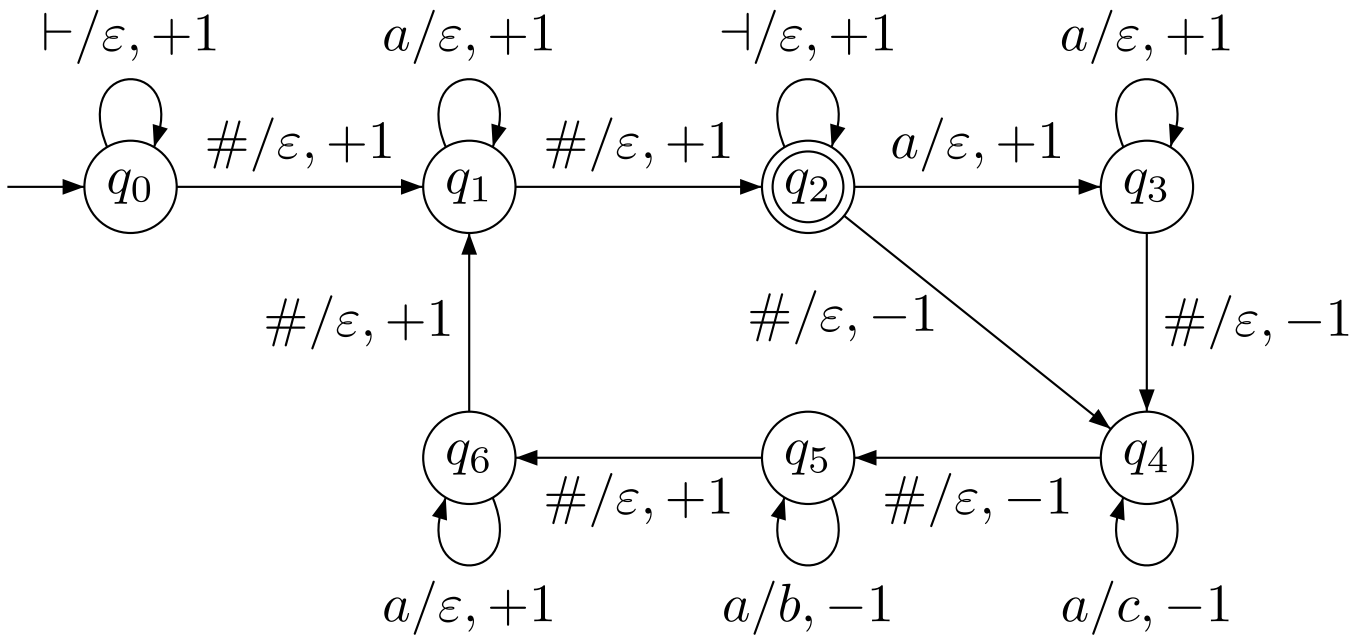
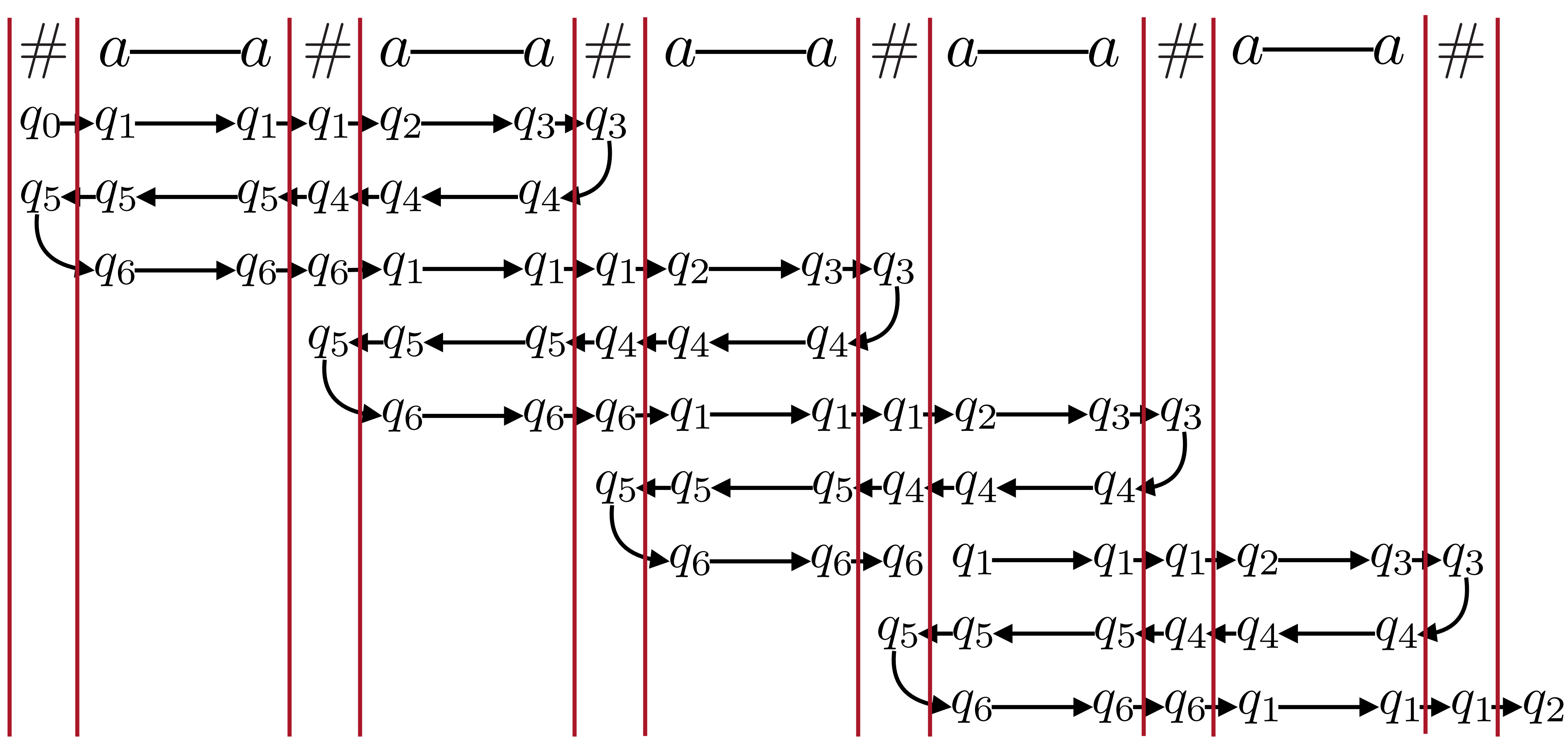




Transition Monoid

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$





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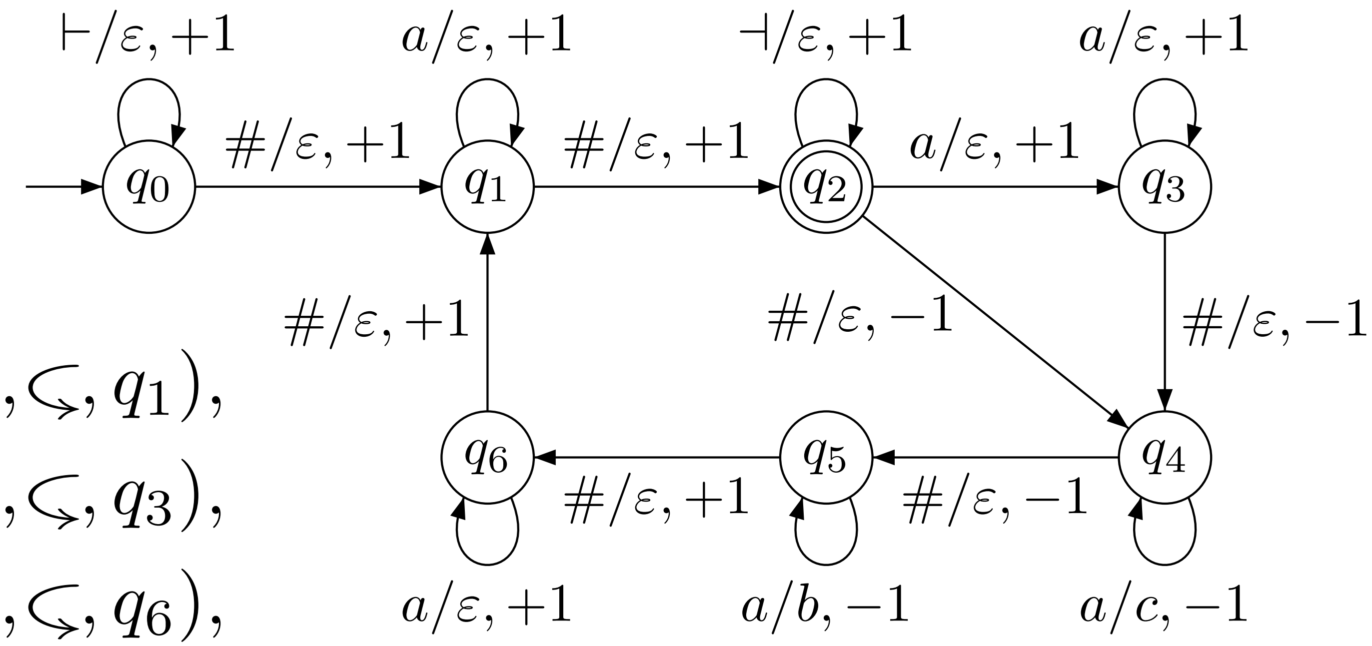
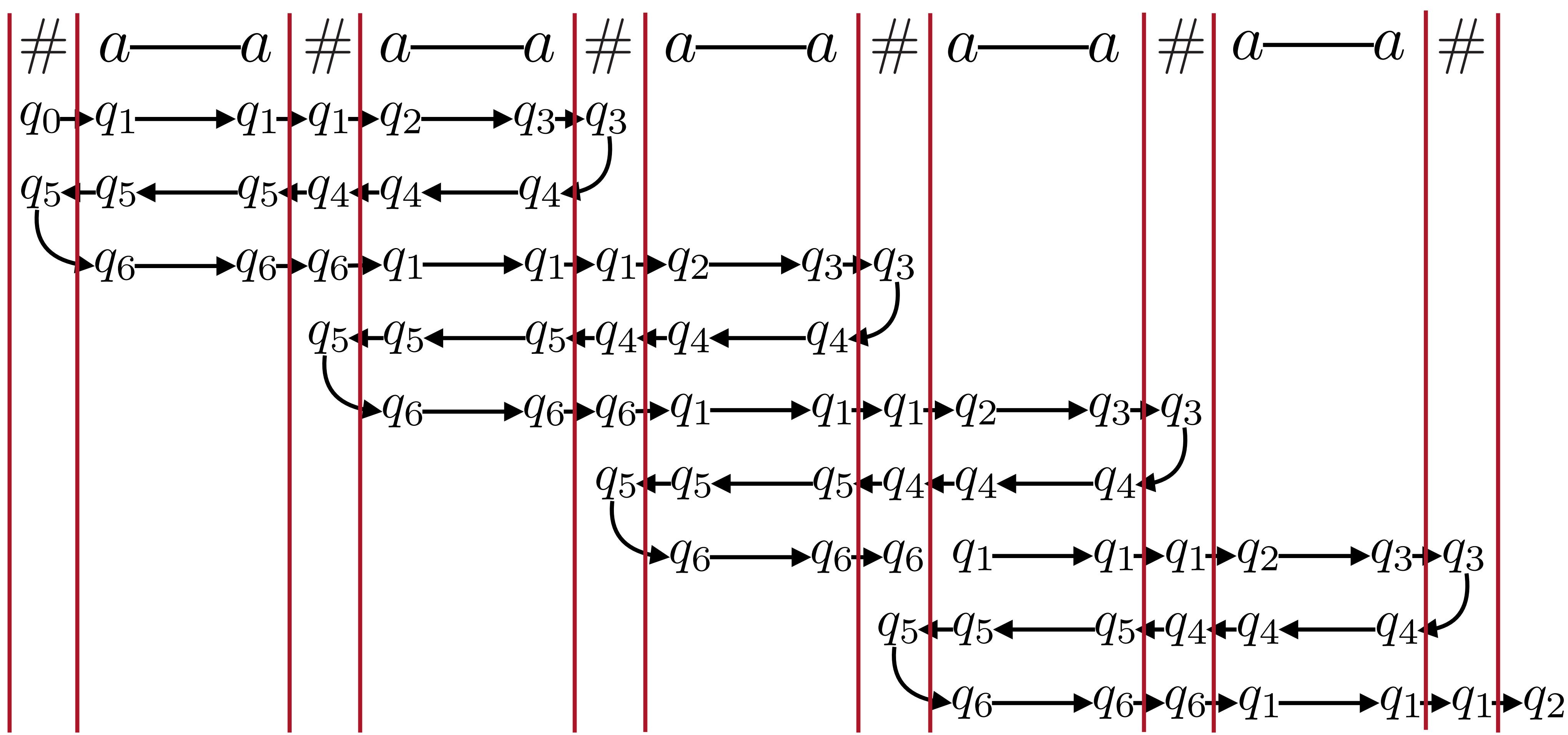


$$\mathsf{Tr}(\#a^+\#a^+) = \{(q_0,\rightarrow,q_3),(q_1,\hookleftarrow,q_5),(q_1,\hookrightarrow,q_1),$$
$$(q_2,\hookleftarrow,q_4),(q_2,\hookrightarrow,q_3),(q_3,\hookleftarrow,q_4),(q_3,\hookrightarrow,q_3),$$
$$(q_4,\hookleftarrow,q_5),(q_4,\hookrightarrow,q_1),(q_5,\rightarrow,q_1),(q_5,\hookrightarrow,q_6),$$
$$(q_6,\rightarrow,q_3),(q_6,\hookrightarrow,q_6)\}$$



