

FO-definable transformations of infinite strings

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Abstract

The theory of regular and aperiodic transformations of finite strings has recently received a lot of interest. These classes can be equivalently defined using logic (Monadic second-order logic and first-order logic), two-way machines (regular two-way and aperiodic two-way transducers), and one-way register machines (regular streaming string and aperiodic streaming string transducers). These classes are known to be closed under operations such as sequential composition and regular (star-free) choice; and problems such as functional equivalence and type checking, are decidable for these classes. On the other hand, for infinite strings these results are only known for ω -regular transformations: Alur, Filiot, and Trivedi studied transformations of infinite strings and introduced an extension of streaming string transducers over ω -strings and showed that they capture monadic second-order definable transformations for infinite strings. In this paper we extend their work to recover connection for infinite strings among first-order logic definable transformations, aperiodic two-way transducers, and aperiodic streaming string transducers.

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1 Introduction

The beautiful theory of regular languages is the cornerstone of theoretical computer science and formal language theory. The perfect harmony among the languages of finite words definable using abstract machines (deterministic finite automata, nondeterministic finite automata, and two-way automata), algebra (regular expressions and finite monoids), and logic (monadic second-order logic (MSO) [7]) set the stage for the generalizations of the theory to not only for the theory of regular languages of infinite words [8, 16], trees [5], partial orders [21], but more recently for the theory of regular transformations of the finite strings [3], infinite strings [4, 1], and trees [2]. For the theory of regular transformations it has been shown that abstract machines (two-way transducers [12] and streaming string transducers [3]) precisely capture the transformations definable via monadic second-order logic transformations [10]. For a detailed exposition on the regular theory of languages and transformations, we refer to the surveys by Thomas [21, 22] and Filiot [13], respectively.

There is an equally appealing and rich theory for first-order logic (FO) definable subclasses of regular languages. McNaughton and Papert [17] observed the equivalence between FO-definability and star-free regular expressions for finite words, while Ladner [15] and Thomas [23] extended this connection to infinite words. The equivalence of star-free regular expressions and languages defined via aperiodic monoids is due to Schützenberger [19] and corresponding extension to infinite words is due to Perrin [18]. For a detailed introduction to FO-definable language we refer the reader to Diekert and Gastin [11].

The results for the theory of FO-definable transformations are relatively recent. While Courcelle's definition of logic based transformations [10] provides a natural basis for FO-definable transformations of finite as well as infinite words, [14] observed that over finite words, streaming string transducers [3] with an appropriate notion of aperiodicity precisely capture the same class of transformations. Carton and Dartois [9] introduced aperiodic two-way transducers for finite words and showed that it precisely captures the notion of

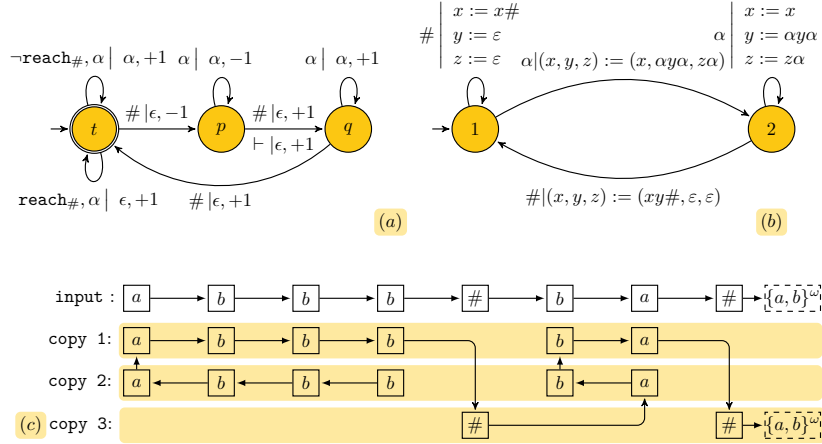


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■ **Figure 1** Transformation f_1 given as (a) two-way transducers with look-ahead (b) streaming string transducers with $F(\{2\}) = xz$ is the output associated with Muller set $\{2\}$, and (c) FO-definable transformation for the string $abb\#ba\#\{a, b\}^\omega$. Here symbol α stands for both symbols a and b , and the predicate $\text{reach}\#$ is the lookahead that checks whether string contains a $\#$ in future.

FO-definability. We consider transformations of infinite strings and generalize these results by showing that appropriate aperiodic restrictions on two-way transducers and streaming string transducers on infinite strings capture the essence of FO-definable transformations. Let us study an example to see how the following ω -transformation can be represented using logic, two-way transducers, and streaming string transducers.

► **Example 1 (Example Transformation).** Let $\Sigma = \{a, b, \#\}$. Consider an ω -transformation $f_1 : \Sigma^\omega \rightarrow \Sigma^\omega$ such that it replaces any maximal $\#$ -free finite string u by $\bar{u}u$, where \bar{u} is the reverse of u . Moreover f_1 is defined only for strings with finitely many $\#$'s, e.g. for all $w = u_1\#u_2\#\dots u_n\#v$ s.t. $u_i \in \{a, b\}^*$ and $v \in \{a, b\}^\omega$, we have $f_1(w) = \bar{u}_1u_1\#\dots\#\bar{u}_nu_n\#v$.

Logic based transformations. Logical descriptions of transformations of structures—as introduced by Courcelle [10]—work by introducing a fixed number of copies of the vertices of the input graph; and the domain, the labels and the edges of the output graph are defined by MSO formulae with zero, one or two free variables, respectively, interpreted over the input graph. Figure 1(c) shows a way to express transformation f_1 using three copies of the input with a) logical formula ϕ_{dom} expressing the domain of the transformation, b) logical formulae $\phi_\alpha^c(i)$ (with one free variable) for every copy $c \in \{1, 2\}$ and letter $\alpha \in \{a, b\}$ expressing the label of a position i for copy c , and c) logical formulae $\phi^{c,d}(i, j)$ with two free variables expressing the edge from position i of copy c to position j of copy d . The formulae ϕ_{dom} , ϕ_α^c , and $\phi^{c,d}$ are interpreted over input structure (in this paper always an infinite string), and it is easy to see that these formulae for our example can easily be expressed in MSO. In this paper we study logical transformations expressible with FO and to cover a larger class of transformations, we use natural order relation $<$ for positions instead of the successor relation. We will later show that the transformation f_1 indeed can be expressed using FO.

Two-Way Transducers. For finite string transformations, Engelfriet and Hoogeboom [12] showed that the finite-state transducers when equipped with a two-way input tape have the same expressive power as MSO transducers, and Carton and Dartois [9] recovered this result for FO transducers and two-way transducers with aperiodicity restriction. A crucial property of two-way finite-state transducers exploited in these proofs [12, 9] is the fact that transitions

capable of regular (star-free) *look-ahead* (i.e., transitions that test the whole input string against a regular property) do not increase the expressiveness of regular (aperiodic) two-way transducers. However, this property does not hold in case of ω -strings. In Figure 1(a), we show a two-way transducer characterizing transformation f_1 . The transducer uses the lookahead $\text{reach}_\#$ to check if the remaining part of the string contains a $\#$ in future. A transition labeled $\langle \phi, \alpha | \beta, +1 \rangle$ of the two-way transducer should be read as: if the current position on the string satisfies the look-ahead ϕ and the current symbol is α then output symbol β and move the input tape head to the right. This transducer works by first checking if the string contains a $\#$ in the future of the current position, if so it moves its head all the way to the position before $\#$ and starts outputting the symbols in reverse, and when it sees the end-marker or a $\#$ it prints the string before the $\#$; however, if there is no $\#$ in future, then the transducer outputs the rest of the string. It is straightforward to verify that this transducer characterizes the transformation f_1 . However, in the absence of the look-ahead a two-way transducer can not express this transformation.

Streaming String Transducers. Alur and Černý [3, 6] proposed a one-way finite-state transducer model, called the *streaming string transducers* (SST), that manipulates a finite set of string variables to compute its output, and showed that they have same expressive power as MSO transducers. SST, instead of appending symbols to the output tape, concurrently update all string variables using a concatenation of string variables and output symbols in a *copyless* fashion, i.e. no variable occurs more than once in each concurrent variable update. The transformation of a string is then defined using an output (partial) function F that associates states with a copyless concatenation of string variables, s.t. if the state q is reached after reading the string and $F(q)=xy$, then the output string is the final valuation of x concatenated with that of y . [4] generalized this by introducing a Muller acceptance condition to give an SST to characterize ω -transitions. Figure 1(b) shows a streaming string transducer accepting the transformation f_1 . It uses three string variables and concurrently prepends and/or appends these variables in a copyless fashion to construct the output. The acceptance set and the output is characterized by a Muller set (here $\{2\}$ and its output xz), such that if the infinitely visiting states set is $\{2\}$ then the output is limit of the values of the concatenation xz . Again, it is easy to verify that SST in Figure 1(b) captures the transformation f_1 .

Contributions and Challenges. Our main contributions include the definition of aperiodic streaming string transducers and aperiodic two-way transducers, and the proof of the following key theorem connecting FO and transducers for transformations of infinite strings.

► **Theorem 2.** *Let $F : \Sigma^\omega \rightarrow \Gamma^\omega$. Then the following assertions are equivalent:*

1. *F is first-order definable.*
2. *F is definable by some aperiodic two-way transducer with star-free look-around.*
3. *F is definable by some aperiodic streaming string transducers.*

We introduce the notion of transition monoids for automata, 2WST, and SST with the Muller acceptance condition; and recover the classical result proving aperiodicity of a language using the aperiodicity of the transition monoid of its underlying automaton. The equivalence between FOT and 2WST with star-free look-around (Section 4), crucially uses the transition monoid with Muller acceptance, which is necessary to show aperiodicity of the underlying language of the 2WST. On the other hand, while going from aperiodic SST to FOT (Section 5), the main difficulty is the construction of the FOT using the aperiodicity of the SST, and while going from 2WST with star-free look-around to SST (Section 6), the hard part is to establish the aperiodicity of the SST. Due to space limitation, we only provide key

definitions and sketches of our results—complete proofs and related supplementary material can be found in the appendix.

2 Preliminaries

A finite (infinite) string over alphabet Σ is a finite (infinite) sequence of letters from Σ . We denote by ϵ the empty string. We write Σ^* for the set of finite strings, Σ^ω for the set of ω -strings over Σ , and $\Sigma^\infty = \Sigma^* \cup \Sigma^\omega$ for the set of finite and ω -strings. A language L over an alphabet Σ is defined as a set of strings, i.e. $L \subseteq \Sigma^\infty$.

For a string $s \in \Sigma^\infty$ we write $|s|$ for its length; note that $|s| = \infty$ for an ω -string s . Let $\text{dom}(s) = \{1, 2, 3, \dots\}$ be the set of positions in s . For all $i \in \text{dom}(s)$ we write $s[i]$ for the i -th letter of the string s . For two ω -strings $s, s' \in \Sigma^\omega$, we define the distance $d(s, s')$ as $\frac{1}{2^j}$ where $j = \min\{k \mid s[k] \neq s'[k]\}$. We say that a string $s \in \Sigma^\omega$ is the limit of a sequence s_1, s_2, \dots of ω -strings $s_i \in \Sigma^\omega$ if for every $\epsilon > 0$, there is an index $n_\epsilon \in \mathbb{N}$ such that for all $i \geq n_\epsilon$, we have that $d(s, s_i) \leq \epsilon$. Such a limit, if exists, is unique and is denoted as $s = \lim_{i \rightarrow \infty} s_i$. For example, $b^\omega = \lim_{i \rightarrow \infty} b^i c^\omega$.

2.1 Aperiodic Monoids for ω -String Languages

A monoid \mathcal{M} is an algebraic structure (M, \cdot, e) with a non-empty set M , a binary operation \cdot , and an identity element $e \in M$ such that for all $x, y, z \in M$ we have that $(x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$, and $x \cdot e = e \cdot x$ for all $x \in M$. We say that a monoid (M, \cdot, e) is *finite* if the set M is finite. A monoid that we will use in this paper is the *free* monoid, $(\Sigma^*, \cdot, \epsilon)$, which has a set of finite strings over some alphabet Σ with the empty string ϵ as the identity.

We define the notion of acceptance of a language via monoids. A morphism (or homomorphism) between two monoids $\mathcal{M} = (M, \cdot, e)$ and $\mathcal{M}' = (M', \times, e')$ is a mapping $h : M \rightarrow M'$ such that $h(e) = e'$ and $h(x \cdot y) = h(x) \times h(y)$. Let $h : \Sigma^* \rightarrow M$, be a morphism from free monoid $(\Sigma^*, \cdot, \epsilon)$ to a finite monoid (M, \cdot, e) . Two strings $u, v \in \Sigma^*$ are said to be similar with respect to h denoted $u \sim_h v$, if for some $n \in \mathbb{N} \cup \{\infty\}$, we can factorize u, v as $u = u_1 u_2 \dots u_n$ and $v = v_1 v_2 \dots v_n$ with $u_i, v_i \in \Sigma^+$ and $h(u_i) = h(v_i)$ for all i . Two ω -strings are h -similar if we can find factorizations $u_1 u_2 \dots$ and $v_1 v_2 \dots$ such that $h(u_i) = h(v_i)$ for all i . Let \cong be the transitive closure of \sim_h . \cong is an equivalence relation. Note that since M is finite, the equivalence relation \cong is of finite index. For $w \in \Sigma^\infty$ we define $[w]_h$ as the set $\{u \mid u \cong w\}$. We say that a morphism h accepts a language $L \subseteq \Sigma^\infty$ if $w \in L$ implies $[w]_h \subseteq L$ for all $w \in \Sigma^\infty$.

We say that a monoid (M, \cdot, e) is *aperiodic* [20] if there exists $n \in \mathbb{N}$ such that for all $x \in M$, $x^n = x^{n+1}$. Note that for finite monoids, it is equivalent to require that for all $x \in M$, there exists $n \in \mathbb{N}$ such that $x^n = x^{n+1}$. A language $L \subseteq \Sigma^\infty$ is said to be aperiodic iff it is recognized by some morphism to a finite and aperiodic monoid (See Appendix A).

2.2 First-Order Logic for ω -String Languages

A string $s \in \Sigma^\omega$ can be represented as a relational structure $\Xi_s = (\text{dom}(s), \preceq^s, (L_a^s)_{a \in \Sigma})$, called the string model of s , where $\text{dom}(s) = \{1, 2, \dots\}$ is the set of positions in s , \preceq^s is a binary relation over the positions in s characterizing the natural order, i.e. $(x, y) \in \preceq^s$ if $x \leq y$; L_a^s , for all $a \in \Sigma$, are the unary predicates that hold for the positions in s labeled with the letter a , i.e., $L_a^s(i)$ iff $s[i] = a$, for all $i \in \text{dom}(s)$. When it is clear from context we will drop the superscript s from the relations \preceq^s and L_a^s .

Properties of string models over the alphabet Σ can be formalized by first-order logic denoted by $\text{FO}(\Sigma)$. Formulas of $\text{FO}(\Sigma)$ are built up from variables x, y, \dots ranging over positions of string models along with *atomic formulae* of the form $x=y, x \preceq y$, and $L_a(x)$ for all $a \in \Sigma$ where formula $x=y$ states that variables x and y point to the same position, the formula $x \preceq y$ states that position corresponding to variable x is not larger than that of y , and the formula $L_a(x)$ states that position x has the label $a \in \Sigma$. Atomic formulae are connected with *propositional connectives* $\neg, \wedge, \vee, \rightarrow$, and *quantifiers* \forall and \exists that range over node variables and we use usual semantics for them. We say that a variable is *free* in a formula if it does not occur in the scope of some quantifier. A *sentence* is a formula with no free variables. We write $\phi(x_1, x_2, \dots, x_k)$ to denote that at most the variables x_1, \dots, x_k occur free in ϕ . For a string $s \in \Sigma^*$ and for positions $n_1, n_2, \dots, n_k \in \text{dom}(s)$ we say that s with valuation $\nu = (n_1, n_2, \dots, n_k)$ satisfies the formula $\phi(x_1, x_2, \dots, x_k)$ and we write $(s, \nu) \models \phi(x_1, x_2, \dots, x_k)$ or $s \models \phi(n_1, n_2, \dots, n_k)$ if formula ϕ with n_i as the interpretation of x_i is satisfied in the string model Ξ_s . The language defined by an FO sentence ϕ is $L(\phi) \stackrel{\text{def}}{=} \{s \in \Sigma^\omega : \Xi_s \models \phi\}$. We say that a language L is FO-definable if there is an FO sentence ϕ such that $L = L(\phi)$. The following is a well known result.

► **Theorem 3.** [17][19] *A language $L \subseteq \Sigma^*$ is FO-definable iff it is aperiodic.*

2.3 Aperiodic Muller Automata for ω -String Languages

A deterministic Muller automaton (DMA) is a tuple $\mathcal{A} = (Q, q_0, \Sigma, \delta, F)$ where Q is a finite set of states, $q_0 \in Q$ is the initial state, Σ is an input alphabet, $\delta : Q \times \Sigma \rightarrow Q$ is a transition function, and $F \subseteq 2^Q$ are the accepting (Muller) sets. For states $q, q' \in Q$ and letter $a \in \Sigma$ we say that (q, a, q') is a transition of the automaton \mathcal{A} if $\delta(q, a) = q'$ and we write $q \xrightarrow{a} q'$. We say that there is a run of \mathcal{A} over a finite string $s = a_1 a_2 \dots a_n \in \Sigma^*$ from state p to state q if there is a finite sequence of transitions $\langle (p_0, a_1, p_1), (p_1, a_2, p_2), \dots, (p_{n-1}, a_n, p_n) \rangle \in (Q \times \Sigma \times Q)^*$ with $p = p_0$ and $q = p_n$. We write $L_{p,q}$ for the set of finite strings w such that there is a run of \mathcal{A} over w from p to q . We say that there is a run of \mathcal{A} over an ω -string $s = a_1 a_2 \dots \in \Sigma^\omega$ if there is a sequence of transitions $\langle (q_0, a_1, q_1), (q_1, a_2, q_2), \dots \rangle \in (Q \times \Sigma \times Q)^\omega$. For an infinite run r , we denote by $\Omega(r)$ the set of states that occur infinitely often in r . We say that an ω -string w is accepted by a Muller automaton \mathcal{A} if the run of \mathcal{A} on w is such that $\Omega(r) \in F$ and we write $L(\mathcal{A})$ for the set of all ω -strings accepted by \mathcal{A} .

A Muller automaton \mathcal{A} is *aperiodic* iff there exists some $m \geq 1$ s.t. $u^m \in L_{p,q}$ iff $u^{m+1} \in L_{p,q}$ for all $u \in \Sigma^*$ and $p, q \in Q$. Another equivalent way to define aperiodicity is using the transition monoid, which, to the best of our knowledge, has not been defined in the literature for Muller automata. Given a DMA $\mathcal{A} = (Q, q_0, \Sigma, \Delta, \{F_1, \dots, F_n\})$, we define the transition monoid $\mathcal{M}_{\mathcal{A}} = (M_{\mathcal{A}}, \times, \mathbf{1})$ of \mathcal{A} as follows: $M_{\mathcal{A}}$ is a set of $|Q| \times |Q|$ square matrices over $(\{0, 1\} \cup 2^Q)^n \cup \{\perp\}$. Matrix multiplication \times is defined for matrices in $M_{\mathcal{A}}$ with identity element $\mathbf{1} \in M_{\mathcal{A}}$, where $\mathbf{1}$ is the matrix whose diagonal entries are $(\emptyset, \emptyset, \dots, \emptyset)$ and non-diagonal entries are all \perp 's. Formally, $M_{\mathcal{A}} = \{M_s : s \in \Sigma^*\}$ is defined using matrices M_s for strings $s \in \Sigma^*$ s.t. $M_s[p][q] = \perp$ if there is no run from p to q over s in \mathcal{A} . Otherwise, let P be the set of states (excluding p and q) witnessed in the unique run from p to q . Then $M_s[p][q] = (x_1, \dots, x_n) \in (\{0, 1\} \cup 2^Q)^n$ where (1) $x_i = 0$ iff $\exists t \in P \cup \{p, q\}, t \notin F_i$; (2) $x_i = 1$ iff $P \cup \{p, q\} = F_i$, and (3) $x_i = P \cup \{p, q\}$ iff $P \cup \{p, q\} \subset F_i$. It is easy to see that $M_\epsilon = \mathbf{1}$, since ϵ takes a state to itself and nowhere else. The operator \times is simply matrix multiplication for matrices in $M_{\mathcal{A}}$, however we need to define addition \oplus and multiplication \odot for elements $(\{0, 1\} \cup 2^Q)^n \cup \{\perp\}$ of the matrices. We have $\alpha_1 \odot \alpha_2 = \perp$ if $\alpha_1 = \perp$ or

$\alpha_2 = \perp$, and if $\alpha_1 = (x_1, \dots, x_n)$ and $\alpha_2 = (y_1, \dots, y_n)$ then $\alpha_1 \odot \alpha_2 = (z_1, \dots, z_n)$ s.t.:

$$z_i = \begin{cases} 0 & \text{if } x_i = 0 \text{ or } y_i = 0 \\ 1 & \text{if } (x_i = y_i = 1) \text{ or if } (x_i, y_i \subset F_i \text{ and } x_i \cup y_i = F_i) \\ 1 & \text{if } (x_i = 1 \text{ and } y_i \subset F_i) \text{ or } (y_i = 1 \text{ and } x_i \subset F_i) \\ x_i \cup y_i & \text{if } x_i, y_i \subset F_i \text{ and } x_i \cup y_i \subset F_i \end{cases} \quad (\star)$$

Due to determinism, we have that for every matrix M_s and every state p there is at most one state q such that $M_s[p][q] \neq \perp$ and hence the only addition rule we need to introduce is $\alpha \oplus \perp = \perp \oplus \alpha = \alpha$. It is easy to see that $(M_{\mathcal{A}}, \times, \mathbf{1})$ is a monoid (a proof is deferred to the Appendix C.1). It is straightforward to see that a Muller automaton is aperiodic if and only if its transition monoid is aperiodic. Appendix C.2 gives a proof showing that a language $L \subseteq \Sigma^\omega$ is aperiodic iff there is an aperiodic DMA accepting it.

3 Aperiodic Transformations

In this section we formally introduce three models to express FO-transformations, and prepare the machinery required to prove their expressive equivalence in the rest of the paper.

3.1 First-Order Logic Definable Transformations

Courcelle [10] initiated the study of structure transformations using MSO logic. His main idea was to define a transformation $(w, w') \in R$ by defining the string model of w' using a finite number of copies of positions of the string model of w . The existence of positions, various edges, and position labels are then given as $\text{MSO}(\Sigma)$ formulas. We study a restriction of his formalism to use first-order logic to express string transformations.

► **Definition 4.** An *FO string transducer* is a tuple $T = (\Sigma, \Gamma, \phi_{\text{dom}}, C, \phi_{\text{pos}}, \phi_{\preceq})$ where:

- Σ and Γ are finite input and output alphabets;
- ϕ_{dom} is a closed $\text{FO}(\Sigma)$ formula characterizing the domain of the transformation;
- $C = \{1, 2, \dots, n\}$ is a finite index set;
- $\phi_{\text{pos}} = \{\phi_\gamma^c(x) : c \in C \text{ and } \gamma \in \Gamma\}$ is a set of $\text{FO}(\Sigma)$ formulae with a free variable x ;
- $\phi_{\preceq} = \{\phi_{\preceq}^{c,d}(x, y) : c, d \in C\}$ is a set of $\text{FO}(\Sigma)$ formulae with two free variables x and y .

The transformation $\llbracket T \rrbracket$ defined by T is as follows. A string s with $\Xi_s = (\text{dom}(s), \preceq, (L_a)_{a \in \Sigma})$ is in the domain of $\llbracket T \rrbracket$ if $s \models \phi_{\text{dom}}$ and the output string w with structure

$M = (D, \preceq^M, (L_\gamma^M)_{\gamma \in \Gamma})$ is such that

- $D = \{v^c : v \in \text{dom}(s), c \in C \text{ and } \phi^c(v)\}$ is the set of positions where $\phi^c(v) \stackrel{\text{def}}{=} \bigvee_{\gamma \in \Gamma} \phi_\gamma^c(v)$;
- $\preceq^M \subseteq D \times D$ is the ordering relation between positions and it is such that for $v, u \in \text{dom}(s)$ and $c, d \in C$ we have that $v^c \preceq^M u^d$ if $w \models \phi_{\preceq}^{c,d}(v, u)$; and
- for all $v^c \in D$ we have that $L_\gamma^M(v^c)$ iff $\phi_\gamma^c(v)$.

Observe that the output is unique and therefore FO transducers implement functions. A string $s \in \Sigma^\omega$ can be represented by its string-graph with $\text{dom}(s) = \{i \in \mathbb{N}\}$, $\preceq = \{(i, j) \mid i \leq j\}$ and $L_a(i)$ iff $s[i] = a$ for all i . From now on, we denote the string-graph of s as s only. We say that an FO transducer is a *string-to-string* transducer if its domain is restricted to string graphs and the output is also a string graph. We say that a string-to-string transformation is FO-definable if there exists an FO transducer implementing the transformation. We write FOT for the set of FO-definable string-to-string ω -transformations.

► **Example 5.** Figure 1(c) shows a transformation for an FOT that implements the transformation $f_1 : \Sigma^* \# \{a, b\}^\omega \rightarrow \Sigma^\omega$, where $\Sigma = \{a, b, \#\}$, by replacing every maximal $\#$ free string u with $\bar{u}u$. Let $\text{is_string}_\#$ be an FO formula that defines a string that contains a $\#$, and let $\text{reach}_\#(x)$ be an FO formula that is true at a position which has a $\#$ at a later position. To define the FOT formally, we have $\phi_{\text{dom}} = \text{is_string}_\#$, $\phi_\gamma^1(x) = \phi_\gamma^2(x) = L_\gamma(x) \wedge \neg L_\#(x) \wedge \text{reach}_\#(x)$, since we only keep the non $\#$ symbols that can “reach” a $\#$ in the input string in the first two copies. $\phi_\gamma^3(x) = L_\#(x) \vee (\neg L_\#(x) \wedge \neg \text{reach}_\#(x))$, since we only keep the $\#$'s, and the infinite suffix from where there are no $\#$'s. The full list of formulae $\phi^{i,j}$ can be seen in Appendix D.

3.2 Two-way Transducers (2WST)

A 2WST is a tuple $T = (Q, \Sigma, \Gamma, q_0, \delta, F)$ where Σ, Γ are respectively the input and output alphabet, q_0 is the initial state, δ is the transition function and $F \subseteq 2^Q$ is the acceptance set. The transition function is given by $\delta : Q \times \Sigma \rightarrow Q \times \Gamma^* \times \{1, 0, -1\}$. A configuration of the 2WST is a pair (q, i) where $q \in Q$ and $i \in \mathbb{N}$ is the current position of the input string. A run r of a 2WST on a string $s \in \Sigma^\omega$ is a sequence of transitions $(q_0, i_0=0) \xrightarrow{a_1/c_1, \text{dir}} (q_1, i_1) \xrightarrow{a_2/c_2, \text{dir}} (q_2, i_2) \cdots$ where $a_i \in \Sigma$ is the input letter read and $c_i \in \Gamma^*$ is the output string produced during a transition and i_j s are the positions updated during a transition for all $j \in \text{dom}(s)$. dir is the direction, $\{1, 0, -1\}$. W.l.o.g. we can consider the outputs to be over $\Gamma \cup \{\epsilon\}$. The output $\text{out}(r)$ of a run r is simply a concatenation of the individual outputs, i.e. $c_1 c_2 \cdots \in \Gamma^\infty$. We say that the transducer reads the whole string s when $\sup \{i_n \mid 0 \leq n < |r|\} = \infty$. The output of s , denoted $T(s)$ is defined as $\text{out}(r)$ only if $\Omega(r) \in F$ and r reads the whole string s . We write $\llbracket T \rrbracket$ for the transformation captured by T .

Transition Monoid. The transition monoid of a 2WST $T = (Q, \Sigma, \Gamma, q_0, \delta, \{F_1, \dots, F_n\})$ is the transition monoid of its underlying automaton. However, since the 2WST can read their input in both directions, the transition monoid definition must allow for reading the string starting from left side and leaving at the left (left-left) and similar other behaviors (left-right, right-left and right-right). Following [9], we define the behaviors $\mathcal{B}_{xy}(w)$ of a string w for $x, y \in \{\ell, r\}$. $\mathcal{B}_{\ell r}(w)$ is a set consisting of pairs (p, q) of states such that starting in state p in the left side of w the transducer leaves w in right side in state q . In the example in figure 1(a), we have $\mathcal{B}_{\ell r}(ab\#) = \{(t, t), (p, t), (q, t)\}$ and $\mathcal{B}_{rr}(ab\#) = \{(q, t), (t, t), (p, q)\}$. Two words w_1, w_2 are “equivalent” if their left-left, left-right, right-left and right-right behaviors are same. That is, $\mathcal{B}_{xy}(w_1) = \mathcal{B}_{xy}(w_2)$ for $x, y \in \{\ell, r\}$. The transition monoid of T is the conjunction of the 4 behaviors, which also keeps track, in addition, the set of states witnessed in the run, as shown for the deterministic Muller automata earlier. For each string $w \in \Sigma^*$, $x, y \in \{\ell, r\}$, and states p, q , the entries of the matrix $M_u^{xy}[p][q]$ are of the form \perp , if there is no run from p to q on word u , starting from the side x of u and leaving it in side y , and is (x_1, \dots, x_n) otherwise, where x_i is defined exactly as in section 2.3. For equivalent words u_1, u_2 , we have $M_{u_1}^{xy}[p][q] = M_{u_2}^{xy}[p][q]$ for all $x, y \in \{\ell, r\}$ and states p, q . Addition and multiplication of matrices are defined as in the case of Muller automata. See Appendix E for more details. Note that behavioral composition is quite complex, due to left-right movements. In particular, it can be seen from the example that $\mathcal{B}_{\ell r}(ab\#a\#) = \mathcal{B}_{\ell r}(ab\#)\mathcal{B}_{\ell \ell}(a\#)\mathcal{B}_{rr}(ab\#)\mathcal{B}_{\ell r}(a\#)$. Since we assume that the 2WST T is deterministic and completely reads the input string $\alpha \in \Sigma^\omega$, we can find a unique factorization $\alpha = [\alpha_0 \dots \alpha_{p_1}][\alpha_{p_1+1} \dots \alpha_{p_2}] \dots$ such that the run of \mathcal{A} on each α -block progresses from left to right, and each α -block will be processed completely. That is, one can find a unique sequence of states q_{p_1}, q_{p_2}, \dots such that the 2WST starting in initial

state q_0 at the left of the block $\alpha_0 \dots \alpha_{p_1}$ leaves it at the right in state q_{p_1} , starts the next block $\alpha_{p_1+1} \dots \alpha_{p_2}$ from the left in state q_{p_1} and leaves it at the right in state q_{p_2} and so on.