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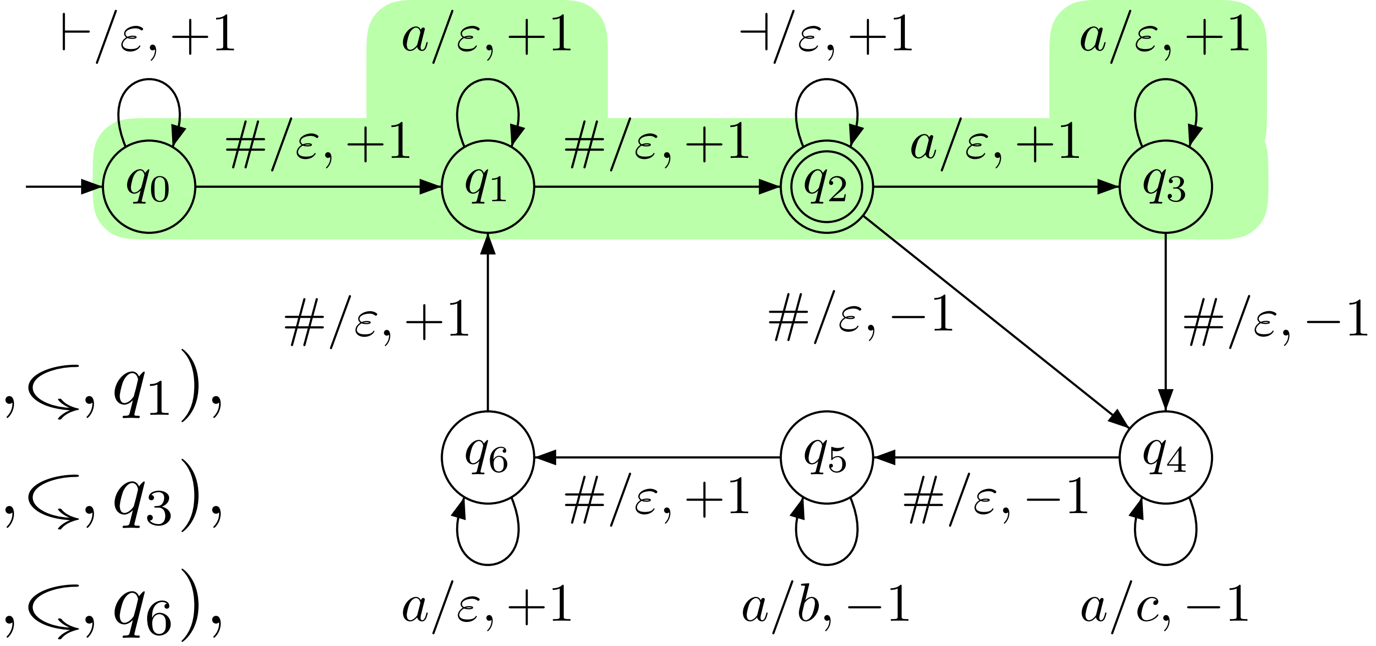

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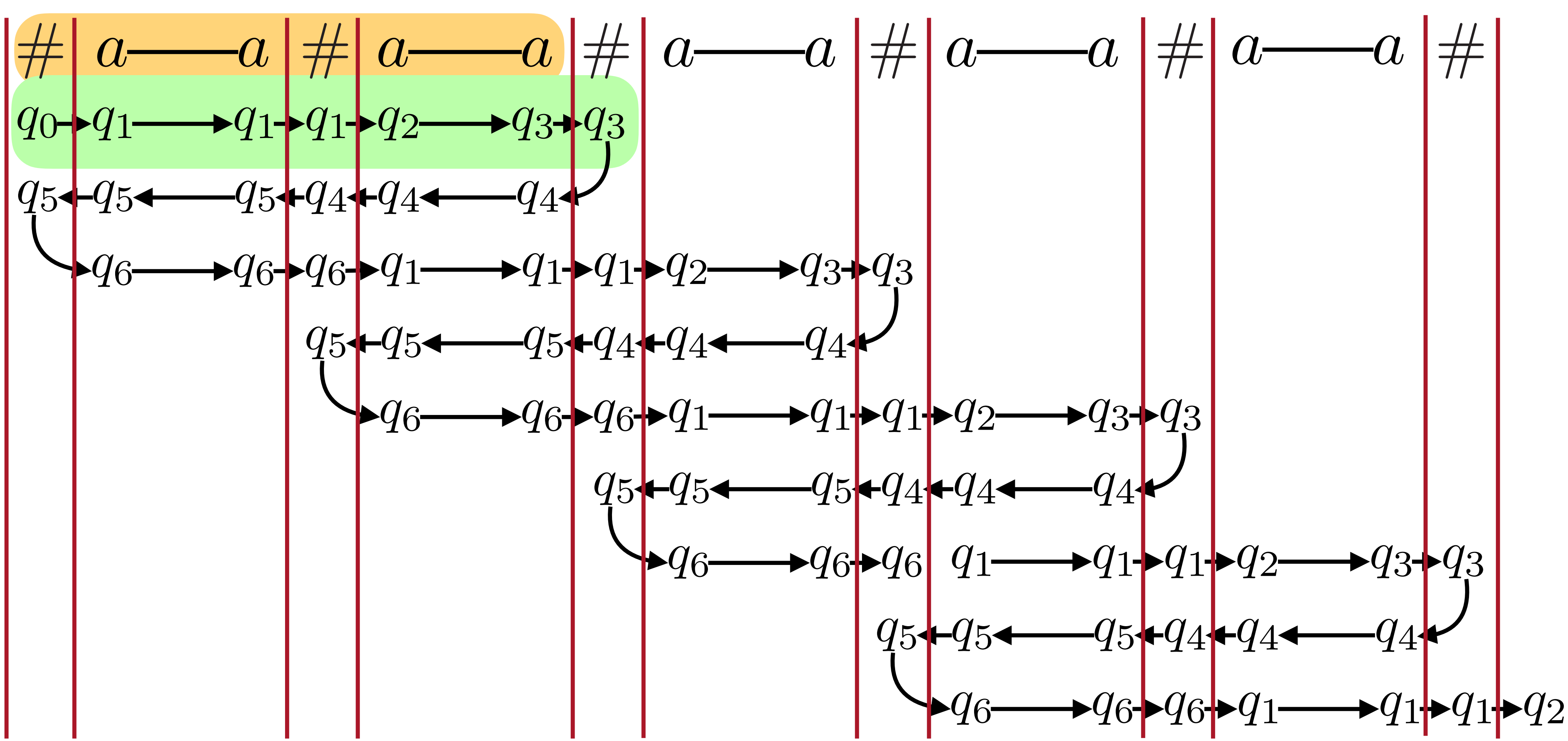

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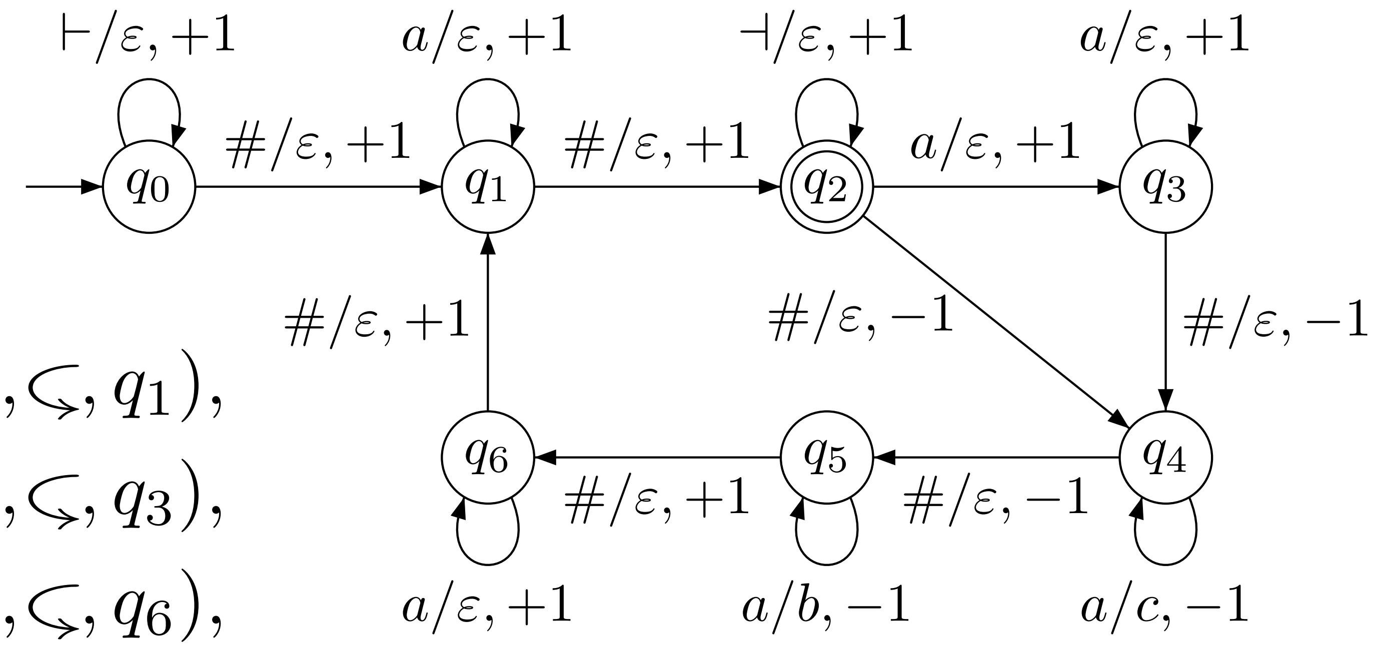

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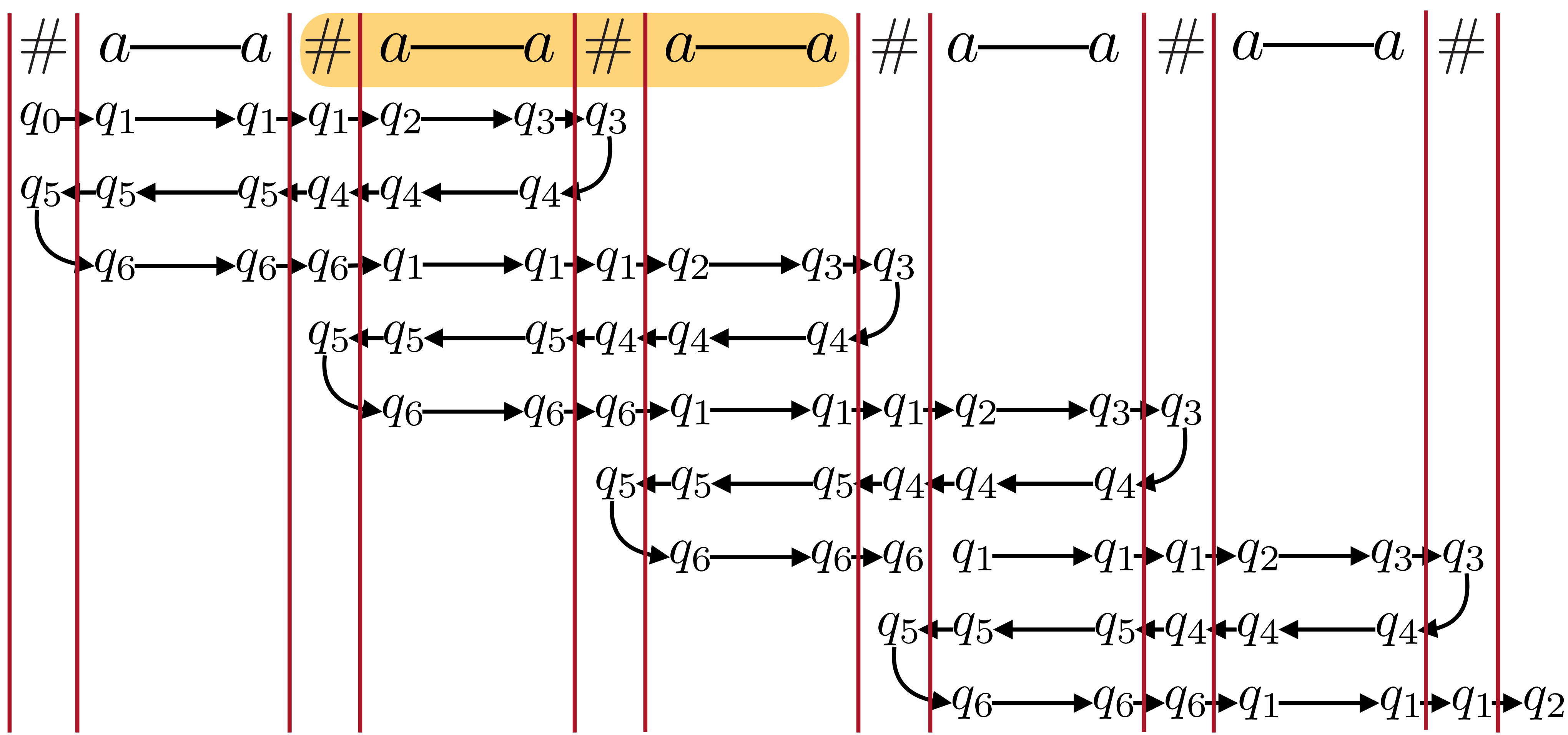