We consider the languages L_{pq}^{xy} for $x, y \in \{\ell, r\}$, where ℓ, r respectively stand for left and right. $L_{pq}^{\ell\ell}$ stands for all strings w such that, starting at state p at the left of w , one leaves the left of w in state q . Similarly, $L_{pq}^{r\ell}$ stands for all strings w such that starting at the right of w in state p , one leaves the left of w in state q . In figure 1(a), note that starting on the right of $ab\#$ in state t , we leave it on the right in state t , while we leave it on the left in state p . So $ab\# \in L_{tt}^{rr}, L_{tp}^{r\ell}$. Also, $ab\# \in L_{pq}^{rr}$.

A 2WST is said to be *aperiodic* iff for all strings $u \in \Sigma^*$, all states p, q and $x, y \in \{\ell, r\}$, there exists some $m \geq 1$ such that $u^m \in L_{pq}^{xy}$ iff $u^{m+1} \in L_{pq}^{xy}$.

Star-Free Lookaround. We wish to introduce aperiodic 2WST that are capable of capturing FO-definable transformations. However, as we discussed earlier (see page 2 in the paragraph on two-way transducers) 2WST without look-ahead are strictly less expressive than MSO transducers. To remedy this we study aperiodic 2WSTs enriched with star-free look-ahead (star-free look-back can be assumed for free).

An aperiodic 2WST with star-free look-around ($2WST_{sf}$) is a tuple (T, A, B) where A is an aperiodic Muller look-ahead automaton and B is an aperiodic look-behind automaton, resp., and $T = (\Sigma, \Gamma, Q, q_0, \delta, F)$ is an aperiodic 2WST as defined earlier except that the transition function $\delta : Q \times Q_B \times \Sigma \times Q_A \rightarrow Q \times \Gamma \times \{-1, 0, +1\}$ may consult look-ahead and look-behind automata to make its decisions. Let $s \in \Sigma^\omega$ be an input string, and $L(A, p)$ be the set of infinite strings accepted by A starting in state p . Similarly, let $L(B, r)$ be the set of finite strings accepted by B starting in state r . We assume that $2WST_{sf}$ are *deterministic* i.e. for every string $s \in \Sigma^\omega$ and every input position $i \leq |s|$, there is exactly one state $p \in Q_A$ and one state $r \in Q_B$ such that $s(i)s(i+1) \dots \in L(A, p)$ and $s(0)s(1) \dots s(i-1) \in L(B, r)$. If the current configuration is (q, i) and $\delta(q, r, s(i), p) = (q', z, d)$ is a transition, such that the string $s(i)s(i+1) \dots \in L(A, p)$ and $s(0)s(1) \dots s(i-1) \in L(B, r)$, then $2WST_{sf}$ writes $z \in \Gamma$ on the output tape and updates its configuration to $(q', i + d)$. Figure 1(a) shows a 2WST with star-free look-ahead $\text{reach}_\#(x)$ capturing the transformation f_1 (details in App. E).

3.3 Streaming ω -String Transducers (SST)

Streaming string transducers(SSTs) manipulate a finite set of string variables to compute their output. In this section we introduce aperiodic SSTs for infinite strings. Let \mathcal{X} be a finite set of variables and Γ be a finite alphabet. A substitution σ is defined as a mapping $\sigma : \mathcal{X} \rightarrow (\Gamma \cup \mathcal{X})^*$. A valuation is defined as a substitution $\sigma : \mathcal{X} \rightarrow \Gamma^*$. Let $\mathcal{S}_{\mathcal{X}, \Gamma}$ be the set of all substitutions $[\mathcal{X} \rightarrow (\Gamma \cup \mathcal{X})^*]$. Any substitution σ can be extended to $\hat{\sigma} : (\Gamma \cup \mathcal{X})^* \rightarrow (\Gamma \cup \mathcal{X})^*$ in a straightforward manner. The composition $\sigma_1\sigma_2$ of two substitutions σ_1 and σ_2 is defined as the standard function composition $\hat{\sigma}_1\hat{\sigma}_2$, i.e. $\hat{\sigma}_1\hat{\sigma}_2(x) = \hat{\sigma}_1(\hat{\sigma}_2(x))$ for all $x \in \mathcal{X}$. We say that a string $u \in (\Gamma \cup \mathcal{X})^*$ is *copyless* (or linear) if each $x \in \mathcal{X}$ occurs at most once in u . A substitution σ is copyless if $\hat{\sigma}(u)$ is copyless, for all linear $u \in (\Gamma \cup \mathcal{X})^*$.

► **Definition 6.** A *streaming ω -string transducer* (SST) is a tuple $T = (\Sigma, \Gamma, Q, q_0, \delta, \mathcal{X}, \rho, F)$

- Σ and Γ are finite input and output alphabets;
- Q is a finite set of states with initial state q_0 ;
- $\delta : Q \times \Sigma \rightarrow Q$ is a transition function and \mathcal{X} is a finite set of variables;
- $\rho : (Q \times \Sigma) \rightarrow \mathcal{S}_{\mathcal{X}, \Gamma}$ is a variable update function to copyless substitutions such that any variable x occurs at most once on the right hand side of a simultaneous substitution;

- $F : 2^Q \rightarrow \mathcal{X}^*$ is an output function such that for all $P \in \text{dom}(F)$ the string $F(P)$ is copyless of form $x_1 \dots x_n$, and for $q, q' \in P$ and $a \in \Sigma$ s.t. $q' = \delta(q, a)$ we have
 - $\rho(q, a)(x_i) = x_i$ for all $i < n$ and $\rho(q, a)(x_n) = x_n u$ for some $u \in (\Gamma \cup \mathcal{X})^*$.

The concept of a run of an SST is defined in an analogous manner to that of a Muller automaton. The sequence $\langle \sigma_{r,i} \rangle_{0 \leq i \leq |r|}$ of substitutions induced by a run $r = q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \dots$ is defined inductively as the following: $\sigma_{r,i} = \sigma_{r,i-1} \rho(q_{i-1}, a_i)$ for $0 < i \leq |r|$ and $\sigma_{r,0} = x \in X \mapsto \varepsilon$. The output $T(r)$ of an infinite run r of T is defined only if $F(r)$ is defined and equals $T(r) \stackrel{\text{def}}{=} \lim_{i \rightarrow \infty} \langle \sigma_{r,i}(F(r)) \rangle$, when the limit exists. If not, we pad \perp^ω to the obtained finite string to get $\lim_{i \rightarrow \infty} \langle \sigma_{r,i}(F(r)) \perp^\omega \rangle$ as the infinite output string.

The assumptions on the output function F in the definition of an SST ensure that this limit exists whenever $F(r)$ is defined. Indeed, when a run r reaches a point from where it visits only states in P , these assumptions enforce the successive valuations of $F(P)$ to be an increasing sequence of strings by the prefix relation. The padding by unique letter \perp ensures that the output is always an ω -string. The output $T(s)$ of a string s is then defined as the output $T(r)$ of its unique run r . The transformation $\llbracket T \rrbracket$ defined by an SST T is the partial function $\{(s, T(s)) : T(s) \text{ is defined}\}$. See Appendix F.1 for an example. We remark that for every SST $T = (\Sigma, \Gamma, Q, q_0, \delta, \mathcal{X}, \rho, F)$, its domain is always an ω -regular language defined by the Muller automaton $(\Sigma, Q, q_0, \delta, \text{dom}(F))$, which can be constructed in linear time. However, the range of an SST may not be ω -regular. For instance, the range of the SST-definable transformation $a^n \#^\omega \mapsto a^n b^n \#^\omega$ ($n \geq 0$) is not ω -regular.

Aperiodic Streaming String Transducers. We define the notion of aperiodic SSTs by introducing an appropriate notion of transition monoid for transducers. The transition monoid of an SST T is based on the effect of a string s on the states as well as on the variables. The effect on variables is characterized by, what we call, flow information that is given as a relation that describes the number of copies of the content of a given variable that contribute to another variable after reading a string s .

Let $T = (\Sigma, \Gamma, Q, q_0, \delta, \mathcal{X}, \rho, F)$ be an SST. Let s be a string in Σ^* and suppose that there exists a run r of T on s . Recall that this run induces a substitution σ_r that maps each variable $X \in \mathcal{X}$ to a string $u \in (\Gamma \cup \mathcal{X})^*$. For string variables $X, Y \in \mathcal{X}$, states $p, q \in Q$, and $n \in \mathbb{N}$ we say that n copies of Y flow to X from p to q if there exists a run r on $s \in \Sigma^*$ from p to q , and Y occurs n times in $\sigma_r(X)$. We extend the notion of transition monoid for the Muller automata as defined in Section 2 for the transition monoid for SSTs to equip it with variables. Formally, the transition monoid $\mathcal{M}_T = (M_T, \times, \mathbf{1})$ of an SST $T = (\Sigma, \Gamma, Q, q_0, \delta, \mathcal{X}, \rho, \{F_1, \dots, F_n\})$ is such that M_T is a set of $|Q \times \mathcal{X}| \times |Q \times \mathcal{X}|$ square matrices over $(\mathbb{N} \times (\{0, 1\} \cup 2^Q)^n) \cup \{\perp\}$ along with matrix multiplication \times defined for matrices in M_T and identity element $\mathbf{1} \in M_T$ is the matrix whose diagonal entries are $(1, (\emptyset, \emptyset, \dots, \emptyset))$ and non-diagonal entries are all \perp 's. Formally $M_T = \{M_s : s \in \Sigma^*\}$ is defined using matrices M_s for strings $s \in \Sigma^*$ s.t. $M_s[(p, X)][(q, Y)] = \perp$ if there is no run from state p to state q over s in T , otherwise $M_s[(p, X)][(q, Y)] = (k, (x_1, \dots, x_n)) \in (\mathbb{N} \times (\{0, 1\} \cup 2^Q)^n)$ where x_i is defined exactly as in section 2.3, and k copies of variable X flow to variable Y from state p to state q after reading s . We write $(p, X) \rightsquigarrow_\alpha^u (q, Y)$ for $M_u[(p, X)][(q, Y)] = \alpha$.

It is easy to see that $M_\varepsilon = \mathbf{1}$. The operator \times is simply matrix multiplication for matrices in M_T , however we need to define addition \oplus and multiplication \odot for elements $(\{0, 1\} \cup 2^Q)^n \cup \{\perp\}$ of the matrices. We have $\alpha_1 \odot \alpha_2 = \perp$ if $\alpha_1 = \perp$ or $\alpha_2 = \perp$, and if $\alpha_1 = (k_1, (x_1, \dots, x_n))$ and $\alpha_2 = (k_2, (y_1, \dots, y_n))$ then $\alpha_1 \odot \alpha_2 = (k_1 \times k_2, (z_1, \dots, z_n))$ s.t. for all $1 \leq i \leq n$ z_i are defined as in (\star) from Section 2.3. Note that due to determinism of the SSTs we have that for every matrix M_s and every state p there is at most one state q such that $M_s[p][q] \neq \perp$ and hence the only addition rules we need to introduce is

$\alpha \oplus \perp = \perp \oplus \alpha = \alpha$, $0 \oplus 0 = 0$, $1 \oplus 1 = 1$ and $\kappa \oplus \kappa = \kappa$ for $\kappa \subseteq Q$. It is easy to see that $(M_T, \times, \mathbf{1})$ is a monoid and we give a proof in Appendix F. We say that the transition monoid M_T of an SST T is 1-bounded if in all entries $(j, (x_1, \dots, x_n))$ of the matrices of M_T , $j \leq 1$. A streaming string transducer is *aperiodic* if its transition monoid is aperiodic.

4 FOTs \equiv Aperiodic $2WST_{sf}$

► **Theorem 7.** *A transformation $f : \Sigma^\omega \rightarrow \Gamma^\omega$ is FOT-definable if and only if it is definable using an aperiodic two way transducer with star-free look-around.*

Proof (Sketch). This proof is in two parts.

■ **Aperiodic $2WST_{sf} \subseteq$ FOT.** We first show that given an aperiodic $2WST_{sf}$ \mathcal{A} , we can effectively construct an FOT that captures the same transduction as \mathcal{A} over infinite words. Let $\mathcal{A} = (Q, \Sigma, \Gamma, q_0, \delta, F)$ be an aperiodic $2WST_{sf}$, where each transition outputs at most one letter. Note that this is without loss of generality, since we can output any longer string by having some extra states. Given \mathcal{A} , we construct the FOT $T = (\Sigma, \Gamma, \phi_{dom}, C, \phi_{pos}, \phi_{\prec})$ that realizes the transduction of \mathcal{A} . The formula ϕ_{dom} is the conjunction of formulae `is_string` and φ where φ is a FO formula that captures the set of accepted strings of \mathcal{A} (obtained by proving $L(\mathcal{A})$ is aperiodic, lemma21, Appendix G) and `is_string` is a FO formula that specifies that the input graph is a string (see Appendix B). The copies of the FOT are the states of \mathcal{A} . For any two positions x, y of the input string, and any two copies q, q' , we need to define $\phi_{\prec}^{q, q'}$. This is simply describing the behaviour of \mathcal{A} on the substring from position x to position y of the input string u , assuming at position x , we are in state q , and reach state q' at position y . The following lemma (proof in Appendix G.1) gives an FO formula $\psi_{q, q'}(x, y)$ describing this.

► **Lemma 8.** *Let \mathcal{A} be an aperiodic $2WST_{sf}$ with the Muller acceptance condition. Then for all pairs of states q, q' , there exists an FO formula $\psi_{q, q'}(x, y)$ such that for all strings $s \in \Sigma^\omega$ and a pair of positions x, y of s , $s \models \psi_{q, q'}(x, y)$ iff there is a run from state q starting at position x of s that reaches position y of s in state q' .*

An edge exists between position x of copy q and position y of copy q' iff the input string $u \models \psi_{q, q'}(x, y)$. The formulae $\phi_{\gamma}^q(x)$ for each copy q specifies the output at position x in state q . We have to capture that position x is reached from the initial position in state q , and also the possible outputs produced while in state q at x . The transition function δ gives us these symbols. The formula $\bigvee_{\delta(q, a) = (q', dir, \gamma)} L_a(x)$ captures the possible output symbols. To state that we reach q at position x , we say $\exists y[\text{first}(y) \wedge \psi_{q_0, q}(y, x)]$. The conjunction of these two formulae gives $\phi_{\gamma}^q(x)$. This completes the FOT T .