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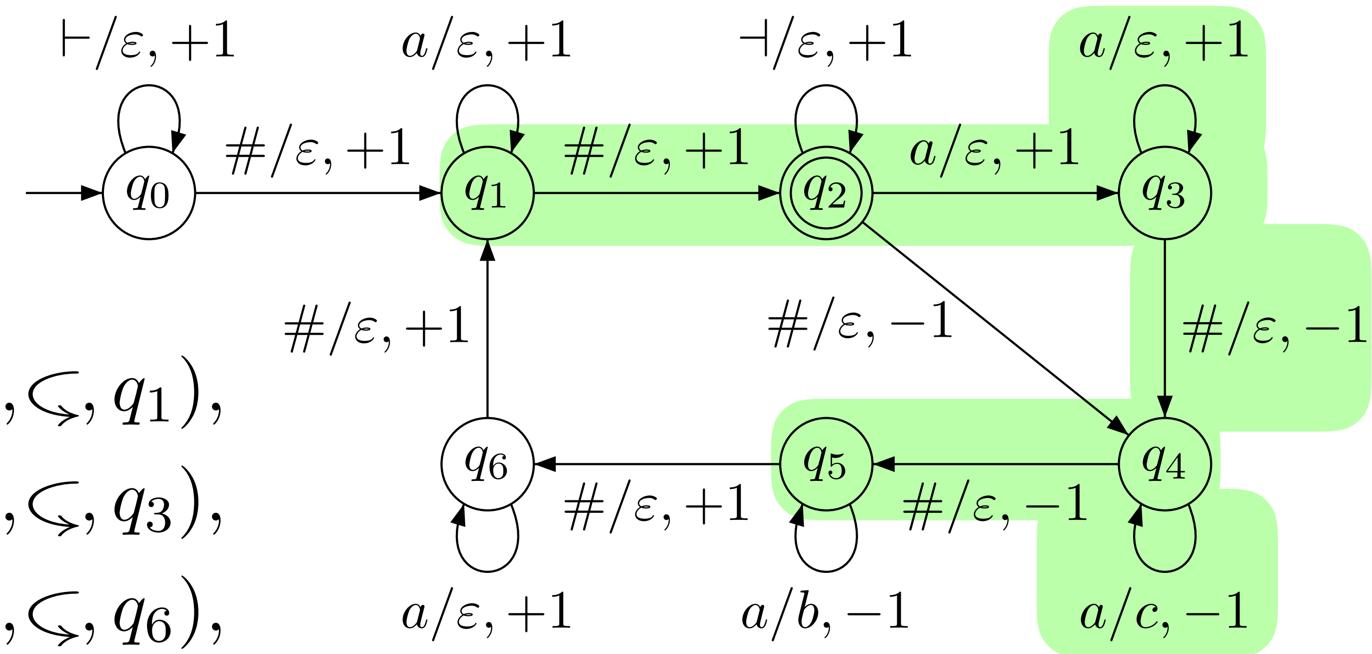

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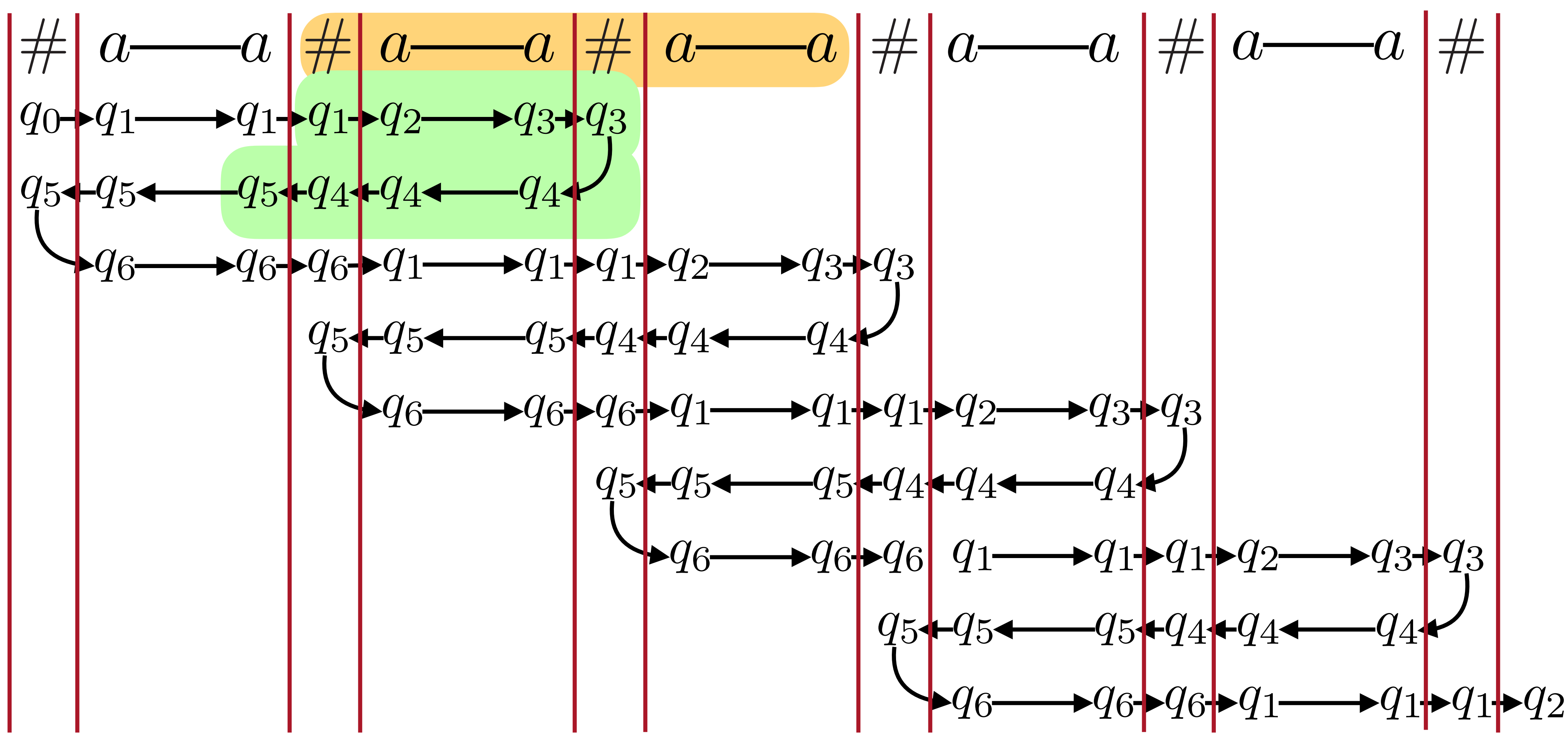

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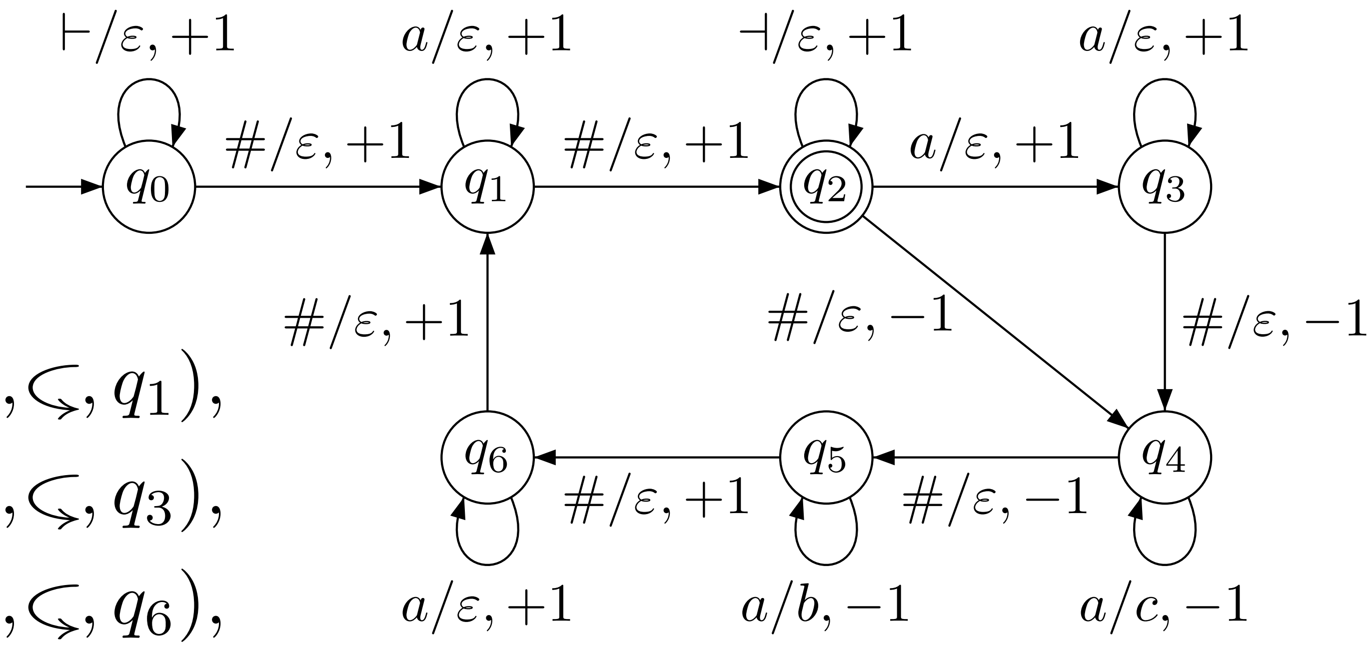

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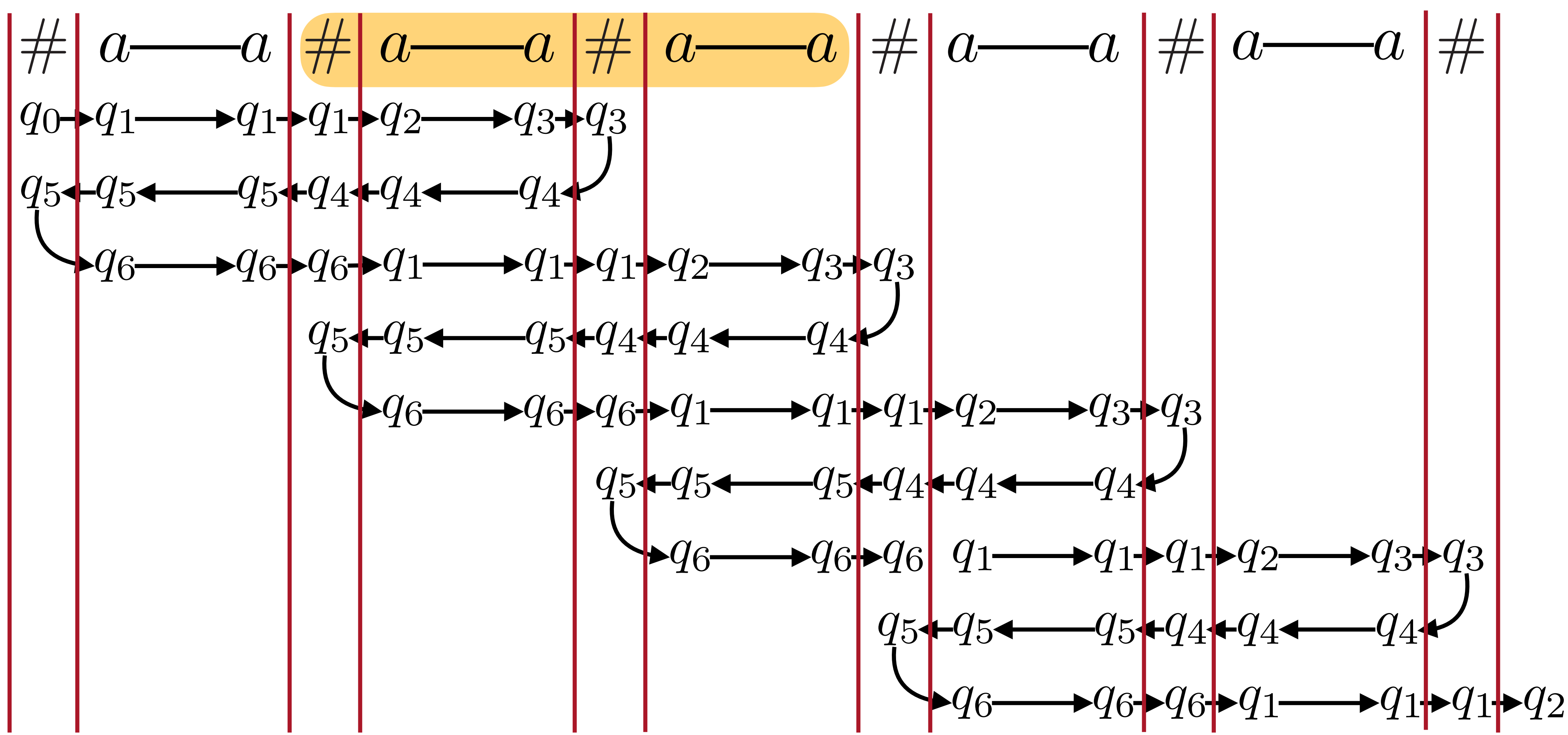

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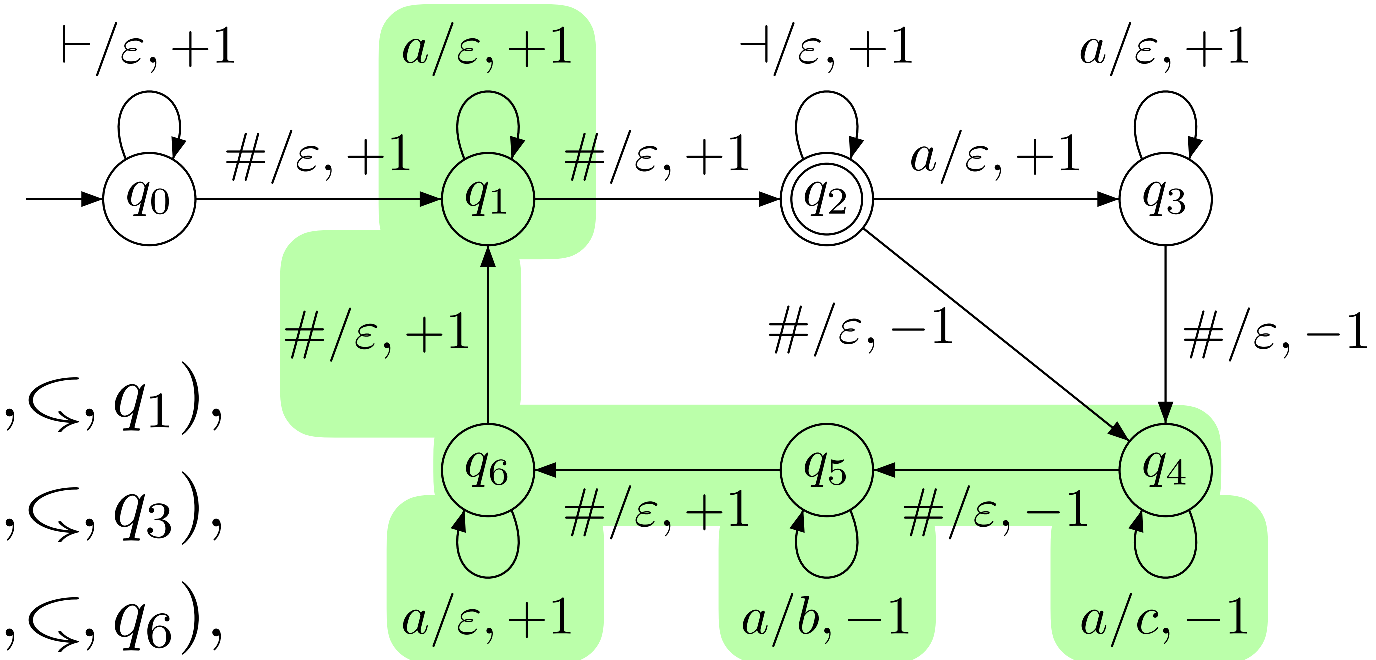
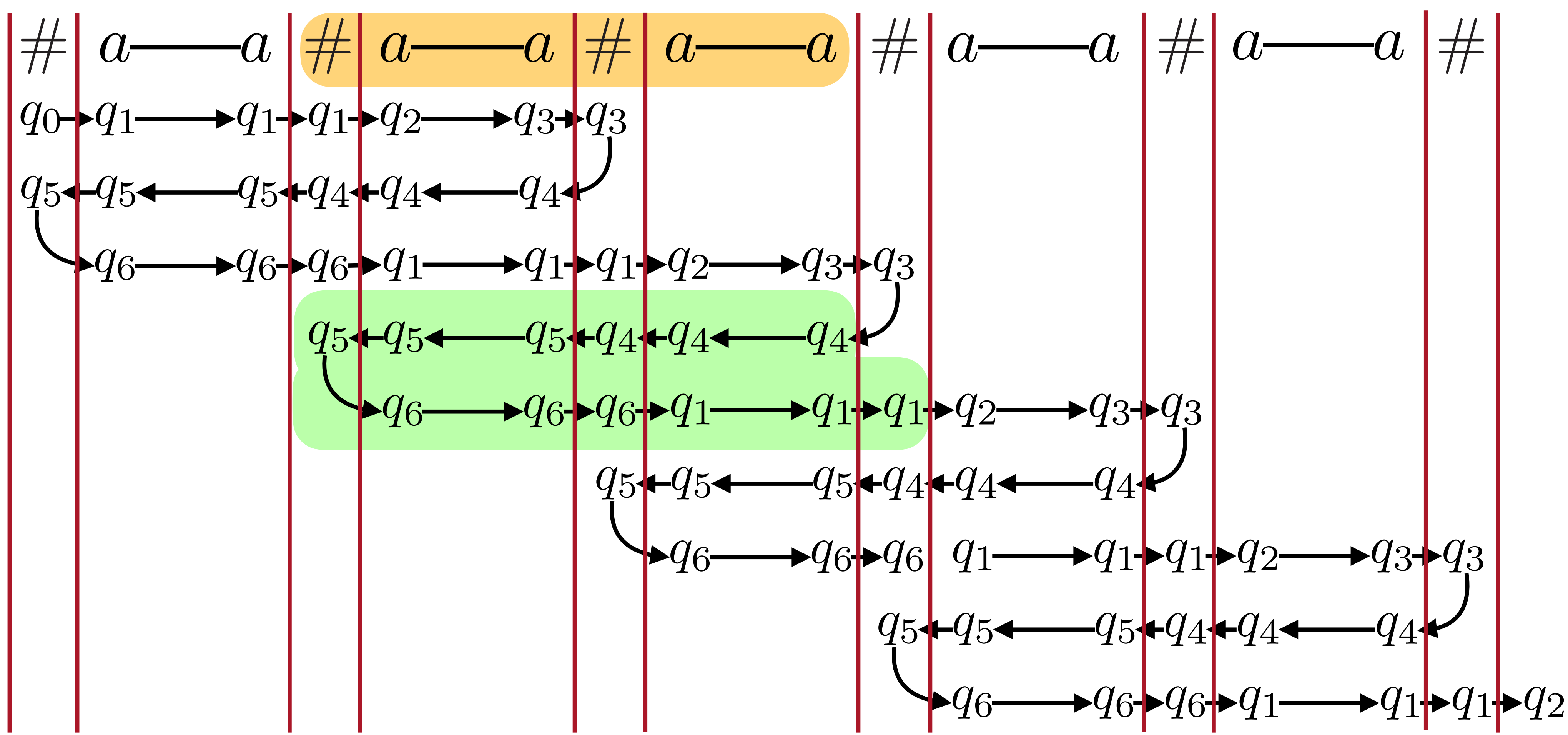


$$\mathsf{Tr}(\#a^+\#a^+) = \{(q_0, \rightarrow, q_3), (q_1, \supsetneq, q_5), (q_1, \subsetneq, q_1), (q_2, \supsetneq, q_4), (q_2, \subsetneq, q_3), (q_3, \supsetneq, q_4), (q_3, \subsetneq, q_3), (q_4, \supsetneq, q_5), (q_4, \subsetneq, q_1), (q_5, \rightarrow, q_1), (q_5, \subsetneq, q_6), (q_6, \rightarrow, q_3), (q_6, \subsetneq, q_6)\}$$



Summary

- Regular Transducer Expressions (RTE)
- Transition Monoid
- Good Rational Expressions
- From 2DFT to RTE
- Extension to Infinite words
- Conclusion

Good Rational Expressions

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For each $s \in S$, there is an ε-free *good* rational expression F_s such that

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Theorem: (Simon 1990)
Every word can be factorized (parsed) with a tree of height at most 9|S|

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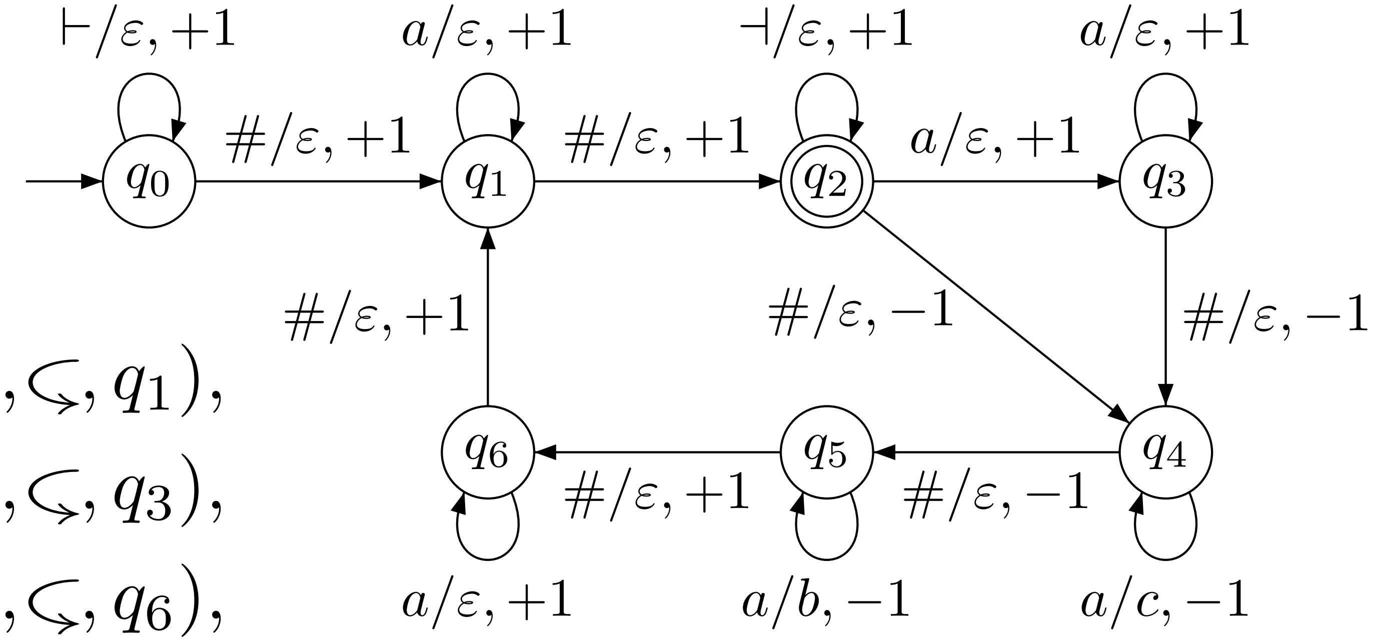
2DFT to RTE

$\mathsf{Tr}\colon \Sigma^* \to \mathsf{TrM}$



$$\mathsf{Tr}(\#a^+\#a^+) = \{(q_0,\rightarrow,q_3),(q_1,\curvearrowleft,q_5),(q_1,\curvearrowright,q_1),(q_2,\curvearrowleft,q_4),(q_2,\curvearrowright,q_3),(q_3,\curvearrowleft,q_4),(q_3,\curvearrowright,q_3),(q_4,\curvearrowleft,q_5),(q_4,\curvearrowright,q_1),(q_5,\rightarrow,q_1),(q_5,\curvearrowright,q_6),(q_6,\rightarrow,q_3),(q_6,\curvearrowright,q_6)\}$$

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2DFT to RTE: atomic

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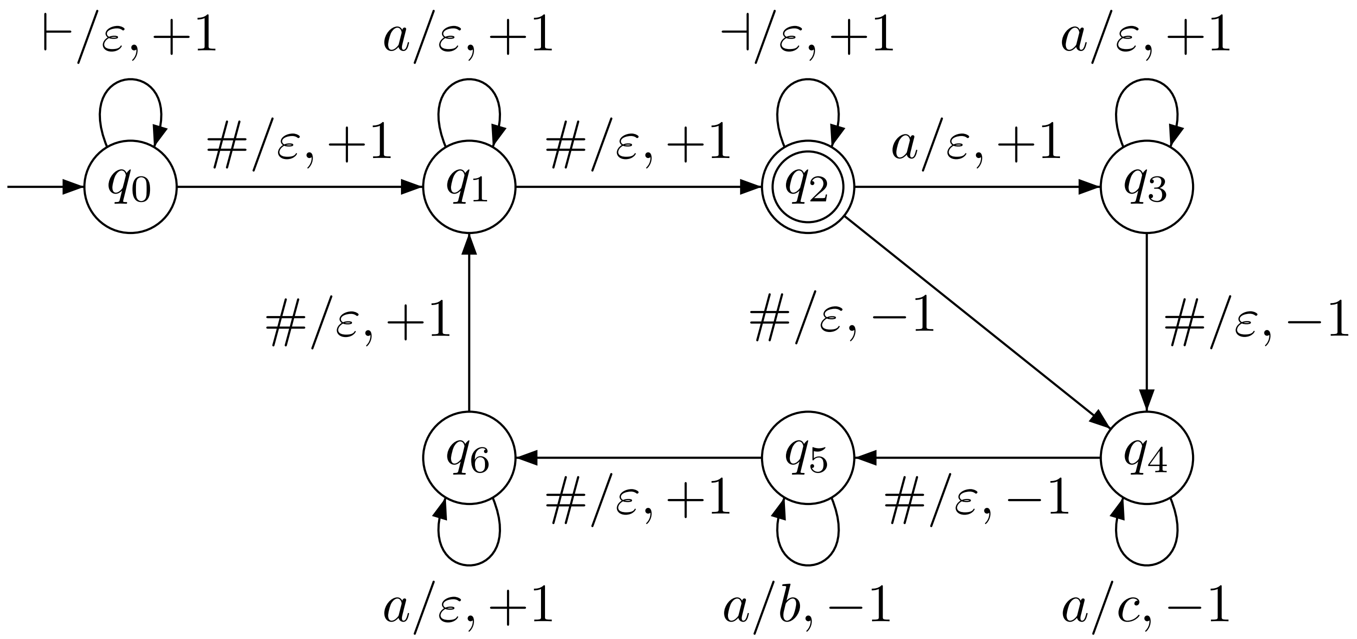