■ **FOT \subseteq Aperiodic $2WST_{sf}$.** Given an FOT, we show that we can construct an aperiodic $2WST$ with star-free look-around capturing the same transduction over ω -words. For this, we first show that given an FOT, we can construct $2WST$ enriched with FO instructions that captures the same transduction as the FOT. The idea of the proof follows [12], where one first defines an intermediate model of aperiodic $2WST$ with FO instructions instead of look-around. Then we show $\text{FOT} \subseteq 2WST_{fo} \subseteq 2WST_{sf}$, to complete the proof.

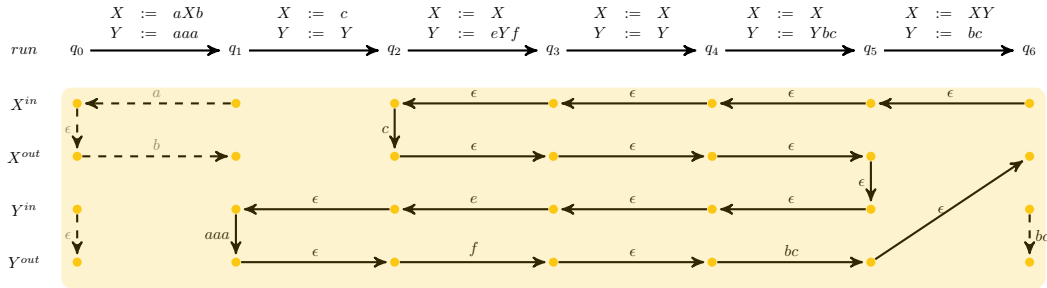
The omitted details can be found in Appendix G.2. ◀

5 Aperiodic SST \subseteq FOT

► **Lemma 9.** *A transformation is FO-definable if it is aperiodic-SST definable.*

We show that every aperiodic 1-bounded SST definable transformation is definable using FO-transducers. A crucial component in the proof of this lemma is to show that the variable

flow in the aperiodic 1-bounded SST is FO-definable (see Appendix H). To construct the FOT, we make use of the output structure for SST. It is an intermediate representation of the output, and the transformation of any input string into its SST-output structure will be shown to be FO-definable. For any SST T and string $s \in \text{dom}(T)$, the SST-output structure of s is a relational structure $G_T(s)$ obtained by taking, for each variable $X \in \mathcal{X}$, two copies of $\text{dom}(s)$, respectively denoted by X^{in} and X^{out} . For notational convenience we assume that these structures are labeled on the edges. A pair (X, i) is *useful* if the content of variable X before reading $s[i]$ will be part of the output after reading the whole string s . This structure satisfies the following *invariants*: for all $i \in \text{dom}(s)$, (1) the nodes (X^{in}, i) and (X^{out}, i) exist only if (X, i) is useful, and (2) there is a directed path from (X^{in}, i) to (X^{out}, i) whose labels are same as variable X computed by T after reading $s[i]$.



We define SST-output structures formally in Appendix H.3, however, the illustration above shows an SST-output structure. We show only the variable updates. Dashed arrows represent variable updates for useless variables, and therefore does not belong the SST-output structure. The path from $(X^{in}, 6)$ to $(X^{out}, 6)$ gives the contents of X ($ceaaa fbc$) after 6 steps. We write O_T for the set of strings appearing in right-hand side of variable updates.

We next show that the transformation that maps an ω -string s into its output structure is FO-definable, whenever the SST is 1-bounded and aperiodic. Using the fact that variable flow is FO-definable, we show that for any two variables X, Y , we can capture in FO, a path from (X^d, i) to (Y^e, j) for $d, e \in \{in, out\}$ in $G_T(s)$ and all positions i, j .

► **Lemma 10.** *Let T be an **aperiodic, 1-bounded** SST T . For all $X, Y \in \mathcal{X}$ and all $d, d' \in \{in, out\}$, there exists an $FO[\Sigma]$ -formula $path_{X,Y,d,d'}(x, y)$ with two free variables such that for all strings $s \in \text{dom}(T)$ and all positions $i, j \in \text{dom}(s)$, $s \models path_{X,Y,d,d'}(i, j)$ iff there exists a path from (X^d, i) to $(Y^{d'}, j)$ in $G_T(s)$.*

The proof of Lemma 10 is in Appendix H.4. As seen in Appendix (in Proposition 4) one can write a formula $\phi_q(x)$ (to capture the state q reached) and formula ψ_P^{Rec} (to capture the recurrence of a Muller set P) in an accepting run after reading a prefix. For each variable $X \in \mathcal{X}$, we have two copies X^{in} and X^{out} that serve as the copy set of the FOT. As given by the SST output-structure, for each step i , state q and symbol a , a copy is connected to copies in the previous step based on the updates $\rho(q, a)$. The full details of the FOT construction handling the Muller acceptance condition of the SST are in Appendix H.5.

6 Aperiodic $2WST_{sf} \subset$ Aperiodic SST

We show that given an aperiodic $2WST \mathcal{A} = (\Sigma, \Gamma, Q, q_0, \delta, F)$ with star-free look around over ω -words, we can construct an aperiodic SST \mathcal{T} that realizes the same transformation.

► **Lemma 11.** *For every transformation definable with an aperiodic 2WST with star-free look around, there exists an equivalent aperiodic 1-bounded SST.*

Proof. While the idea of the construction is similar to [4], the main challenge is to eliminate the star-free look-around for infinite strings from the SST, preserving aperiodicity. As an intermediate model we introduce streaming ω -string transducers with star-free look-around SST_{sf} that can make transitions based on some star-free property of the input string. We first show that for every aperiodic 2WST_{sf} one can obtain an aperiodic SST_{sf} , and then prove that the star-free look arounds can be eliminated from the SST_{sf} .

■ ($2\text{WST}_{sf} \subset \text{SST}_{sf}$). One of the key observations in the construction is that a 2WST_{sf} can move in either direction, while SST_{sf} cannot. Since we start with a deterministic 2WST_{sf} that reads the entire input string, it is clear that if a cell i is visited in a state q , then we never come back to that cell in the same state. We keep track in each cell i , with current state q , the state $f(q)$ the 2WST_{sf} will be in, when it moves into cell $i + 1$ for the first time. The SST_{sf} will move from state q in cell i to state $f(q)$ in cell $i + 1$, keeping track of the output produced in the interim time; that is, the output produced between q in cell i and $f(q)$ in cell $i + 1$ must be produced by the SST_{sf} during the move. This output is stored in a variable X_q . The state of the SST_{sf} at each point of time thus comprises of a pair (q, f) where q is the current state of the 2WST_{sf} , and f is the function which computes the state that q will evolve into, when moving to the right, the first time. In each cell i , the state of the SST will coincide with the state the 2WST_{sf} is in, when reading cell i for the first time. In particular, in the SST_{sf} , we define $\delta((q, f), r, a, p) = (f'(q), f')$ where $f'(q) = f'(f(t))$ if in the 2WST_{sf} we have $\delta(q, r, a, p) = (t, \gamma, -1)$. $f'(q)$ gives the state in which the 2WST_{sf} will move to the right of the current cell, but clearly this depends on $f(t)$, the state in which the 2WST_{sf} will move to the right from the previous cell. The variables of the SST_{sf} are of the form X_q , where q is the current state of the SST_{sf} . Update of X_q depends on whether the 2WST_{sf} moves left, right or stays in state q . For example, X_q is updated as $X_t \rho(X_{f(t)})$ if in the 2WST , $\delta(q, r, a, p) = (t, \gamma, -1)$ and $f(t)$ is defined. The definition is recursive, and X_t handles the output produced from state t in cell $i - 1$. We allow all subsets of Q as Muller sets of the SST_{sf} , and keep any checks on these, as part of the look-ahead.

A special variable O is used to define the output of the Muller sets, by simply updating it as $O := O\rho(X_q)$ corresponding to the current state q of the 2WST_{sf} (and (q, f) is the state of the SST_{sf}). The details of the correctness of construction are in Appendix I.

■ ($\text{SST}_{sf} \subset \text{SST}$). An aperiodic SST with star-free lookahead is a tuple (T, B, A) where $A = (P_A, \Sigma, \delta_A, P_f)$ is an aperiodic, deterministic Muller automaton called a look-ahead automaton, $B = (P_B, \Sigma, \delta_B)$ is an aperiodic automaton called the look-behind automaton, and T is a tuple $(\Sigma, \Gamma, Q, q_0, \delta, \mathcal{X}, \rho, F)$ where $\Sigma, \Gamma, Q, q_0, \mathcal{X}, \rho$, and F are defined in the same fashion as for ω -SSTs, and $\delta : Q \times P_B \times \Sigma \times P_A \rightarrow Q$ is the transition function. On a string $a_1 a_2 \dots$, while processing symbol a_i , we have in the SST_{sf} , $\delta((q, p_B, p_A), a_i) = q'$, (and the next transition is $\delta((q', p'_B, p'_A), a_{i+1})$) if (i) the prefix $a_1 a_2 \dots a_i \in L(p_A)$, (ii) the suffix $a_{i+1} a_{i+2} \dots \in L(p_B)$, where $L(p_A)$ ($L(p_B)$) denotes the language accepted starting in state p_A (p_B). We further assume that the look-aheads are mutually exclusive, i.e. for all symbols $a \in \Sigma$, all states $q \in Q$, and all transitions $q' = \delta(q, r, a, p)$ and $q'' = (q, r', a, p')$, we have that $L(A_p) \cap L(A_{p'}) = \emptyset$ and $L(B_r) \cap L(B_{r'}) = \emptyset$. In Appendix J.1, we show that for any input string, there is atmost one useful, accepting run in the SST_{sf} , while in Lemma 29 in Appendix J.1, we show that adding (aperiodic) look-around to SST does not increase their expressiveness.

The proof sketch is now complete. ◀

7 Conclusion

We extended the notion of aperiodicity from finite string transformations to that on infinite strings. We have shown a way to generalize transition monoids for deterministic Muller automata to streaming string transducers and two-way finite state transducers that capture the FO definable global transformations. An interesting and natural next step is to investigate LTL-definable transformations, their connection with FO-definable transformations, and their practical applications in verification and synthesis.

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Appendix

A Example of an Aperiodic Monoid Recognizing a Language

The language $L = (ab)^\omega$ is aperiodic, since it is recognized by the morphism $h : \{a, b\}^* \rightarrow M$ where M is an aperiodic monoid. $M = (\{1, 2, 3\}, \cdot, 1)$ with $2 \cdot 3 = 1$, $3 \cdot 2 = 3$, $2 \cdot 2 = 3$, $3 \cdot 3 = 3$ and $x \cdot 1 = 1 \cdot x = x$ for all $x \in M$. Define $h(\epsilon) = 1$, $h(a) = 2$, $h(b) = 3$. Then M is aperiodic, as $x^n = x^{n+1}$ for all $x \in M$ and $n \geq 3$. It is clear that h recognizes L .

B First-Order Logic: Examples

We define the following useful FO-shorthands.

- $x \succ y \stackrel{\text{def}}{=} \neg(x \preceq y)$ and $x \prec y \stackrel{\text{def}}{=} (x \preceq y) \wedge \neg(x = y)$,
- $\text{first}(x) = \neg \exists y (y \prec x)$
- is_string is defined as

$$\forall x (\text{u_succ}(x) \wedge \text{u_pred}(x)) \wedge \exists y [\text{first}(y) \wedge \forall z [\text{first}(z) \rightarrow z = y]] \wedge \forall x \exists y [x \prec y \wedge x \neq y]$$

where $\text{u_succ}(x) = \{\exists y [x \prec y \wedge \neg \exists z [x \prec z \wedge z \prec y]] \wedge \exists y' [x \prec y' \wedge \neg \exists z [x \prec z \wedge z \prec y']] \rightarrow (y = y')\} \wedge \exists y [x \prec y \wedge \neg \exists z [x \prec z \wedge z \prec y]]$ and

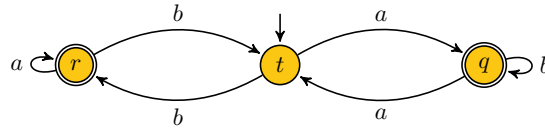
$\text{u_pred}(x) = \{\exists y [y \prec x \wedge \neg \exists z [y \prec z \wedge z \prec x]] \wedge \exists y' [y' \prec x \wedge \neg \exists z [y' \prec z \wedge z \prec x]] \rightarrow (y = y')\} \wedge \exists y [y \prec x \wedge \neg \exists z [y \prec z \wedge z \prec x]]$ characterize that the position x has unique predecessor and successor.

It is easy to see that a structure satisfying is_string characterizes a string.

C Proofs from Section 2.3**C.1 Transition Monoid of Muller Automata**

We start with an example for a transition monoid. The Muller automaton given in figure 2 has two Muller acceptance sets $\{q\}, \{r\}$. Consider the strings ab and bb . The transition monoids are

$$M_{ab} = \begin{matrix} & q & r & t \\ \begin{matrix} q \\ r \\ t \end{matrix} & \begin{pmatrix} \perp & (0,0) & \perp \\ \perp & \perp & (0,0) \\ (0,0) & \perp & \perp \end{pmatrix} \end{matrix} \quad M_{bb} = \begin{matrix} & q & r & t \\ \begin{matrix} q \\ r \\ t \end{matrix} & \begin{pmatrix} (1,0) & \perp & \perp \\ \perp & (0,0) & \perp \\ \perp & \perp & (0,0) \end{pmatrix} \end{matrix}$$



■ **Figure 2** Muller accepting set = $\{\{q\}, \{r\}\}$

► **Lemma 12.** $(M_T, \times, \mathbf{1})$ is a monoid, where \times is defined as matrix multiplication and the identity element $\mathbf{1}$ is the matrix with diagonal elements $(\emptyset, \emptyset, \dots, \emptyset)$ and all non-diagonal elements being \perp .

Proof. Consider any matrix M_s where $s \in \Sigma^*$. Let there be m states $\{p_1, \dots, p_m\}$ in the Muller automaton. Consider a row corresponding to some p_i . Only one entry can be different from \perp . Let this entry be $[p_i][p_j] = (\kappa_1, \dots, \kappa_n)$, where each $\kappa_h \in \{0, 1, 2^Q\}$, $1 \leq h \leq n$.

Consider $M_s \times \mathbf{1}$. The $[p_i][p_j]$ entry of the product are obtained from the p_i th row of M_s and the p_j th column of $\mathbf{1}$. The p_j th column of $\mathbf{1}$ has exactly one entry $[p_j][p_j] = (\emptyset, \dots, \emptyset)$, while all other elements are \perp . Then the $[p_i][p_j]$ entry for the product matrix $M_s \times \mathbf{1}$ is of the form $\perp \oplus \dots \oplus \perp \oplus (\kappa_1, \dots, \kappa_n) \odot (\emptyset, \dots, \emptyset) \oplus \perp \oplus \dots \oplus \perp$. Clearly, this is equal to $(\kappa_1, \dots, \kappa_n)$, since $\kappa \odot \emptyset = \kappa \cup \emptyset = \kappa$ and $\perp \oplus \kappa = \kappa$.