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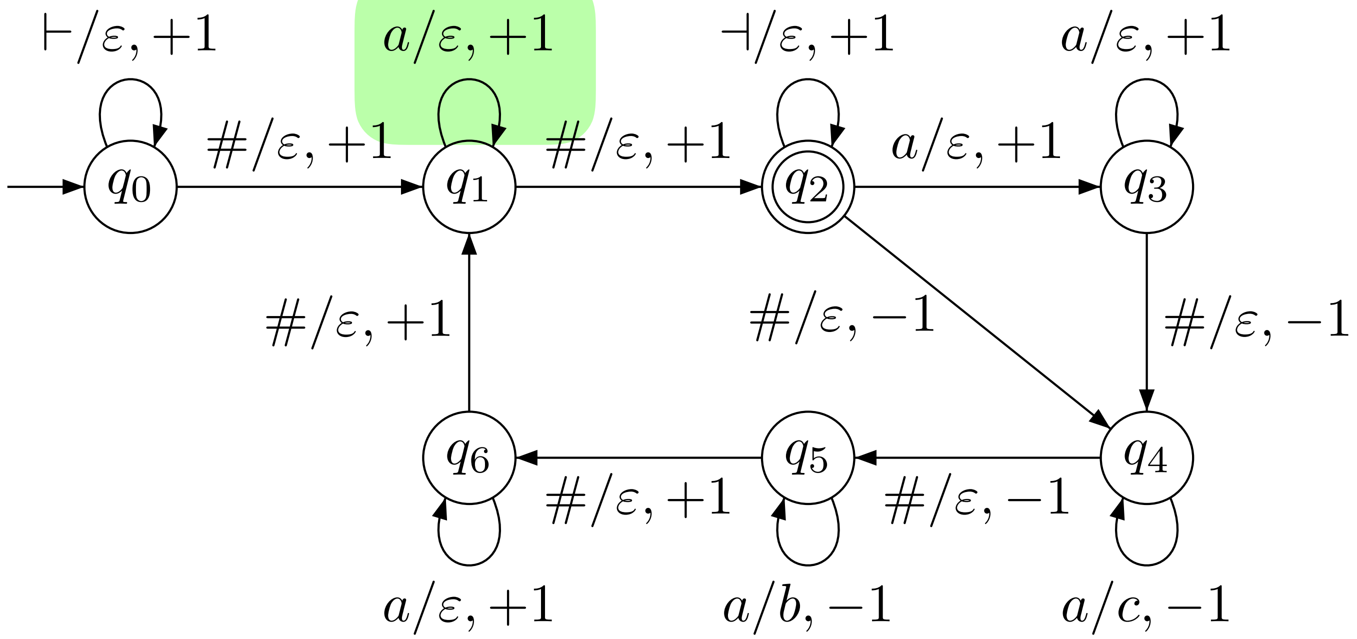

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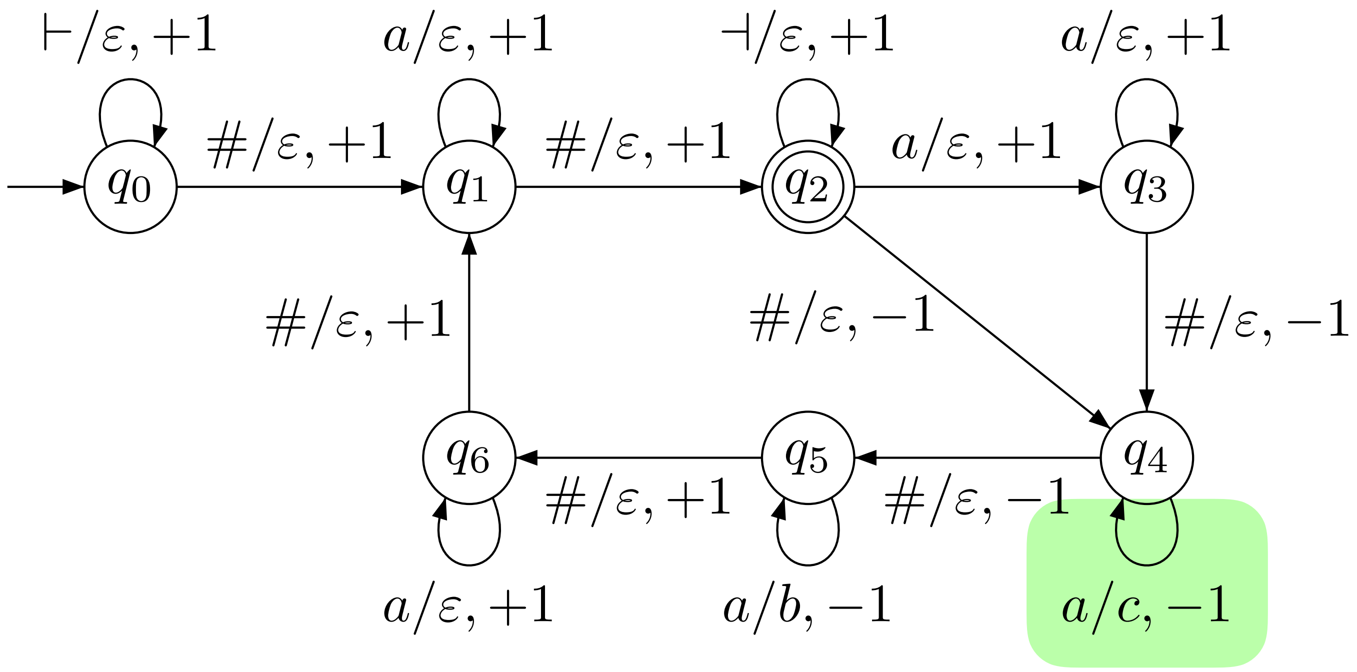

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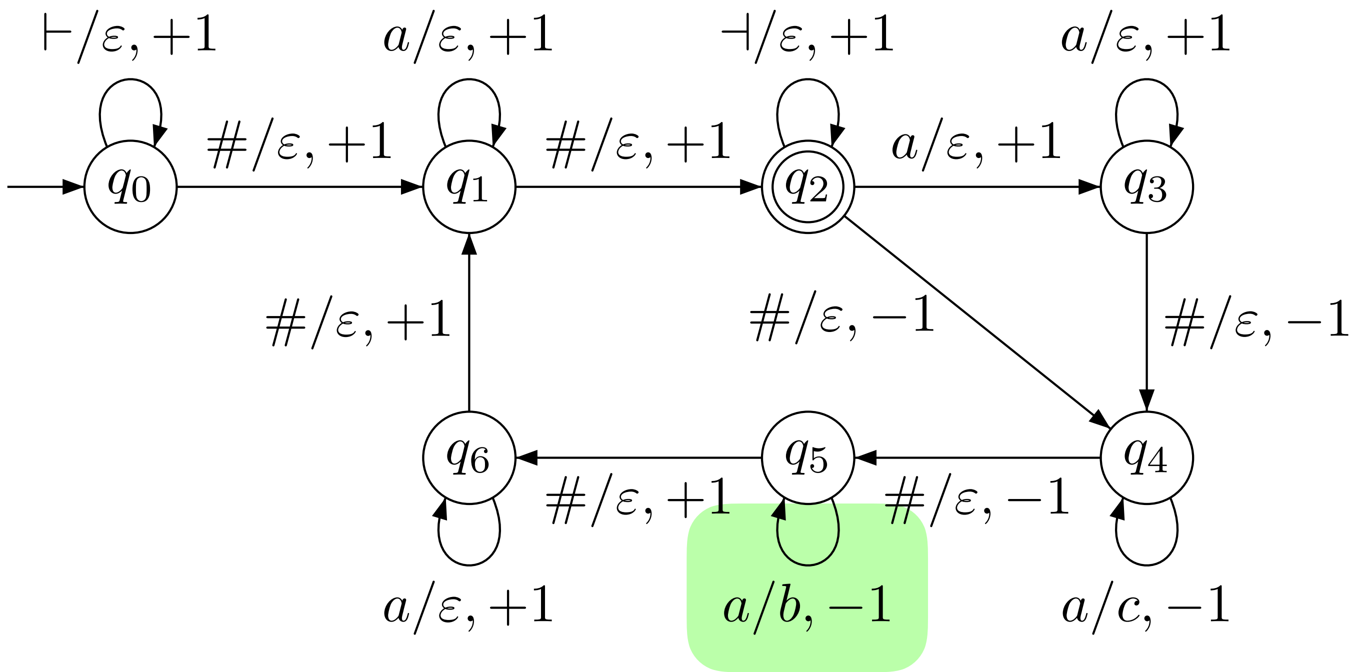

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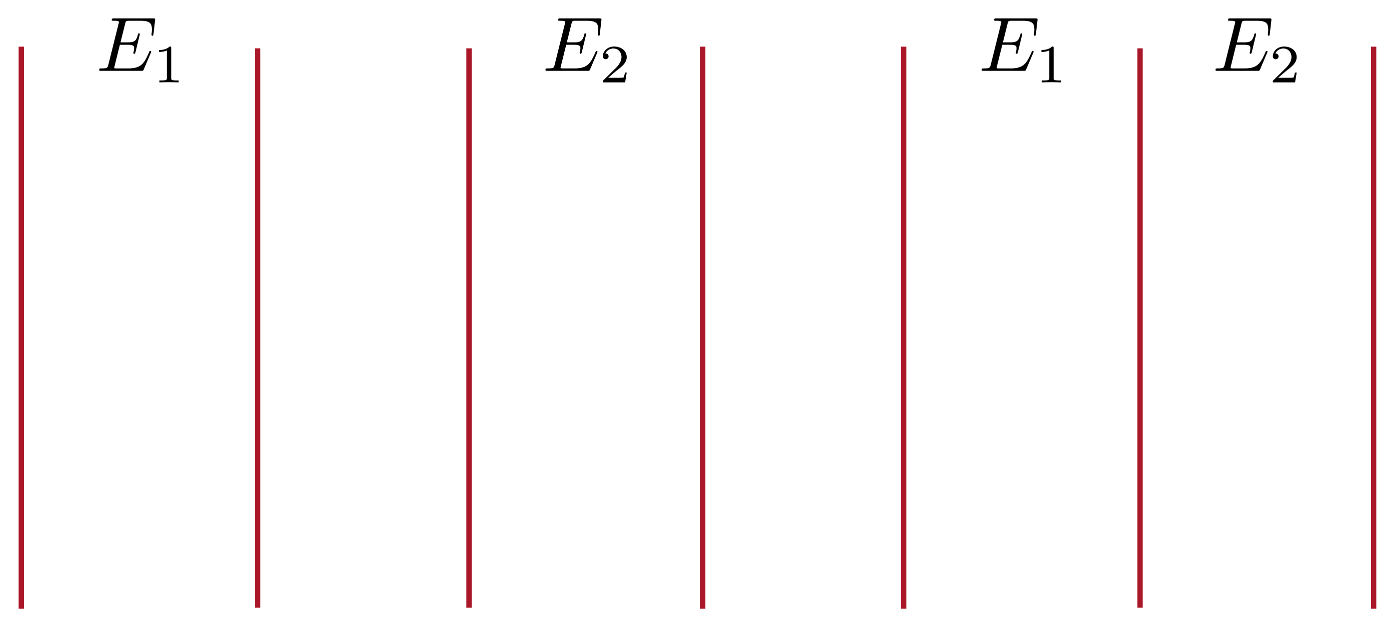

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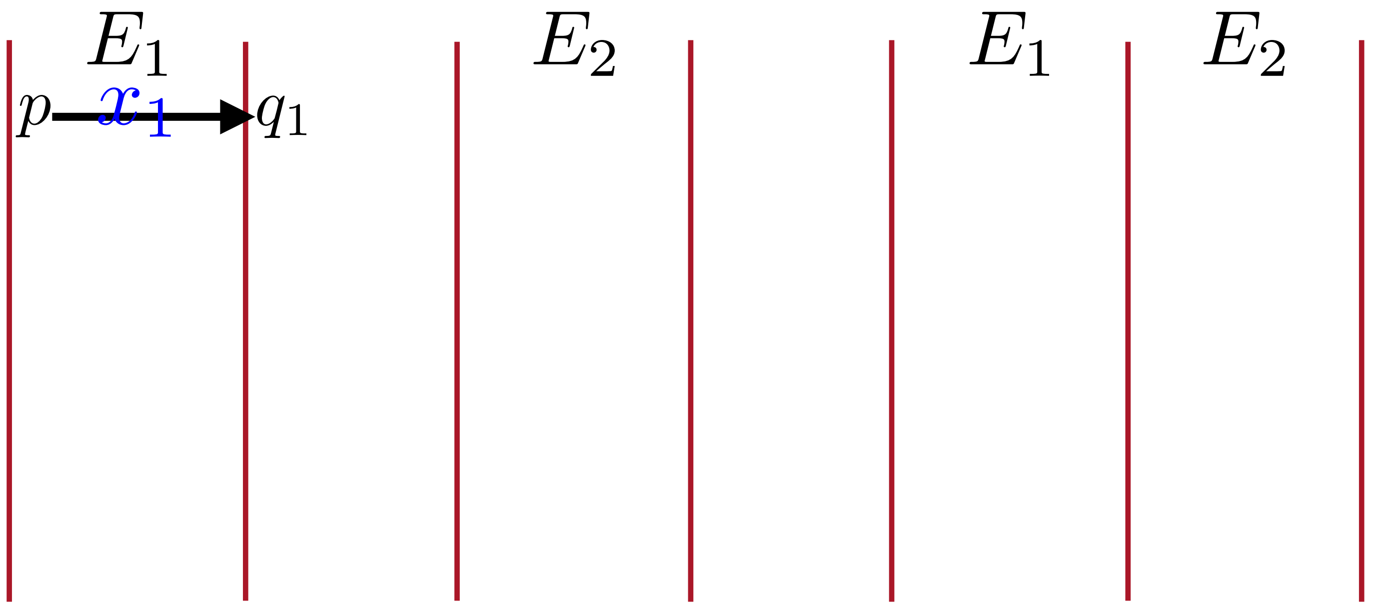