Similarly, it can be shown that the $[p_i][p_j]$ th entry of M_s is preserved in $\mathbf{1} \times M_s$ as well. Associativity of matrix multiplication follows easily. ◀

We exploit the following lemma, proved in [11], in our proofs.

► **Lemma 13.** *Let $h : \Sigma^* \rightarrow M$ be a morphism to a finite monoid M and let $w = u_0 u_1 \dots$ be an infinite word with $u_i \in \Sigma^+$ for $i \geq 0$. Then there exist $s, e \in M$ and an increasing sequence $0 < p_1 < p_2 < \dots$ such that*

1. $se = s$ and $e^2 = e$
2. $h(u_0 \dots u_{p_1-1}) = s$ and $h(u_{p_i} \dots u_{p_j}) = e$ for all $0 < i < j$.

Using Lemma 13, we prove the following lemma, which is used in the proof of Lemma 17.

► **Lemma 14.** *Let \mathcal{A} be a DMA. The mapping h which maps any string s to its transition matrix M_s , is a morphism from $(\Sigma^*, \cdot, \epsilon)$ to $(M_T, \times, \mathbf{1})$. Hence, h recognizes $L(\mathcal{A})$.*

Proof. It is easy to see that h is a morphism : $h(s_1 s_2)$ is by definition, $M_{s_1 s_2}$. This is clearly equal to $M_{s_1} M_{s_2} = h(s_1)h(s_2)$.

Let $w \in L(\mathcal{A})$ and let $w = w_1 w_2 w_3 w_4 \dots$ be a factorization of w , with $w_i \in \Sigma^+$ for all i . Consider another string $w' \in \Sigma^\omega$ with a factorization $w' = w'_1 w'_2 w'_3 w'_4 \dots$, such that $h(w_i) = h(w'_i)$ for all i . We have to show that $w' \in L(\mathcal{A})$.

Since $w \in L(\mathcal{A})$ we know that after some position i , the run r of w will contain only and all states from some muller set F_k . Assume that the factorization of w is such that, after the factor w_i of w , the run visits only states from F_k . We can write w as $uv_1 v_2 v_3 \dots$ such that $u = w_1 w_2 \dots w_i$, $v_1 = w_{i+1} w_{i+2} \dots w_{i+j_1}$, $v_2 = w_{i+j_1+1} w_{i+j_1+2} \dots w_{i+j_1+j_2}$ and so on, such that each of v_1, v_2, \dots witness all states of F_k in the run. Since $h(w_i) = h(w'_i)$ for all i , we know that $M_{w_i} = M_{w'_i}$ for all i . So we can factorize w' as $u'v'_1 v'_2 \dots$ such that $h(u) = h(u')$, $h(v_i) = h(v'_i)$ for all i . Hence, w' also has a run that also witnesses the muller set F_k from some point onwards. Hence, $w' \in L(\mathcal{A})$. Thus, h recognizes $L(\mathcal{A})$ since $w \in L(\mathcal{A}) \rightarrow [w]_h \subseteq L(\mathcal{A})$. ◀

C.2 Aperiodicity of DMA $\mathcal{A} \equiv$ Aperiodicity of $L(\mathcal{A})$

Note that a result similar to Theorem 15 (whose proof is below) has been proved for Büchi automata in [11], where aperiodicity of Büchi automata was defined using transition monoids.

► **Theorem 15.** *A language $L \subseteq \Sigma^\omega$ is aperiodic iff there exists an aperiodic Muller automaton \mathcal{A} such that $L = L(\mathcal{A})$.*

Proof. We obtain the proof of Theorem 15 by proving the following two lemmas.

► **Lemma 16.** *Let $L \subseteq \Sigma^\omega$ be an aperiodic language. Then there is an aperiodic Muller automaton accepting L .*

Proof. Let $L \subseteq \Sigma^\omega$ be an aperiodic language. Then by definition, L is recognized by a morphism $h : \Sigma^* \rightarrow M$ where M is a finite aperiodic monoid. We first show that we can construct a counter-free Muller automaton \mathcal{A} such that $L = L(\mathcal{A})$. A counter-free automaton is one having the property that $u^m \in L_{pp} \rightarrow u \in L_{pp}$ for all $u \in \Sigma^*$, states p and $m \geq 1$.

It can be shown [11] that the aperiodic language L can be written as the finite union of languages UV^ω where U, V are aperiodic languages of finite words. Moreover, for any $u_0u_1u_2\cdots \in \Sigma^\omega$, there is an increasing sequence $0 < p_1 < p_2 \dots$ of natural numbers such that for the morphism $h : \Sigma^* \rightarrow M$ recognizing L , we have $h(u_0 \dots u_{p_1}) = s \in M$ and $h(u_{p_i} \dots u_{p_j}) = e \in M$ for all $0 < i < j$. The $e \in M$ is an idempotent element in M . Then we have $h^{-1}(e) = V$ and $h^{-1}(s) = U$.

Since V is an aperiodic language $\subseteq \Sigma^*$, there is a minimal DFA \mathcal{D} that accepts V . The initial state of this DFA is $[\epsilon]$, and the states are of the form $[x]$, $x \in \Sigma^*$, such that $xw \in L$ iff there is a run from $[x]$ on w to an accepting state. Accepting states have the form $[w]$ with $w \in V$ and the transition function is $\delta([x], a) = [xa]$. For all $x, y, v \in \Sigma^*$, $(xv \in V \text{ iff } yv \in V) \Rightarrow [x] = [y]$. We first show that \mathcal{D} is counter-free.

Since V is aperiodic, there is a morphism $g : V \rightarrow M_V$ recognizing V , for an aperiodic monoid M_V . As M_V is aperiodic, there exists some $m \geq 1$ such that for all $x \in M_V$, $x^m = x^{m+1}$. If $[u] = [uv^m]$ for some $u, v \in \Sigma^*$, then $g(u) = g(uv^m) = g(uv^{m+1}) = g(uv^m v) = g(uv)$. If $[u] \neq [uv]$, then we can find some string w such that $uw \in V$ but $uvw \notin V$ or viceversa. This contradicts the hypothesis that V is recognized by a morphism $g : \Sigma^* \rightarrow M_V$, since $uw \in V$ and $g(uw) = g(uvw)$ implies either both uw, uvw belong to V or neither. Thus, $[u] = [uv^m]$ implies $[u] = [uv]$ for all $u, v \in \Sigma^*$. That is, whenever $\delta^*([u], v^m) = [uv^m] = [u]$, we have $\delta^*([u], v) = [uv] = [u]$ as well, which shows that the minimal DFA \mathcal{D} for V is counter-free.

If we interpret \mathcal{D} as a Buchi automaton, and consider $\alpha \in L(\mathcal{D})$, then α has infinitely many prefixes $v_1 < v_2 < \dots$ such that each $v_i \in V$. We have to show that $\alpha \in V^\omega$; that is we have to show that $\alpha = \alpha_1\alpha_2\dots$ with $\alpha_i \in V$ for all i . We know $v_2 = v_1v'_1$, $v_3 = v_2v'_2\dots$ with $v_1, v_2, \dots \in V$. Then $\alpha = v_1v'_1v'_2v'_3\dots$. If $v'_i \in V$ for all i , we are done, since in that case, \mathcal{D} will be a counter-free Buchi automaton for V^ω .

To obtain the infinitely many prefixes $v_1 < v_2 < v_3 < \dots$ such that $v_{i+1} = v_i v'_i$ with $v_i, v'_i \in V$, we consider the language $W = V.Prefree(V)$ where $Prefree(V)$ is the set of all strings in V which do not contain a proper prefix also lying in V . Since V is aperiodic, W is also aperiodic. Let \mathcal{E} be the minimal counter free DFA accepting W . We interpret \mathcal{E} as a Buchi automaton, as we did for the case of \mathcal{D} , and show that \mathcal{E} is a counter-free Buchi automaton accepting V^ω . Clearly, $L(\mathcal{E})$ is the set of strings which has infinitely many prefixes from W .

If $w \in V^\omega$, then $w \in L(\mathcal{E})$ since w has infinitely many prefixes from W . Conversely, let $w \in \Sigma^\omega$ be such that infinitely many prefixes $w_1 < w_2 < w_3 < \dots$ of w are in W . Then we have to show that $w \in V^\omega$. Let $w_i = x_i y_i$ where $x_i \in V, y_i \in Prefree(V)$ for each i . Then we have

$$x_1 < x_1 y_1 < x_2 < x_2 y_2 < x_3 < \dots$$

Note that if $w_1 \neq w_2$ that is, $x_1 y_1 \neq x_2 y_2$ and $x_1 = x_2$, then y_1 is a prefix of y_2 . Since $y_1, y_2 \in Prefree(V)$, this is not possible. Thus, for any two $w_i \neq w_j$, we have $x_i < x_j$. Let $x_{i+1} = x_i y_i y'_i$ for some y'_i . Recall that, using the morphism $h : \Sigma^* \rightarrow M$ recognizing L , we have $h^{-1}(e) = V$ for some idempotent $e \in M$. $h(x_{i+1}) = h(x_i y_i y'_i) = e.e.h(y'_i) = e.h(y'_i) = h(y_i y'_i)$. Hence we get $w = x_1 y_1 y'_1 y_2 y'_2 y_3 y'_3 \dots = x_1 x_2 x_3 \dots \in V^\omega$. Thus, \mathcal{E} is a counter-free Buchi automaton accepting V^ω . Let \mathcal{E}' be the counter-free Muller automaton obtained from \mathcal{E} .

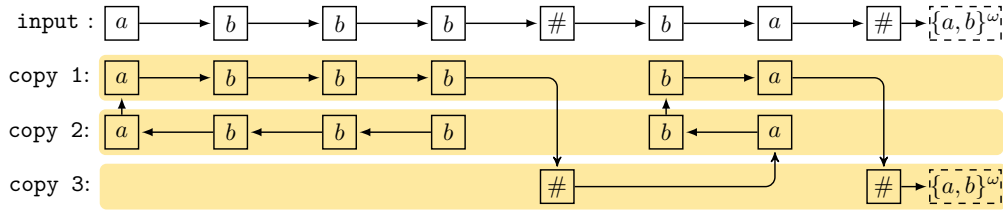
Since $U \subseteq \Sigma^*$ is aperiodic, we can construct as for V , the minimal counter-free DFA \mathcal{U} for U . The concatenation $\mathcal{U}\mathcal{E}'$ is then counter-free. The finite union of such automata are also counter-free.

Finally we show that counter-free automata are aperiodic. Let $x^n \in L_{pq}$ for any two states p, q , for a large n . We can decompose x^n as x^{k+l+m} such that $x^k \in L_{ps}, x^l \in L_{ss}$ and $x^m \in L_{sq}$, with $l \geq 2$. Then we have $x \in L_{ss}$ by the counter-freeness. Then we obtain $x^{n-1} \in L_{pq}$. Similarly, we can show that $x^{n-1} \in L_{pq} \Rightarrow x^n \in L_{pq}$. This shows that we have an aperiodic Muller automata accepting the finite union UV^ω that represents L . Thus, starting from the assumption that L is an aperiodic language, we have obtained an aperiodic Muller automaton that accepts L . ◀

► **Lemma 17.** *Let \mathcal{A} be Muller automaton whose transition monoid is aperiodic. Then $L(\mathcal{A})$ is aperiodic.*

Proof. From Lemma 14, we know that we can construct a morphism mapping $(\Sigma^*, \cdot, \epsilon)$ to the transition monoid of \mathcal{A} which recognizes $L(\mathcal{A})$. Hence, $L(\mathcal{A})$ is aperiodic. ◀

D Example of FOT



■ **Figure 3** Transformation f_1 given as FO-definable transformation for the string $abbb\#ba\#\{a, b\}^\omega$.

We give the full list of $\phi^{c,d}$ here. Let $btw(x, y, z) = (y \prec z \prec x) \vee (x \prec z \prec y)$ be a shorthand that says that z lies between x, y . Let $btw(x, y, \gamma) = \exists z(L_\gamma(z) \wedge btw(x, y, z))$ for $\gamma \in \Gamma$ and let $reach_\#(x) = \exists y(x \prec y \wedge L_\#(y))$ be a shorthand which says there is a $\#$ that is ahead of x .

1. $\phi_{dom} = \text{is_string}_\#$,
2. $\phi_\gamma^1(x) = \phi_\gamma^2(x) = L_\gamma(x) \wedge \neg L_\#(x) \wedge \text{reach}_\#(x)$, since we only keep the non $\#$ symbols that can “reach” a $\#$ in the input string in the first two copies.
3. $\phi_\gamma^3(x) = L_\#(x) \vee (\neg L_\#(x) \wedge \neg \text{reach}_\#(x))$, since we only keep the $\#$'s, and the infinite suffix from where there are no $\#$'s.

The transitive closure of the output successor relation is given by :

1. $\phi_\prec^{1,1}(x, y) = (x \prec y) = \phi_\prec^{3,3}(x, y)$,
2. $\phi_\prec^{2,2}(x, y) = [\neg btw(x, y, \#) \rightarrow (y \prec x)] \wedge [btw(x, y, \#) \rightarrow (x \prec y)]$ since we are reversing the arrows within a $\#$ -free block from which a $\#$ is reachable.
3. $\phi_\prec^{1,3}(x, y) = L_\#(y) \wedge (x \prec y) = \phi_\prec^{2,3}(x, y)$
4. $\phi_\prec^{1,2}(x, y) = x \prec y \wedge btw(x, y, \#)$ since each position in a $\#$ -free block is related to each position in a $\#$ -free block that comes later.
5. $\phi_\prec^{3,2}(x, y)$ and $\phi_\prec^{3,1}(x, y)$ are given by $L_\#(x) \wedge (x \prec y)$

6. $\phi_{\prec}^{2,1}(x, y)$ expresses that each position x in the i -th $\#$ -free block is related to position y appearing in the j -th $\#$ -free block, $j > i$. Within a $\#$ -free block, the arrows are reversed. This translates simply to $[x < y \wedge btw(x, y, \#)] \vee \{\neg btw(x, y, \#) \wedge [(y < x) \vee y = x]\}$.

E More details for section 3.2

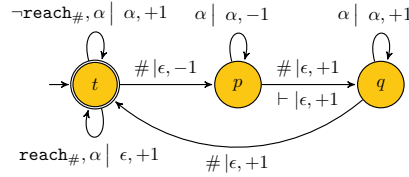


Figure 4 Transformation f_1 given as two-way transducers with look-ahead. Here symbol α stands for both symbols a and b , and the predicate $\mathbf{reach}_{\#}$ is the lookahead that checks whether string contains a $\#$ in future. $\{t\}$ is the only Muller set.

First we give some examples of transition monoids for the 2WST in Figure 4. Consider the string $ab\#$. The transition monoid is obtained by using all 4 behaviours as shown below. Note that on reading $ab\#$, on state t , when we reach symbol $\#$, the look-ahead $\neg \mathbf{reach}_{\#}$ evaluates to true.

$$M_{ab\#}^{\ell r} = \begin{matrix} & t & q & p \\ \begin{matrix} t \\ q \\ p \end{matrix} & \begin{pmatrix} (0) & \perp & \perp \\ (0) & \perp & \perp \\ \perp & \perp & \perp \end{pmatrix} \end{matrix}, \quad M_{ab\#}^{r \ell} = \begin{matrix} & t & q & p \\ \begin{matrix} t \\ q \\ p \end{matrix} & \begin{pmatrix} \perp & \perp & (0) \\ \perp & \perp & \perp \\ \perp & \perp & \perp \end{pmatrix} \end{matrix}$$

$$M_{ab\#}^{\ell \ell} = \begin{matrix} & t & q & p \\ \begin{matrix} t \\ q \\ p \end{matrix} & \begin{pmatrix} \perp & \perp & (0) \\ \perp & \perp & \perp \\ \perp & \perp & (0) \end{pmatrix} \end{matrix}, \quad M_{ab\#}^{rr} = \begin{matrix} & t & q & p \\ \begin{matrix} t \\ q \\ p \end{matrix} & \begin{pmatrix} (0) & \perp & \perp \\ (0) & \perp & \perp \\ \perp & (0) & \perp \end{pmatrix} \end{matrix}$$

If we consider the string $ab\#ab\#$, then we can compute for instance, $M_{ab\#ab\#}^{\ell r}$ using $M_{ab\#}^{\ell r}$, $M_{ab\#}^{\ell \ell}$ and $M_{ab\#}^{rr}$. It can be checked that we obtain $M_{ab\#ab\#}^{\ell r}$ same as $M_{ab\#}^{\ell r}$.

The transition monoid of the two-way automaton is obtained by using all the four matrices M_s^{xy} for a string s . In particular, given a string s , we consider the matrix

$$M_s = \begin{pmatrix} M^{\ell \ell}(s) & M^{\ell r}(s) \\ M^{r \ell}(s) & M^{rr}(s) \end{pmatrix}$$

as the transition matrix of s in the two-way automaton. The identity element is $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$ where $\mathbf{1}$ is the $n \times n$ matrix whose diagonal entries are $(\emptyset, \emptyset, \dots, \emptyset)$ and non-diagonal entries are all \perp 's. The matrix corresponding to the empty string ϵ is $\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$.

Given a word $w \in \Sigma^*$, we can find a decomposition of w into w_1 and w_2 such that we can write all behaviours of w in terms of behaviours of w_1 and w_2 , denoted by $M_{w_1 w_2}$. We enumerate the possibilities in each kind of traversal, for a successful decomposition.

1. For $w = w_1w_2$, and a left-left traversal, we can have $M^{\ell\ell}(w) = M^{\ell\ell}(w_1)$ or $M^{\ell\ell}(w) = M^{\ell r}(w_1) \times (M^{\ell\ell}(w_2) \times M^{rr}(w_1))^* \times M^{\ell\ell}(w_2) \times M^{r\ell}(w_1)$. We denote these cases as LL_1, LL_2 respectively.
2. For $w = w_1w_2$ and a left-right traversal, we have $M^{\ell r}(w) = M^{\ell r}(w_1) \times (M^{\ell\ell}(w_2) \times M^{rr}(w_1))^* \times M^{\ell r}(w_2)$. We denote this case as LR .
3. For $w = w_1w_2$ and a right-left traversal, we have $M^{r\ell}(w) = M^{r\ell}(w_2) \times (M^{rr}(w_1) \times M^{\ell\ell}(w_2))^* \times M^{r\ell}(w_1)$. We denote this case as RL .
4. For $w = w_1w_2$ and a right-right traversal, we have $M^{rr}(w) = M^{rr}(w_2)$ or $M^{rr}(w) = M^{r\ell}(w_2) \times (M^{rr}(w_1) \times M^{\ell\ell}(w_2))^* \times M^{rr}(w_1) \times M^{\ell r}(w_2)$. We denote these cases as RR_1, RR_2 respectively.

With these, for a correct decomposition of w as w_1w_2 , $M_{w_1w_2}$ is one of the four matrices as given below for $i, j \in \{1, 2\}$.

$$M_{w_1w_2} = \begin{pmatrix} LL_i & LR \\ RL & RR_j \end{pmatrix}$$

The multiplication of matrices $M^{xy}(w)$ using the \times operator is as defined for DMA using the multiplication \odot and addition \oplus of elements in $(\{0, 1\} \cup 2^Q)^n \cup \perp$. We define a new operation \otimes , which takes M_{w_1}, M_{w_2} , and for a ‘‘correct’’ decomposition of w using w_1, w_2 , denoted $w_1 \otimes w_2$, we obtain $M_{w_1} \otimes M_{w_2} = M_{w_1 \otimes w_2}$.

It can be seen that with the \otimes operation, the transition matrix of a string decomposed as $w_1.w_2$ correctly follows from the left-right behaviours of the strings w_1, w_2 . Let $\mathcal{T}(\mathcal{A})$ be the transition monoid of the two-way automaton \mathcal{A} . Let $|Q| = n$, the number of states of \mathcal{A} , and let there be m muller sets. For ease of notation, we do not include the states of the look-behind automaton and set of states of the look-ahead automaton in the transition monoid. Thus, $\mathcal{T}(\mathcal{A})$ contains matrices of the above form, with identity and the binary operation as defined above. It can be seen that recursively we can find decompositions of w as w_1 followed by w_2w_3 , and w_1w_2 followed by w_3 such that $M_w = M_{w_1} \otimes M_{w_2w_3} = M_{w_1w_2} \otimes M_{w_3}$.

► **Lemma 18.** *Let \mathcal{A} be a two-way (aperiodic) Muller automaton. The mapping h which maps any string s to its transition monoid $\mathcal{T}(\mathcal{A})$ is a morphism that recognizes $L(\mathcal{A})$. The language $L(\mathcal{A})$ is then aperiodic.*

Proof. The proof of Lemma 18 is similar to Lemma 14. As seen in the case of Lemma 14, it can then be proved that $\mathcal{T}(\mathcal{A})$ is a monoid. To define a morphism from Σ^* to $\mathcal{T}(\mathcal{A})$, we first define a mapping $h : (\Sigma^*, \cdot, \epsilon) \rightarrow (\Sigma^* \times \Sigma^*, R, (\epsilon, \epsilon))$ as $h(s) = (s_1, s_2)$ iff $s = s_1 \otimes s_2$ is a ‘‘correct breakup’’; that is, $M_s = M_{s_1} \otimes M_{s_2}$. The operator R is defined as $(a_1, a_2)R(b_1, b_2) = (a_1a_2, b_1b_2)$, and $h(\epsilon) = (\epsilon, \epsilon)$. It is easy to see that h is a morphism, and $(\Sigma^* \times \Sigma^*, R, (\epsilon, \epsilon))$ is a monoid.

Next, we define another map g which works on the range of h . $g : (\Sigma^* \times \Sigma^*, R, (\epsilon, \epsilon)) \rightarrow (\mathcal{T}_A, \otimes, \mathbf{1})$ as $g((s_1, s_2)) = M_{s_1s_2}$, $g([(s_1, s_2)R(s_3, s_4)]) = M_{s_1s_2} \otimes M_{s_3s_4}$, which by the correct breakup condition, will equal $M_{s_1s_2s_3s_4}$. Also, $g((\epsilon, \epsilon)) = \mathbf{1}$. Again, $(\mathcal{T}_A, \otimes, \mathbf{1})$ is a monoid and the composition of g and h is a morphism from $(\Sigma^*, \cdot, \epsilon)$ to $\mathcal{T}(\mathcal{A})$.

Further, one can show using a similar argument to Lemma 14 that $L(\mathcal{A})$ is recognized by the morphism $g \circ h$.

It remains to show that $L(\mathcal{A})$ is aperiodic. By definition, we know that the languages L_{pq}^{xy} are aperiodic. This means that there is some m such that $w^m \in L_{pq}^{xy}$ iff $w^{m+1} \in L_{pq}^{xy}$ for all strings w . This implies that $M^{xy}(w^m) = M^{xy}(w^{m+1})$ for all strings w . This would in turn imply (since we have the result for all four quadrants) that we obtain for the matrices of $\mathcal{T}(\mathcal{A})$, $M_w^m = M_w^m = M_{w^{m+1}} = M_w^{m+1}$, where M^m stands for the iterated product using \otimes of matrix M m times. This implies that there is an m such that for all matrices M in $\mathcal{T}(\mathcal{A})$, $M^m = M^{m+1}$. Hence, $\mathcal{T}(\mathcal{A})$ is aperiodic. Since h is a morphism to an aperiodic monoid recognizing $L(\mathcal{A})$, we obtain that $L(\mathcal{A})$ is aperiodic. ◀

F Proofs from Section 3.3

F.1 Example of SST

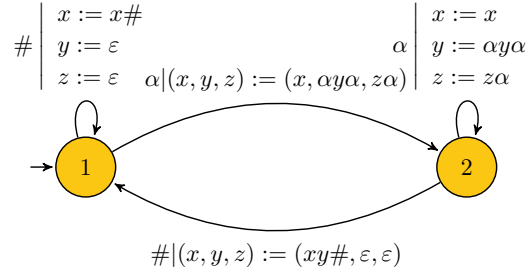


Figure 5 Transformation f_1 given as streaming string transducers with $F(\{2\}) = xz$ is the output associated with Muller set $\{2\}$. α stands for a, b .

Example 19. The transformation f_1 introduced is definable by the SST in Figure 5. Consider the successive valuations of x, y , and z upon reading the string $ab\#a^\omega$.

	a	b	$\#$	a	a	\dots
x	ϵ	ϵ	ϵ	$baab\#$	$baab\#$	$baab\#$
y	ϵ	aa	$baab$	ϵ	aa	$aaaa$
z	ϵ	a	ab	ϵ	a	aa

Notice that the limit of xz exists and equals $baab\#a^\omega$.

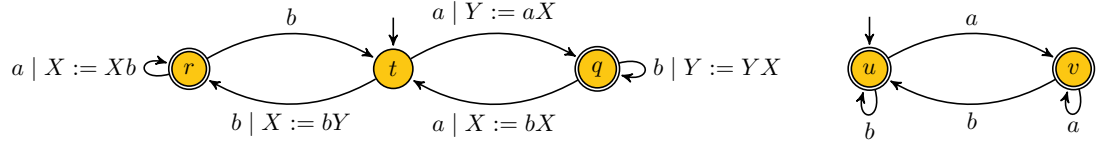
F.2 Transition Monoid for SSTs

Lemma 20. $(M_T, \times, \mathbf{1})$ is a monoid, where \times is defined as matrix multiplication and the identity element $\mathbf{1}$ is the matrix with diagonal elements $(\emptyset, \emptyset, \dots, \emptyset)$ and all non-diagonal elements being \perp .

Proof. Consider any matrix M_s where $s \in \Sigma^*$. Assume the SST has g variables X_1, \dots, X_g , and m states $\{p_1, \dots, p_m\}$. Let there be an ordering $(p_i, X_j) < (p_i, X_l)$ for $j < l$ and $(p_i, X_j) < (p_k, X_j)$ for $i < k$ used in the transition monoid. Consider a row corresponding to some (p, X_i) . The only entries that are not \perp in this row are of the form $[(p, X_i)(r, X_1)], \dots, [(p, X_i)(r, X_g)]$, for some state r , such that there is a run of s from p to r , $p, r \in \{p_1, \dots, p_m\}$. The (x_1, \dots, x_n) component of all entries $[(p, X_i)(r, X_1)], \dots, [(p, X_i)(r, X_g)]$ are same. Let it be $(\kappa_1, \dots, \kappa_n)$. Let $[(p, X_i)(r, X_j)] = k_{ij}(\kappa_1, \dots, \kappa_n)$.

Consider $M_s \cdot \mathbf{1}$. The $[(p, X)(r, -)]$ entries of the product are obtained from the (p, X) th row of M_s and the $(r, -)$ th column of $\mathbf{1}$. The $(r, -)$ th column of $\mathbf{1}$ has exactly one $1(\emptyset, \dots, \emptyset)$, while all other elements are \perp . Let $p = p_c$ and $r = p_h$, with $c < h$ ($h < c$ is similar). Then the $[(p, X)(r, Y)]$ entry for $X = X_i, Y = X_j$ is of the form $\perp + \dots + \perp + k_{ij}(\kappa_1, \dots, \kappa_n) \cdot 1(\emptyset, \dots, \emptyset) + k_{i+1 \ j}(\kappa_1, \dots, \kappa_n) \cdot \perp + \dots + k_{gj}(\kappa_1, \dots, \kappa_n) \cdot \perp + \dots + \perp$. Clearly, this is equal to $k_{ij}(\kappa_1, \dots, \kappa_n)$. Similarly, it can be shown that the $[(p, X_i)(r, X_j)]$ th entry of M_s is preserved in $\mathbf{1} \cdot M_s$. Associativity can also be checked easily. \blacktriangleleft

The mapping M_\bullet , which maps any string s to its transition matrix M_s , is a morphism from $(\Sigma^*, \cdot, \epsilon)$ to $(M_T, \times, \mathbf{1})$. We say that the transition monoid M_T of an SST T is n -bounded if in all entries $(j, (x_1, \dots, x_n))$ of the matrices of M_T , $j \leq n$. Clearly, any n -bounded transition monoid is finite. A streaming string transducer is *aperiodic* if its transition monoid is aperiodic. An ω -streaming string transducer with n Muller acceptance sets is 1-bounded if its transition monoid is 1-bounded. That is, for all strings s , and all pairs $(p, Y), (q, X)$, $M_s[p, Y][q, X] \in \{\perp\} \cup \{(i, (x_1, \dots, x_n)) \mid i \leq 1\}$.



■ **Figure 6** Muller accepting set of SST on the left = $\{\{q\}, \{r\}\}$. Also, $F(q) = XY$, $F(r) = X$. Muller accepting set for SST on the right = $\{u, v\}$; $X := X$ and $Y := Y$ on all edges, $F(\{u, v\}) = X$.