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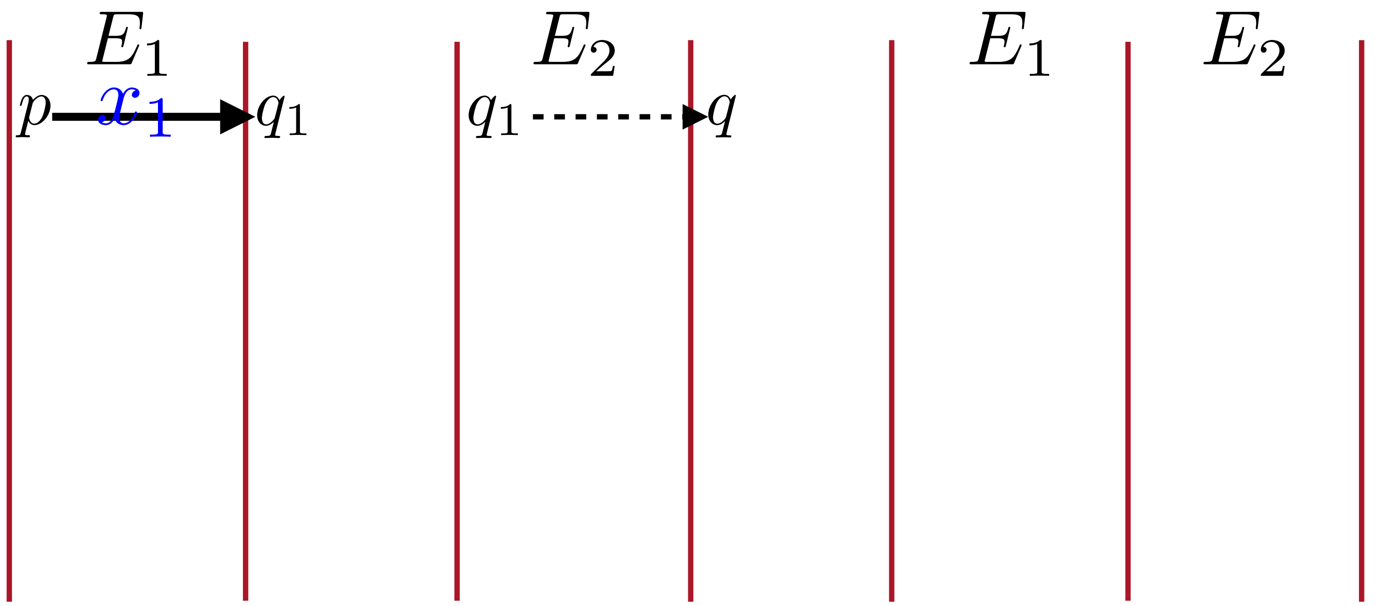

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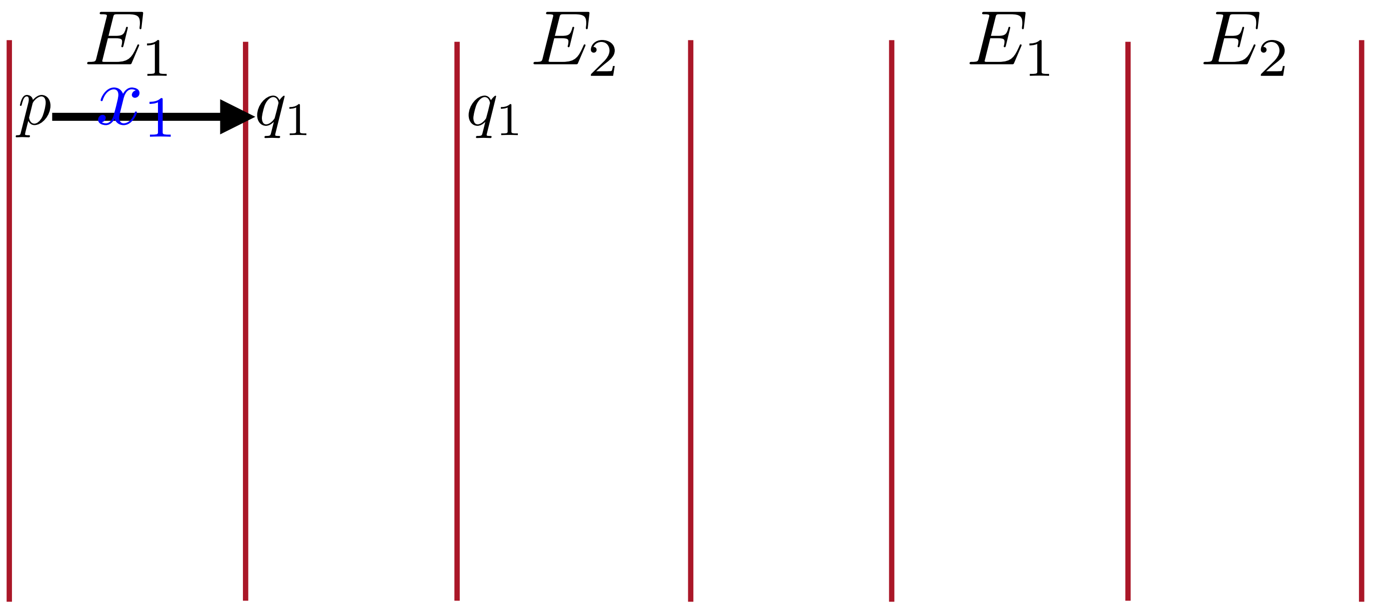

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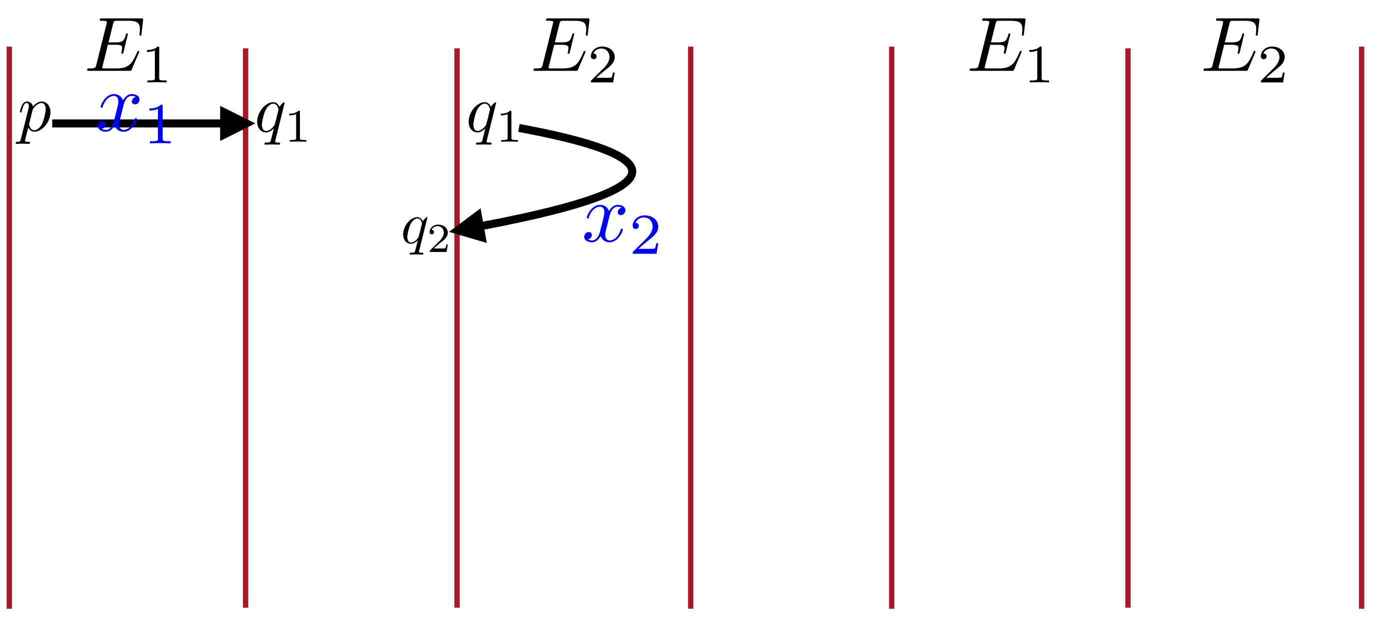

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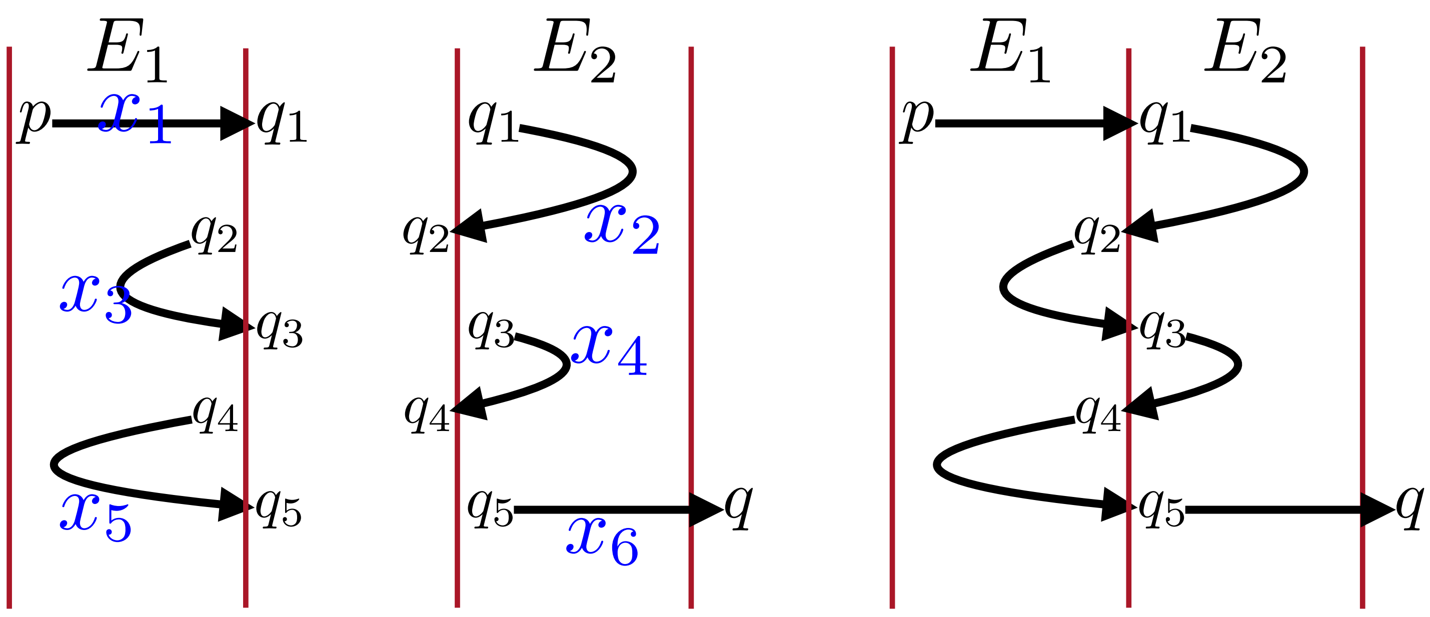

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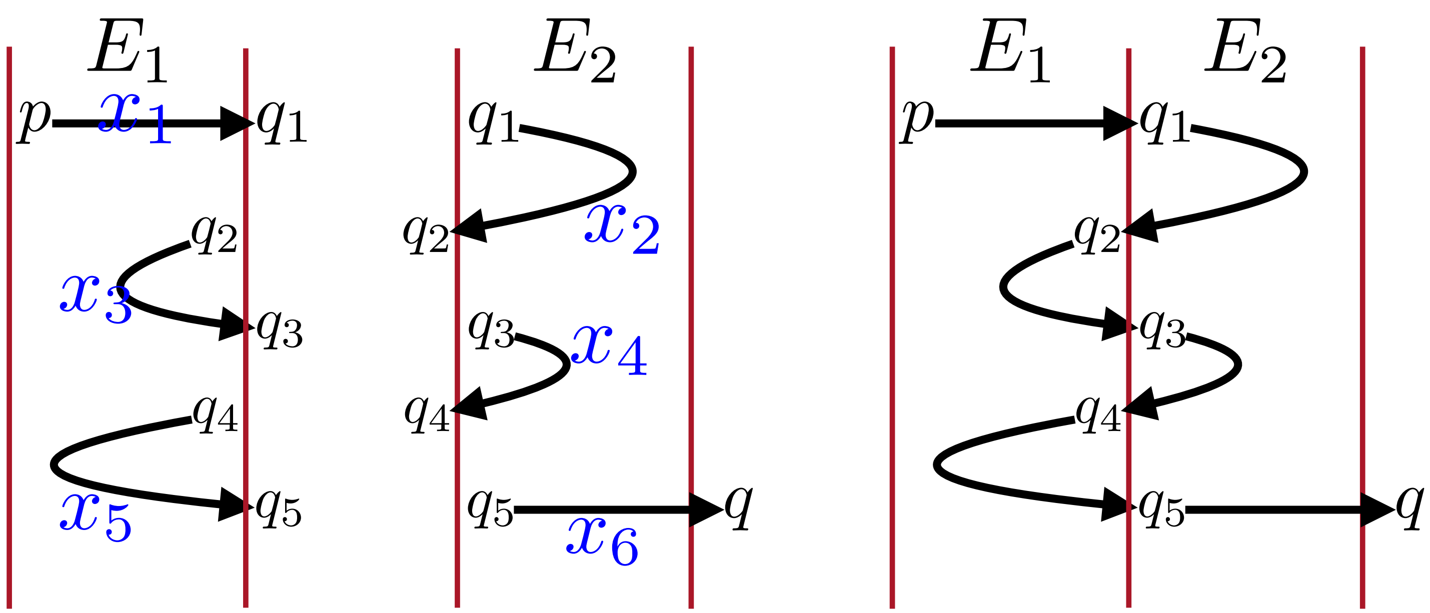