F.3 Example of Transition Monoid

In figure 6 consider the strings ab and bb for the automaton on the left. The transition monoids are

$$M_{ab} = \begin{matrix} & \begin{matrix} (t,X) & (t,Y) & (q,X) & (q,Y) & (r,X) & (r,Y) \end{matrix} \\ \begin{matrix} (t,X) \\ (t,Y) \\ (q,X) \\ (q,Y) \\ (r,X) \\ (r,Y) \end{matrix} & \begin{pmatrix} \perp & \perp & 1, (0,0) & 2, (0,0) & \perp & \perp \\ \perp & \perp & 0, (0,0) & 0, (0,0) & \perp & \perp \\ \perp & \perp & \perp & \perp & 0, (0,0) & 0, (0,0) \\ \perp & \perp & \perp & \perp & 1, (0,0) & 1, (0,0) \\ 1, (0,0) & 0, (0,0) & \perp & \perp & \perp & \perp \\ 0, (0,0) & 1, (0,0) & \perp & \perp & \perp & \perp \end{pmatrix} \end{matrix}$$

$$M_{bb} = \begin{matrix} & \begin{matrix} (t,X) & (t,Y) & (q,X) & (q,Y) & (r,X) & (r,Y) \end{matrix} \\ \begin{matrix} (t,X) \\ (t,Y) \\ (q,X) \\ (q,Y) \\ (r,X) \\ (r,Y) \end{matrix} & \begin{pmatrix} 0, (0,0) & 0, (0,0) & \perp & \perp & \perp & \perp \\ 1, (0,0) & 1, (0,0) & \perp & \perp & \perp & \perp \\ \perp & \perp & 1, (1,0) & 2, (1,0) & \perp & \perp \\ \perp & \perp & 0, (1,0) & 1, (1,0) & \perp & \perp \\ \perp & \perp & \perp & \perp & 0, (0,0) & 0, (0,0) \\ \perp & \perp & \perp & \perp & 1, (0,0) & 1, (0,0) \end{pmatrix} \end{matrix}$$

It can be checked that $M_{abbb} = M_{ab}M_{bb}$. Likewise, the transition monoid for a, b for the automaton on the right is

$$M_a = \begin{matrix} & \begin{matrix} (u,X) & (u,Y) & (v,X) & (v,Y) \end{matrix} \\ \begin{matrix} (u,X) \\ (u,Y) \\ (v,X) \\ (v,Y) \end{matrix} & \begin{pmatrix} \perp & \perp & 1(1) & 0(1) \\ \perp & \perp & 0(1) & 1(1) \\ \perp & \perp & 1(v) & 0(v) \\ \perp & \perp & 0(v) & 1(v) \end{pmatrix} \end{matrix}$$

$$M_b = \begin{matrix} & \begin{matrix} (u,X) & (u,Y) & (v,X) & (v,Y) \end{matrix} \\ \begin{matrix} (u,X) \\ (u,Y) \\ (v,X) \\ (v,Y) \end{matrix} & \begin{pmatrix} 1(u) & 0(u) & \perp & \perp \\ 0(u) & 1(u) & \perp & \perp \\ 1(1) & 0(1) & \perp & \perp \\ 0(1) & 1(1) & \perp & \perp \end{pmatrix} \end{matrix}$$

► **Proposition 1.** The domain of an aperiodic SST is FO-definable.

Proof. Let $T = (\Sigma, \Gamma, Q, q_0, Q_f, \delta, \mathcal{X}, \rho, F)$ be an aperiodic SST and M_T its (aperiodic) transition monoid. Let us define a function φ which associates with each matrix $M \in M_T$, the $|Q| \times |Q|$ Boolean matrix $\varphi(M)$ defined by $\varphi(M)[p][q] = (x_1, \dots, x_n)$ iff there exist $X, Y \in \mathcal{X}$ such that $M_T[p, X][q, Y] = (k, (x_1, \dots, x_n))$, with $k \geq 0$. Clearly, $\varphi(M_T)$ is the transition monoid of the underlying input automaton of T (ignoring the variable updates). The result follows, since the homomorphic image of an aperiodic monoid is aperiodic. ◀

G Proofs from Section 4

► **Lemma 21.** *The proof of Lemma 18 works when \mathcal{A} is a two-way, aperiodic Muller automaton with star-free look-around. That is, $L(\mathcal{A})$ is aperiodic.*

Proof. When we allow star-free look-around, we also have aperiodic Muller look-ahead automaton A and aperiodic look-behind automata B along with \mathcal{A} . Consider a string in $w \in \Sigma^\omega$ with a factorization $w = w_1 w_2 w_3 \dots$. Whenever we consider M_{w_i} for some w_i in $\mathcal{T}(\mathcal{A})$ (recall this is the monoid for \mathcal{A} without the look-around), we also consider the transition matrix of $w_1 \dots w_i$ with respect to B , and the matrices $M_{w_j}, j > i$ with respect to A . A string $w' = w'_1 w'_2 \dots$ is then equivalent to w if the respective matrices $M_{w_i}, M_{w'_i}$ match in \mathcal{A} , and so do the others (prefixes upto w_i, w'_i for look-behind B and $M_{w_j}, M_{w'_j}$ for look-ahead A). We know that the transition monoids of A, B are aperiodic. That is, there exists $m_A, m_B \in \mathbb{N}$ such that for all strings y , and all pairs of states p, q in $(A \text{ or } B)$, $y^{m_x} \in L_{pq}$ iff $y^{m_x+1} \in L_{pq}$ for $x \in \{A, B\}$.

In the presence of look-around, we keep track of the transition monoids in A, \mathcal{A} and B at the same time. It can be seen that for two infinite strings w, w' as above, the map h from Σ^* to the transition monoids with respect to B, \mathcal{A}, A will be a morphism : this is seen by considering the respective morphisms h_1, h_2, h_3 where h_1 is the morphism from Σ^* to the transition monoid of B (a DFA), h_2 is the morphism from Σ^* to the transition monoid of \mathcal{A} (the 2WST), and h_3 is the morphism from Σ^* to the transition monoid of A (a DMA). Clearly, equivalent strings w, w' (equivalent with respect to h) are either both accepted or rejected by the $2WST_{sf}$. The aperiodicity of the combined monoid follows from the aperiodicity of the respective monoids of B, \mathcal{A} and A . Thus, $L(\mathcal{A})$ is recognized by a morphism to an aperiodic monoid, and hence is an aperiodic language. ◀

G.1 Lemma 8

Proof. Lets look at the underlying two-way Muller automaton of the $2WST_{sf}$ \mathcal{A} . Since \mathcal{A} is aperiodic, so is the underlying automaton. Let $s = s[1 \dots x'-1]s[x' \dots y']s[y' \dots]$ be a decomposition of s . Let x, y be any two positions in s . Depending on x, y , we have 4 cases. If $x = y$ then there is a substring s_1 of s such that $s_1 \in L_{qq}^{rr}$ or $s_1 \in L_{qq'}^{ll}$. Similarly, if $x < y$, then there is a substring s_1 of s such that $s_1 \in L_{qq'}^{lr}$. Likewise if $x > y$, then there is a substring s_2 of s such that $s_2 \in L_{qq'}^{rl}$. If we can characterize $L_{qq'}^{xy}$ in FO for all cases, we are done, since $\psi_{q,q'}(x, y)$ will then be the disjunction of all these formulae.

Using lemma 21, the underlying input language $L(\mathcal{A}) \subseteq \Sigma^\omega$ can be shown to be aperiodic, by constructing the morphism $h : \Sigma^* \rightarrow M$ recognizing $L(\mathcal{A})$, where M is the aperiodic monoid described in lemma21. It is known [11] that every aperiodic language $\subseteq \Sigma^\omega$ is FO-definable.

We now show that h recognizes $L_{qq'}^{xy}$. Consider $w \in L_{qq'}^{xy}$. For the morphism $h : \Sigma^* \rightarrow M$ as above, let $w' \in \Sigma^+$ be such that $h(w) = h(w')$. Then it is easy to see that $w' \in L_{qq'}^{xy}$. Then h is a morphism to a finite aperiodic monoid that recognizes $L_{qq'}^{xy}$. Hence, $L_{qq'}^{xy}$ is aperiodic and hence FO-definable. For each case, $x < y, x > y, x = y$, let $\exists x \exists y \psi_{q,q'}(x, y)$ be the FO formula that captures $L_{qq'}^{xy}$.

For a particular assignment of positions x, y , it can be seen that all words u which have a run starting at position x in state q , to state q' in position y will satisfy $\psi_{q,q'}(x, y)$. ◀

G.2 FOT \subseteq Aperiodic $2WST_{sf}$

► **Definition 22.** A 2WST with FO instructions ($2WST_{fo}$) is a tuple $\mathcal{A} = (\Sigma, \Gamma, Q, q_0, \delta, F)$ such that Σ, Γ, Q, q_0 and F are as defined in section 3.2 and $\delta : Q \times \Sigma \times \phi_1 \rightarrow Q \times \Gamma \times \phi_2$ is the transition function where ϕ_1 is a set of FO formulae over Σ with one free variable defining the guard of the transition, and ϕ_2 is a set of FO formulae over Σ with two free variables defining the jump of the input head.

Given a state q and position x of the input string s , the transition $\delta(q, \phi_1) = (q', b, \phi_2)$ is enabled if $s \models \phi_1(x)$, and as a result b is written on the output, and the input head moves to position y such that $s \models \phi_2(x, y)$. Note that the jump is deterministic, at each state q and each position x , the formula $\phi_2(x, y)$ is such that there is a unique position y to which the reading head will jump.

A $2WST_{fo}$ is aperiodic if the underlying input language accepted is aperiodic.

As in the case of $2WST$, we assume that the entire input is read by the $2WST_{fo}$, failing which the output is not defined.

► **Lemma 23.** ($FOT \subseteq 2WST_{fo}$) *Any ω -transformation captured by an FOT is also captured by a $2WST_{fo}$.*

Proof. Let $T = (\Sigma, \Gamma, \phi_{dom}, C, \phi_{pos}, \phi_{\prec})$ be a FOT. We define the $2WST_{fo}$ $\mathcal{A} = (\Sigma, \Gamma, Q, q_0, \delta, F)$ such that $\llbracket T \rrbracket = \llbracket \mathcal{A} \rrbracket$. The states Q of \mathcal{A} correspond to the copies in T . So, $Q = C$. Given a state q and a position x of the input string, the transition that checks a guard at x , and decides the jump to position y , after writing a symbol b on the output is obtained from the formulae is_string , $\phi_b^q(x)$ and $\phi^{q,q'}(x, y)$. is_string describes the input string, $\phi_b^q(x)$ is an FO formula that captures the position x in copy q (and also asserts that the output is $b \in \Gamma$), $\phi^{q,q'}(x, y)$ is an FO formula that enables the \prec relation between positions x, y of copies q, q' respectively. In \mathcal{A} , this amounts to evaluating the guard $\phi_b^q(x)$ at position x in state q , outputting b , and jumping to position y of the input in state q' if the input string satisfies $\phi^{q,q'}(x, y)$. Thus, we write the transition as $\delta(q, a, \phi_b^q(x)) = (q', b, \phi^{q,q'}(x, y))$. The initial state q_0 of \mathcal{A} is the copy c which has its first position y such that $x \not\prec y$ for all positions $y \neq x$. Since the FOT is string-to-string, we will have such a unique copy. Thus, $q_0 = d$ where d is a copy satisfying the formula $\exists y[first^d(y)]$. The set F of Muller states of the constructed $2WST_{fo}$ is all possible subsets of Q , since we have captured the transitions between copies (now states) correctly.

Now we have to show that \mathcal{A} is aperiodic. This amounts to showing that there exists some integer n such that for all pairs of states p, q , $v^n \in L_{pq}^{xy}$ iff $v^{n+1} \in L_{pq}^{xy}$ for all strings $v \in \Sigma^*$, and $x, y \in \{l, r\}$. Recall that the states of the automaton correspond to the copies of the FOT, and $v^n \in L_{pq}^{xy}$ means that $v^n \models \phi^{p,q}(x, y)$. Thus we have to show that $v^n \models \phi^{p,q}(x, y)$ iff $v^{n+1} \models \phi^{p,q}(x, y)$ for all strings $v \in \Sigma^*$.

Note that since the domain of an FOT is aperiodic, for any strings u, v, w , there exists an integer n such that $uv^n w$ is in the domain of T iff $uv^{n+1} w$ is. In particular, for any pair of positions x, y in v, w respectively, and copies p, q , the formula $\varphi^{p,q}(x, y)$ which asserts the existence of a path from position x of copy p to position y of copy q is such that $uv^n w \models \varphi^{p,q}(x, y)$ iff $uv^{n+1} w \models \varphi^{p,q}(x, y)$ (note that this is true since these formulae evaluate in the same way for all strings in the domain, and the domain is aperiodic). This means that $uv^n w \in L_{pq}^{xy}$ iff $uv^{n+1} w \in L_{pq}^{xy}$ for all u, v, w . In particular, for $u = w = \epsilon$, we obtain $v^n \in L_{pq}^{xy}$ iff $v^{n+1} \in L_{pq}^{xy}$ for all strings $v \in \Sigma^*$, showing that \mathcal{A} is aperiodic. ◀

Before we show $2WST_{fo} \subseteq 2WST_{sf}$, we need the next two lemmas.

► **Lemma 24.** *Let Δ, Δ' be disjoint subsets of Σ , and let $L \subseteq \Sigma^\omega$ be an aperiodic language such that each string in L contains exactly one occurrence of a symbol from Δ and one occurrence of a symbol from Δ' . Then L can be written as the finite union of disjoint languages $R_\ell.a.R_m.b.R_r$ where $(a, b) \in (\Delta \times \Delta') \cup (\Delta' \times \Delta)$, $R_\ell, R_m \subseteq (\Sigma - \Delta - \Delta')^*$, and $R_r \subseteq (\Sigma - \Delta - \Delta')^\omega$. Moreover, R_ℓ, R_m and R_r are aperiodic.*

Proof. Let \mathcal{A} be a deterministic, aperiodic Muller automaton accepting L . From our assumption, it follows that each path in \mathcal{A} from the initial state q_0 passes through exactly one transition labeled with a symbol from Δ and one transition labeled with a symbol from Δ' . Let $(q, a, q') \in Q \times \Delta \times Q, (p, b, p') \in Q \times \Delta' \times Q$ be two such transitions. Let R_ℓ consist of all finite strings from q_0 to q , and let R_m consist of all finite strings from q' to p , and let R_r be the set of all strings from p' which continuously witnesses some Muller set from some point onwards. Since the underlying automaton is aperiodic, it is easy to see that the restricted automata for R_ℓ, R_m, R_r are also aperiodic. Hence, R_ℓ, R_m are aperiodic languages $\subseteq \Sigma^*$ while R_r is an aperiodic ω -language. This breakup using R_ℓ, R_m and R_r accounts for strings where the symbol from Δ occurs first, and the symbol from Δ' occurs later.

Symmetrically, we can define R'_ℓ, R'_m and R'_r to be the breakup for strings of L where a symbol of Δ' is seen first, followed by a symbol of Δ . Clearly, L is the finite union of languages $R_\ell.a.R_m.b.R_r$ and $R'_\ell.a.R'_m.b.R'_r$, where $R_\ell, R'_\ell, R_m, R'_m, R_r$ and R'_r are all aperiodic. ◀

► **Lemma 25.** *Let $\Delta \subseteq \Sigma$, and let $L \subseteq \Sigma^\omega$ be an aperiodic language such that each string in L contains exactly one occurrence of a symbol from Δ . Then L can be written as the finite union of disjoint languages $R_\ell.a.R_r$ where $a \in \Delta$, $R_\ell \subseteq (\Sigma - \Delta)^*$, and $R_r \subseteq (\Sigma - \Delta)^\omega$. Moreover, R_ℓ and R_r are aperiodic.*

The proof of Lemma 25 is similar to that of Lemma 24.