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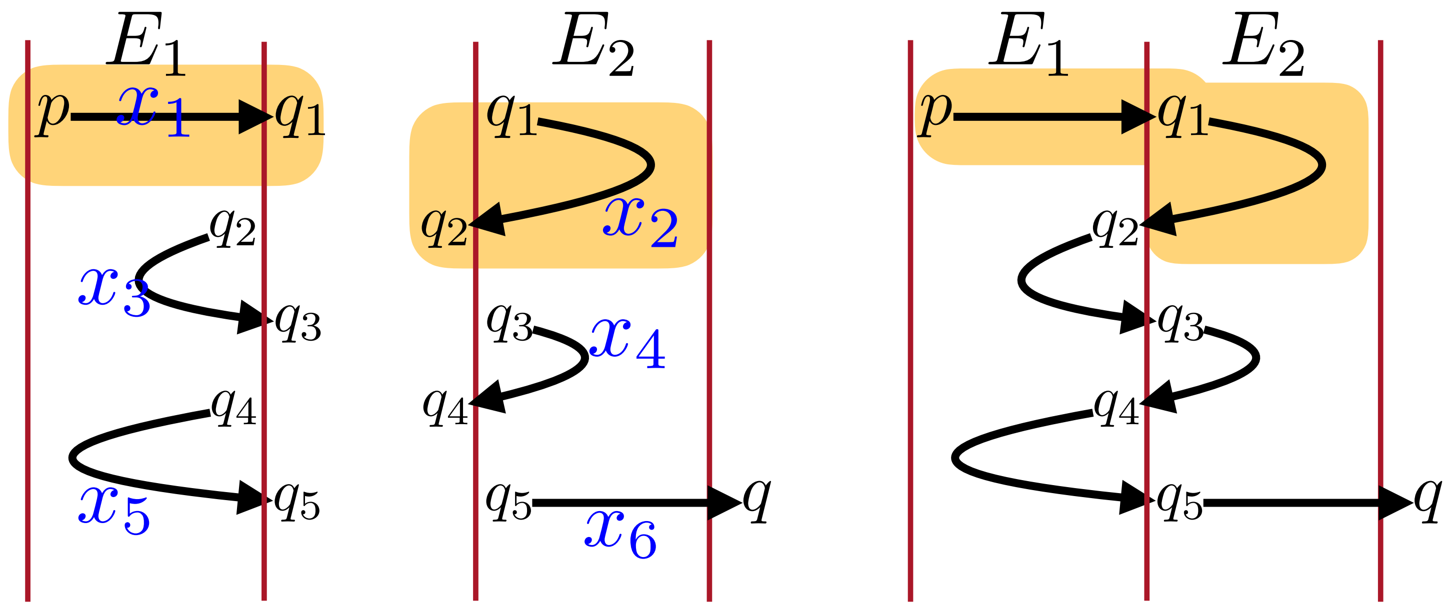

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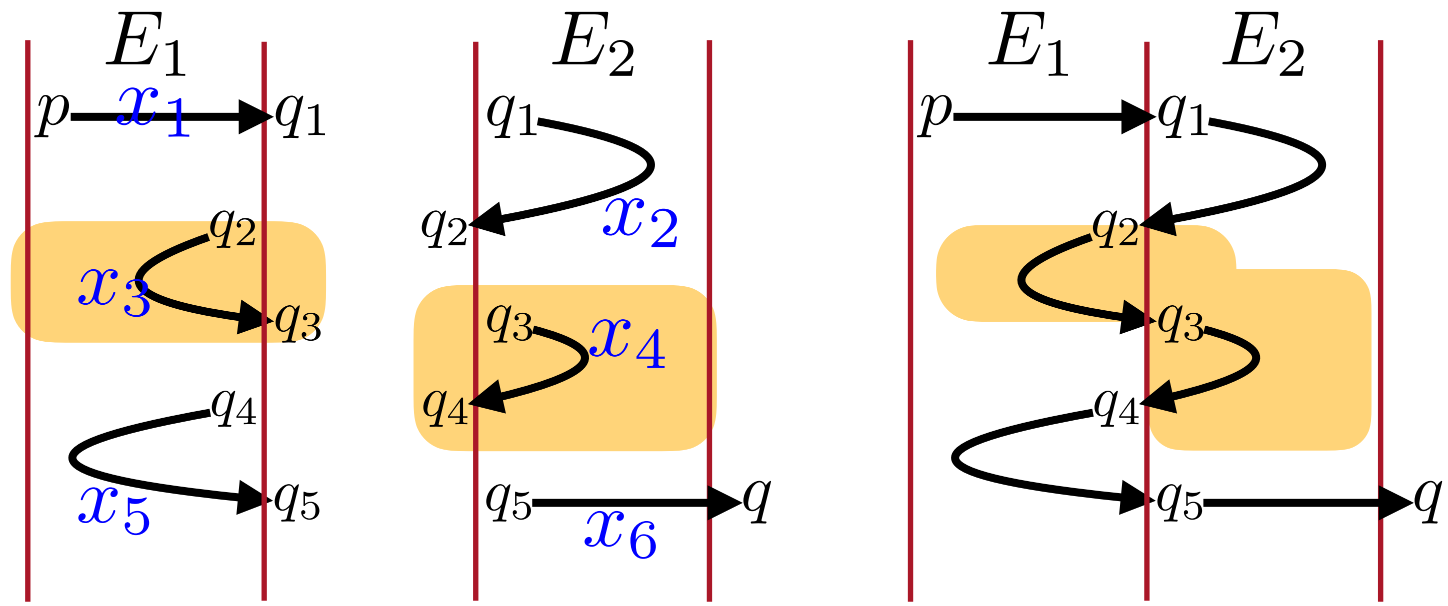

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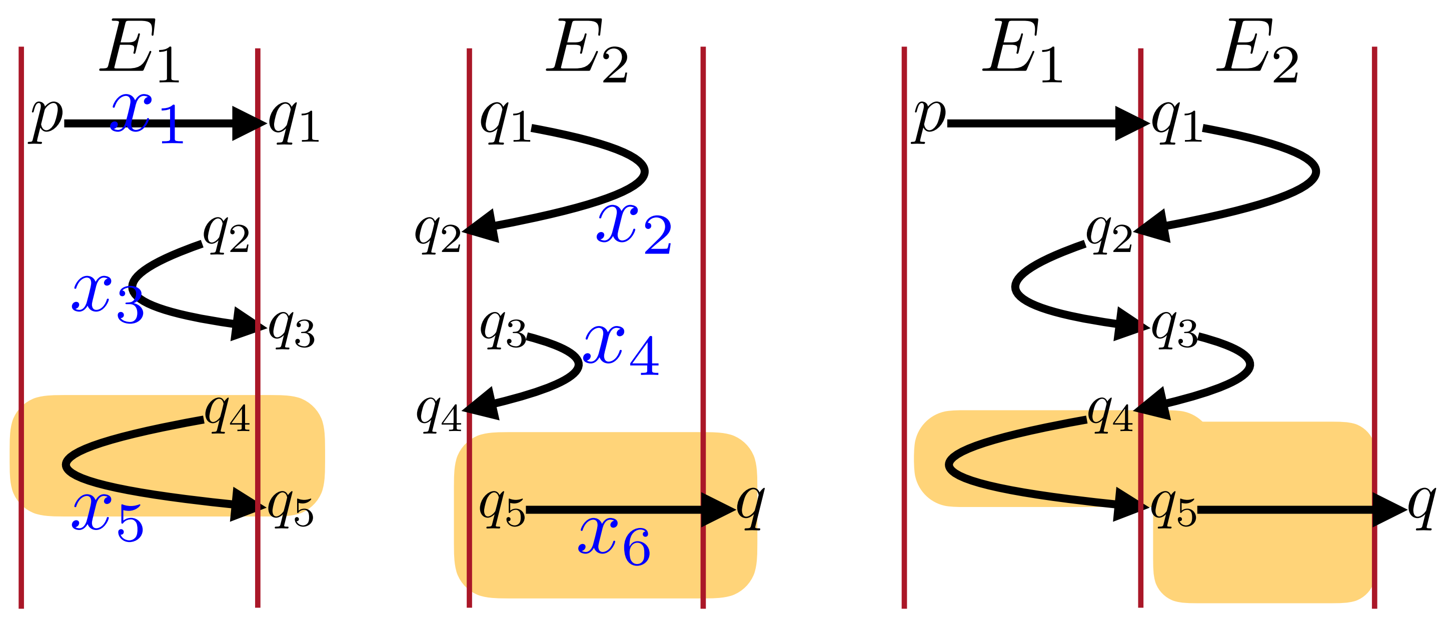

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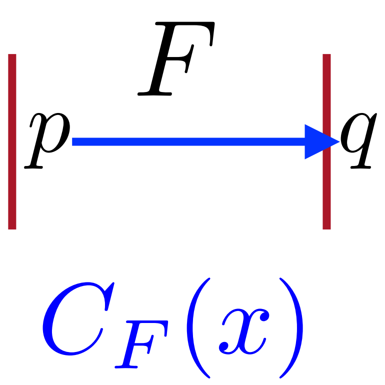
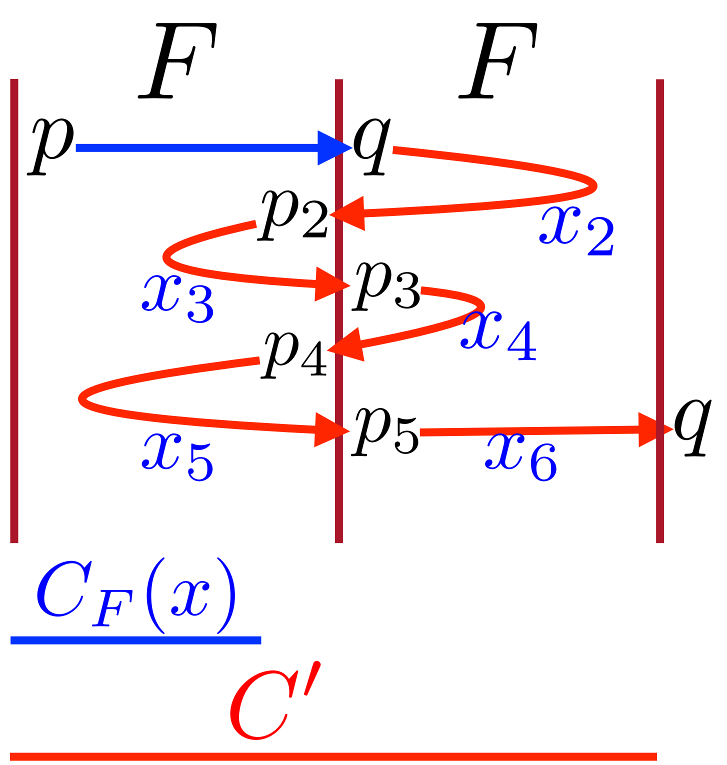
2DFT to RTE: Kleene-plus





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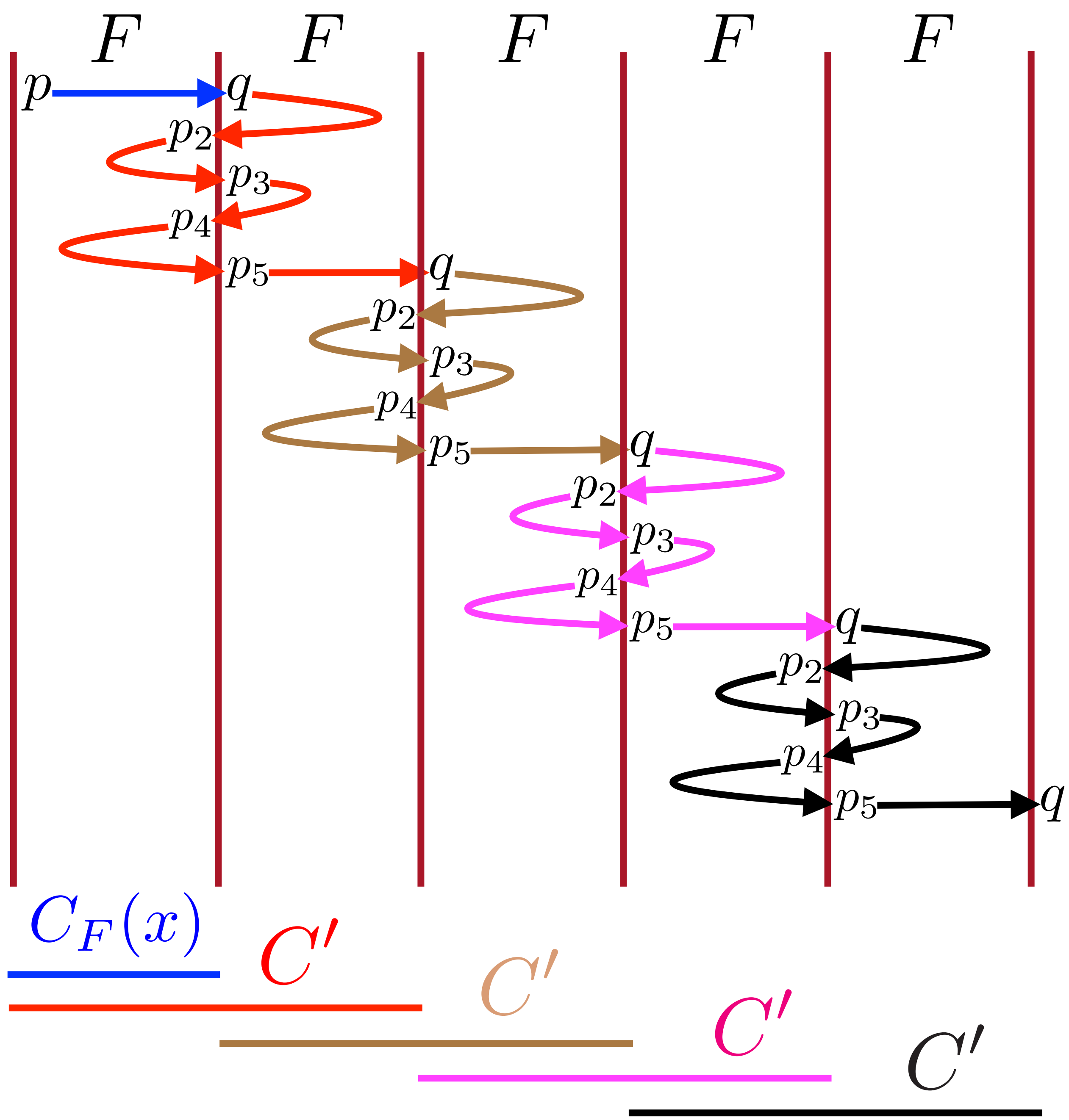

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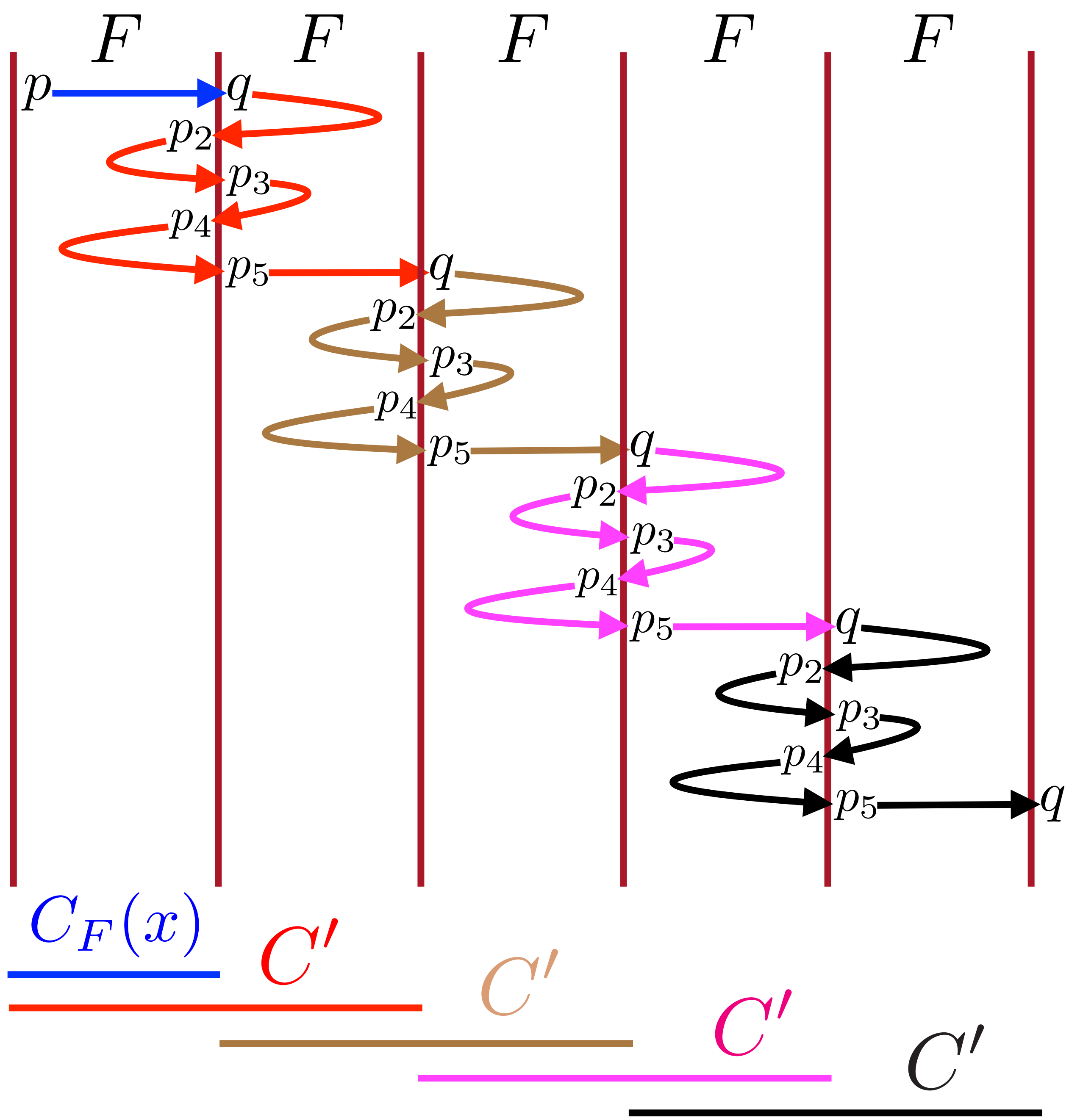
$$x_4 = (p_3, \curvearrowleft, p_4)$$

$$x_5 = (p_4, \curvearrowright, p_5)$$

$$x_6 = (p_5, \rightarrow, q)$$

2DFT to RTE: Kleene-plus



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$$C' = \big((F\,?\,\varepsilon : \bot) \boxdot C_F(x_2)\big) \odot (C_F(x_3) \boxdot C_F(x_4)) \odot (C_F(x_5) \boxdot C_F(x_6))$$

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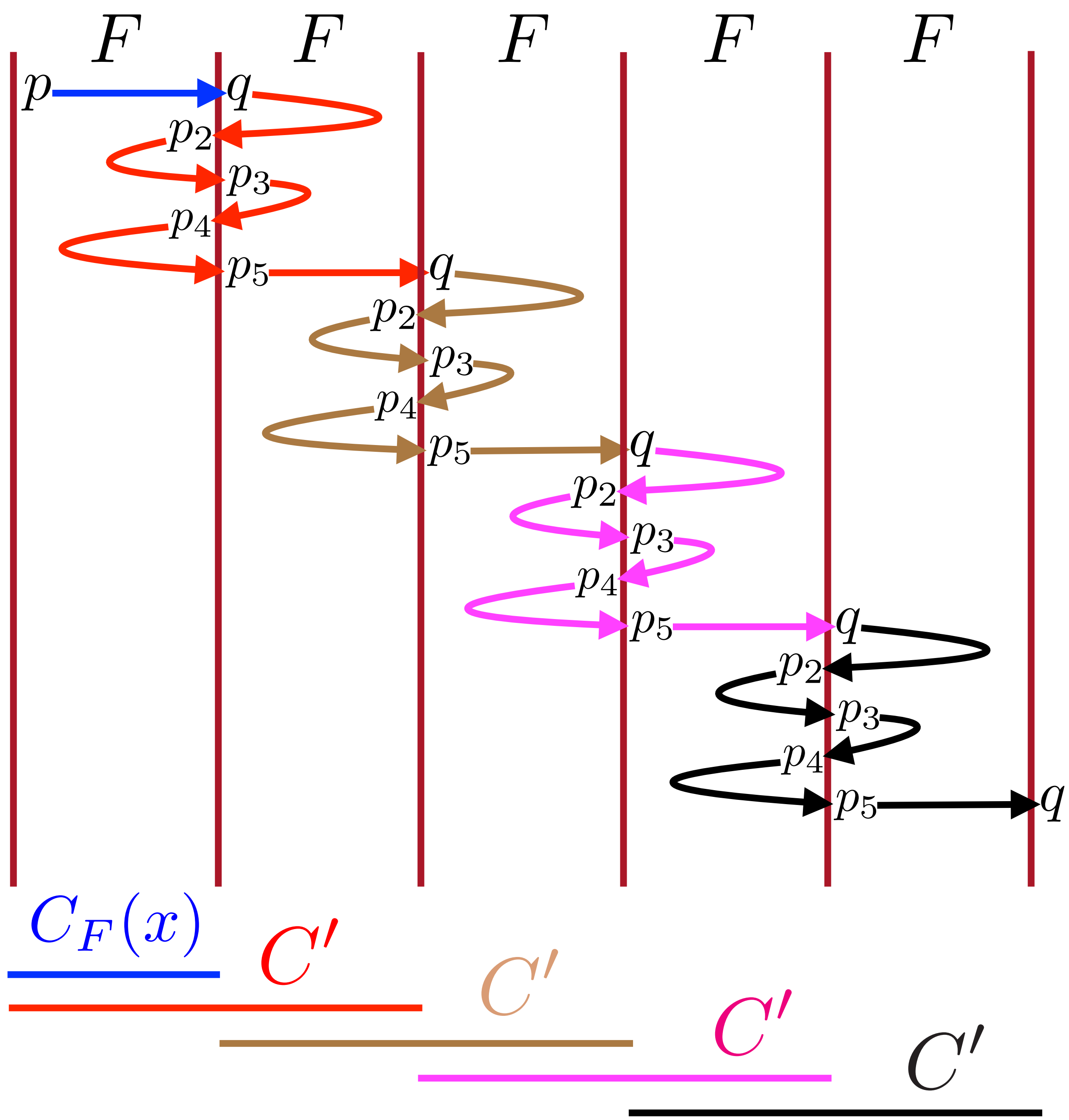

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$$C_{F+}(x) = \big(C_F(x) \boxdot (F^* \,?\, \varepsilon : \bot)\big) \odot [F, C']^{2\boxplus}$$

$$C' = \big((F \,?\, \varepsilon : \bot) \boxdot C_F(x_2)\big) \odot (C_F(x_3) \boxdot C_F(x_4)) \odot (C_F(x_5) \boxdot C_F(x_6))$$

Summary

- Regular Transducer Expressions (RTE)
- Transition Monoid
- Good Rational Expressions
- From 2DFT to RTE
- Extension to Infinite words
- Conclusion

Theorem: (Paul Gastin, S.Krishna)

For each $s \in S$, there is an ε-free *good* rational expression F_s such that

$$\mathcal{L}(F_s) = \varphi^{-1}(s) \setminus \{\varepsilon\} \subseteq \Sigma^+$$

Therefore, $G = \varepsilon \cup \bigcup_{s \in S} F_s$ is an *unambiguous* rational expression over Σ such that $\mathcal{L}(G) = \Sigma^*$.

Main Lemma: $F ::= \emptyset \mid \varepsilon \mid a \mid F \cup F \mid F \cdot F \mid F^+$

Let F be an ε-free Tr-good rational expression with $\mathsf{Tr}(F) = s_F$. We can construct a map $C_F \colon s_F \to \mathsf{RTE}$ such that for each step $x = (p, d, q) \in s_F$:

1. $\mathsf{dom}(C_F(x)) = \mathcal{L}(F)$,
2. for each $u \in \mathcal{L}(F)$, $[\![C_F(x)]\!](u)$ is the output produced by $\mathcal{A}$ when running step x on u (i.e., running $\mathcal{A}$ on u from p to q following direction d).

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If $\mathsf{TrM} = \{s_1, s_2, \ldots, s_m\}$

$$C_{\mathcal{A}} = \varepsilon \,?\, C_\varepsilon : (\mathsf{Tr}^{-1}(s_1) \,?\, C_{F_{s_1}} : (\mathsf{Tr}^{-1}(s_2) \,?\, C_{F_{s_2}} : \cdots$$
$$(\mathsf{Tr}^{-1}(s_{m-1}) \,?\, C_{F_{s_{m-1}}} : C_{F_{s_m}}))) \,.$$

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Regular Transducer Expressions over ω-words

$d \in \Gamma^* \uplus \{\bot\}$

$K \subseteq \Sigma^*$ regular

$$C ::= d \mid K \,?\, C : C \mid C \odot C \mid C \boxdot C \mid C \overleftarrow{\boxdot} C \mid C^{\boxplus} \mid C^{\overleftarrow{\boxplus}} \mid [K, C]^{2\boxplus} \mid [K, C]^{\overleftarrow{2\boxplus}}$$

$$E ::= L \,?\, E : E \mid E \odot E \mid C \boxdot E \mid C^{\omega} \mid [K, C]^{2\omega}$$

$L \subseteq \Sigma^{\omega}$ regular

If then else

$$(L \,?\, g : h)(w) = \begin{cases} g(w) & \text{if } w \in L \\ h(w) & \text{otherwise} \end{cases}$$

Unambiguous ω-iteration

$$f^{\omega}(w) = f(u_1) f(u_2) \cdots \in \Gamma^{\infty}$$

If $w = u_1 u_2 \cdots$ with $u_i \in \mathsf{dom}(f)$

Unambiguous Cauchy product

$$(f \boxdot g)(w) = f(u) \cdot g(v)$$

If $w = u \cdot v$ with $u \in \mathsf{dom}(f)$ and $v \in \mathsf{dom}(g)$

Unambiguous 2-chained ω-iteration

$$[K, f]^{2\omega}(w) = f(u_1 u_2) f(u_2 u_3) \cdots$$

$w = u_1 u_2 \cdots$ with $u_i \in K \; \forall i$

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$$E ::= L\,?\,E : E \mid E \odot E \mid C \boxdot E \mid C^{\omega} \mid [K, C]^{2\omega}$$

$L \subseteq \Sigma^\omega$ regular

Hadamard product

$$(g \odot h)(w) = g(w) \cdot h(w)$$

If $w \in \mathsf{dom}(g) \cap \mathsf{dom}(h)$ with $g(w) \in \Gamma^*$

Extension to Infinite words

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$\mathrm{Tr}\colon \Sigma^* \longrightarrow \mathrm{TrM}$ 🤔

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$\text{Tr}\colon \Sigma^* \longrightarrow \text{TrM}$ 🤔

$$\Sigma^\omega = \bigcup_{k=1}^{m} F_k \cdot G_k^\omega$$

$F_k, G_k -$ good

$G_k \rightarrow$ idempotent

Extension to Infinite words

Tr : $\Sigma^* \to$ TrM 🤔

$$\Sigma^\omega = \bigcup_{k=1}^{m} F_k \cdot G_k^\omega$$

$F_k, G_k -$ good

$G_k \to$ idempotent

C_{FG^ω} ✔

Conclusion

Regular Transducer Expressions

Finite Transducers Deterministic, two-way

New proof technique
Works directly with 2DFT
Extension to infinite words

Conclusion

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Transducer Expression for Aperiodic Transformation? 🤔