► **Lemma 26.** *($2\text{WST}_{fo} \subseteq 2\text{WST}_{sf}$) An ω -transformation captured by an aperiodic 2WST_{fo} is also captured by an aperiodic 2WST_{sf} .*

Proof. To prove this, we show that $\text{Aperiodic } 2\text{WST}_{fo} \subseteq \text{Aperiodic } 2\text{WST}_{sf}$. Let $\mathcal{T} = (\Sigma, \Gamma, Q, q_0, \delta, F)$ be an aperiodic 2WST_{fo} . We define an aperiodic 2WST_{sf} (with star free look around) (T, A, B) where $T = (\Sigma, \Gamma, Q, q_0, \delta, F)$ capturing the same transformation. A transition of the 2WST_{fo} is of the form $\delta(q, a, \varphi_1) = (q', z, \varphi_2)$. $\varphi_1(x)$ is an FO formula that acts as the guard of the transition, while $\varphi_2(x, y)$ is an FO formula that deterministically decides the jump to a position y from the current position x . The languages of these formulae, $L(\varphi_1)$ and $L(\varphi_2)$ are aperiodic, and one can construct aperiodic Muller automata accepting $L(\varphi_1)$ and $L(\varphi_2)$. From Lemma 25, we can write $L(\varphi_1)$ as a finite union of disjoint languages $R'_\ell.(a, 1).R'_r$ with $R'_\ell \subseteq [\Sigma \times \{0\}]^*$ and $R'_r \subseteq [\Sigma \times \{0\}]^\omega$. It is easy to see that one can construct aperiodic automata accepting R_ℓ and R_m , the projections of R'_ℓ and R'_m on Σ .

We have to now show how to simulate the jumps of φ_2 by moving one cell at a time. First we augment $\varphi_2(x, y)$ in such a way that we know whether $y < x$ or $y > x$ or $y = x$. This is done by rewriting $\varphi_2(x, y)$ as $\exists y(y \sim x \wedge \varphi_2(x, y))$ where $\sim \in \{<, =, >\}$. Clearly, the new formula is also FO, and the language of the formula is aperiodic. If $y < x$, by Lemma 24, the language of $\varphi_2(x, y)$ can be written as $R'_\ell(a, 1, 0)R'_m(b, 0, 1)R'_r$, where $R'_\ell, R'_m \subseteq [\Sigma \times \{0\} \times \{0\}]^*$ and $R'_r \subseteq [\Sigma \times \{0\} \times \{0\}]^\omega$. Let R_ℓ, R_m and R_r be the projections of R'_ℓ, R'_m and R'_r to Σ . We can construct an aperiodic look-behind automata for $R_\ell a R_m b$ and an aperiodic look ahead Muller automaton for R_r . This is possible since R_ℓ, R_m are aperiodic. To walk cell by cell, instead of jumping, the 2WST_{sf} does the following. (1) construct the aperiodic automaton that accepts the reverse of R_m (this is possible since the reverse of an aperiodic language is aperiodic), (2) simulate this reverse automaton on each transition, and remember the state reached in this automaton in the finite control, while moving left cell by cell each time, (3) when an accepting state is reached in the reverse automaton, check the look-behind R_ℓ , and look-ahead $R_m b R_r$. Note that there is an aperiodic look-ahead Muller automaton for $R_m b R_r$. If indeed at the position where we are at an accepting state of the reverse of R_m automaton, the look-ahead and look-behind are satisfied, then we can stop moving left.

A similar construction works when $y > x$. Clearly, each transition of the aperiodic 2WST_{fo} can be simulated by a 2WST_{sf} with star-free look around. The aperiodicity of 2WST_{sf} follows from the fact the input language accepted by the 2WST_{sf} is same as that of the 2WST_{fo} , and is aperiodic. ◀

H Proofs from Section 5

► **Proposition 2.** (FO-definability of variable flow) Let T be an aperiodic, 1-bounded SST T with set of variables \mathcal{X} . For all variables $X, Y \in \mathcal{X}$, there exists an FO-formula $\phi_{X \rightsquigarrow Y}(x, y)$ with two free variables such that, for all strings $s \in \text{dom}(T)$ and any two positions $i \leq j \in \text{dom}(s)$, $s \models \phi_{X \rightsquigarrow Y}(i, j)$ iff $(q_i, X) \rightsquigarrow_1^{s[i+1:j]} (q_j, Y)$, where $q_0 \dots q_n \dots$ is the accepting run of T on s .

Let $X \in \mathcal{X}$, $s \in \text{dom}(T)$, $i \in \text{dom}(s)$. Let $P \in 2^Q$ be a Muller accepting set with output $F(P) = X_1 \dots X_n$. Let $r = q_0 \dots q_n \dots$ be an accepting run of T on s . Then there is a position j such that $\forall k > j$, only states in P are visited. On all transitions beyond j , X_1, \dots, X_{n-1} remain unchanged, while X_n is updated as $X_n u$ for some $u \in (\mathcal{X} \cup \Gamma)^*$. We say that the pair (X, i) is *useful* if the content of variable X before reading $s[i]$ will be part of the output after reading the whole string s . Formally, for an accepting run $r = q_0 \dots q_j q_{j+1} \dots$, such that only states from the Muller set P are seen after position j , we say that (X, i) is useful for s if $(q_{i-1}, X) \rightsquigarrow_1^{s[i:j]} (q, X_k)$ for some variable $X_k \in F(P)$, $k \leq n$, or if $(q_{i-1}, X) \rightsquigarrow_1^{s[i:\ell]} (q, X_n)$, with $q \in P$ and $\ell > j$. Thanks to Proposition 2, this property is FO-definable.

► **Proposition 3.** For all $X \in \mathcal{X}$, there exists an FO-formula $\text{useful}_X(i)$ s.t. for all strings $s \in \text{dom}(T)$ and all positions $i \in \text{dom}(s)$, $s \models \text{useful}_X(i)$ iff (X, i) is useful for string s .

Proofs of propositions 2 and 3 are below.

H.1 Proof of Proposition 2

First, we show that reachable states in accepting runs of aperiodic SST are FO-definable:

► **Proposition 4.** Let T be an aperiodic SST T . For all states q , there exists an FO-formula $\phi_q(x)$ such that for all strings $s \in \Sigma^\omega$, for all positions i , $s \models \phi_q(i)$ iff $s \in \text{dom}(T)$ and the state of the (unique) accepting run of T before reading the i -th symbol of s is q .

Proof. Let A be the underlying (deterministic) aperiodic, Muller automaton of the SST T . Since T is aperiodic, so is A . For all states q , let L_q be the set of strings $s \in \Sigma^+$ such that there exists a run of T on string s that ends in state q . Clearly, L_q can be defined by some aperiodic finite state automaton A_q obtained by setting the set of final states of A to $\{q\}$. Here, A_q is interpreted as a DFA. Therefore L_q is definable by some FO-formula ψ_q^L . Let $L_{q,p}$ be the set of strings that have a run in T from state q to state p . This also is obtained from T by considering q as the start state and p as the accepting state. Let $\psi_{q,p}^L$ be the FO formula which captures this.

For $P \in 2^Q$, let R_P be the set of strings $s \in \Sigma^\omega$ such that there exists a run of T on s from some state $p \in P$, which stays in the set of states P , and all states of P are witnessed infinitely often. Clearly, $u \in \text{dom}(T)$ iff there exists $p \in P$ with $v_1 \in L_q, v_2 \in L_{q,p}$ and $v_3 \in R_P$ such that $u = v_1 v_2 v_3$. To capture v_3 , we have the FO-formula ψ_P^{Rec}

$$\bigvee_{p \in P} L_p(x) \wedge \forall y (y \geq x \rightarrow \bigvee_{q \in P} L_q(y)) \wedge \forall y \exists k_1 \dots \exists k_{|P|} (k_i \geq y \wedge \bigwedge_{i \neq j} (k_i \neq k_j) \wedge \bigwedge_{q_i \in P} L_{q_i}(k_i)).$$

Then, $\phi_q(x)$ is defined as

$$\phi_q(x) = [\psi_q^L]_{\prec x} \wedge [\psi_{q,p}^L]_{x \prec y} \wedge [L_p(y) \wedge \psi_P^{Rec}]_{y \preceq}$$

where $[\psi_q^L]_{\prec x}$ is the formula ψ_q^L in which all quantifications of any variable z is guarded by $z \prec x$, $[\psi_{q,p}^L]_{x \prec y}$ is the formula where all variables lie in between x, y , and finally, $[\psi_P^{Rec}]_{y \preceq}$ is the formula ψ_P^{Rec} in which all quantifications of any variable z is guarded by $y \preceq z$. Therefore, $s \models \phi_q(i)$ iff $s[1:i) \in L_q, s[i:j) \in L_{q,p}$ and $s[j:\infty) \in R_P$. ◀

Now we start the proof of Proposition 2.

Proof. For all states $p, q \in Q$, let $L_{(p,X) \rightsquigarrow (q,Y)}$ be the language of strings u such that $(p, X) \rightsquigarrow_1^u (q, Y)$. We show that $L_{(p,X) \rightsquigarrow (q,Y)}$ is an aperiodic language. It is indeed definable by an aperiodic non-deterministic automaton A that keeps track of flow information when reading u . It is constructed from T as follows. Its state set Q' are pairs $(r, Z) \in 2^{Q \times X}$. Its initial state is $\{(p, X)\}$ and final states are all states P such that $(q, Y) \in P$. There exists a transition $P \xrightarrow{a} P'$ in A iff for all $(p_2, X_2) \in P'$, there exists $(p_1, X_1) \in P$ and a transition $p_1 \xrightarrow{a|\rho} p_2$ in T such that $\rho(X_2)$ contains an occurrence of X_1 . Note that by definition of A , there exists a run from a state P to a state P' on some $s \in \Sigma^*$ iff for all $(p_2, X_2) \in P'$, there exists $(p_1, X_1) \in P$ such that $(p_1, X_1) \rightsquigarrow_1^s (p_2, X_2)$ (Remark \star).

Clearly, $L(A) = L_{(p,X) \rightsquigarrow (q,Y)}$. It remains to show that A is aperiodic, i.e. its transition monoid M_A is aperiodic. Since T is aperiodic, there exists $m \geq 0$ such that for all matrices $M \in M_T$, $M^m = M^{m+1}$. For $s \in \Sigma^*$, let $\Phi_A(s) \in M_A$ (resp. $\Phi_T(s)$) the square matrix of dimension $|Q'|$ (resp. $|Q|$) associated with s in M_A (resp. in M_T). We show that $\Phi_A(s^m) = \Phi_A(s^{m+1})$, i.e. $(P, P') \in \Phi_A(s^m)$ iff $(P, P') \in \Phi_A(s^{m+1})$, for all $P, P' \in Q'$.

First, suppose that $(P, P') \in \Phi_A(s^m)$, and let $(p_2, X_2) \in P'$. By definition of A , there exists $(p_1, X_1) \in P$ such that $(p_1, X_1) \rightsquigarrow_1^{s^m} (p_2, X_2)$, and by aperiodicity of T , it implies that $(p_1, X_1) \rightsquigarrow_1^{s^{m+1}} (p_2, X_2)$. Since it is true for all $(p_2, X_2) \in P'$, it implies by Remark (\star) that there exists a run of A from P to P' on s^{m+1} , i.e. $(P, P') \in \Phi_A(s^{m+1})$. The converse is proved similarly.

We have just proved that $L_{(p,X)\rightsquigarrow(q,Y)}$ is aperiodic. Therefore it is definable by some FO-formula $\phi_{(p,X)\rightsquigarrow(q,Y)}$. To capture the variable flow in an accepting run, we also need to have some conditions on the states p and q . $\phi_{X\rightsquigarrow Y}(x, y)$ is defined by

$$\phi_{X\rightsquigarrow Y}(x, y) \equiv x \preceq y \wedge \bigvee_{q,p \in Q} \{[\phi_{(q,X)\rightsquigarrow(p,Y)}]^{x \preceq \cdot \preceq y} \wedge [\psi_q^L]_{\prec x} \wedge [\psi_{q,p}^L]_{x \prec y} \wedge \exists z. \{[\psi_{p,r}^L]_{y \prec z} \wedge [L_r(z) \wedge \psi_P^{Rec}]_{z \preceq}\}\}$$

where $[\psi_q^L]_{\prec x}, [\psi_{q,p}^L]_{x \prec y}, [\psi_p^R]_{y \preceq}$ were defined in Proposition 4 and $[\phi_{(p,X)\rightsquigarrow(q,Y)}]^{x \preceq \cdot \preceq y}$ is obtained from $\phi_{(p,X)\rightsquigarrow(q,Y)}$ by guarding all the quantifications of any variable by $x \preceq z' \preceq y$. The Muller set P starts from some position z ahead of y , in some state $r \in P$. In the case when $p \in P$, consider $r = p$ in the above formula. ◀

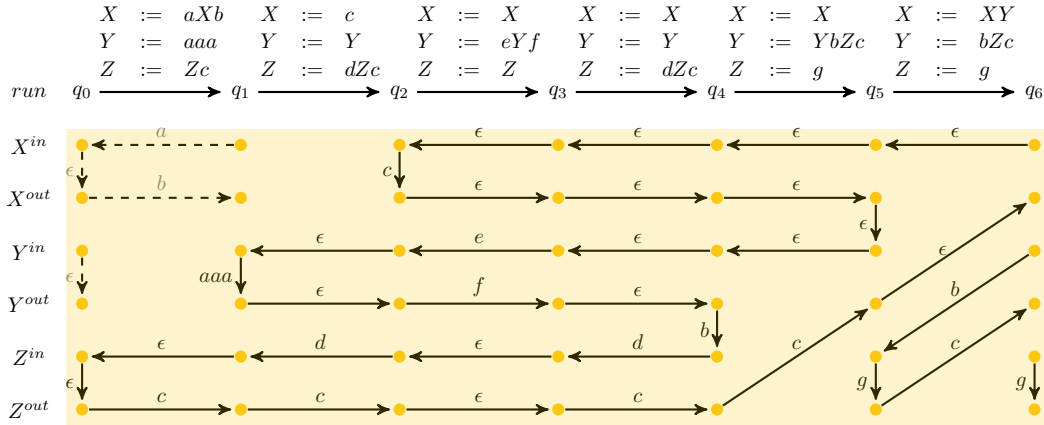
H.2 Proof of Proposition 3

Proof. The formula $\text{useful}_X(x)$ is defined by

$$\text{useful}_X(x) = \exists y. [\bigvee_{P \in 2Q} \psi_P^{Rec}(y) \wedge \bigvee_{p \in P} \psi_p^L(y) \wedge \bigvee_{\{X_1, \dots, X_n \mid F(P) = X_1 \dots X_n\}} \Phi_{X \rightsquigarrow X_i}(x, y)]$$

where $\psi_P^{Rec}(y)$ defines a position y from where the Muller set P is visited continuously. $\psi_p^L(y)$ is defined in proposition 4 and $\Phi_{X \rightsquigarrow Y}(x, y)$ in proposition 2. ◀

H.3 Definition of SST-output graphs



■ **Figure 7** SST-output structure depicting a partial run.

Figure 7 gives an example of SST-output structure. We show only the variable updates. Dashed arrows represent variable updates for useless variables, and therefore does not belong to the SST-output structure. Initially the variable content of Z is equal to ϵ . It is represented by the ϵ -edge from $(Z^{in}, 0)$ to $(Z^{out}, 0)$ in the first column. Then, variable Z is updated to Zc . Therefore, the new content of Z starts with ϵ (represented by the ϵ -edge from $(Z^{in}, 1)$ to $(Z^{in}, 0)$), which is concatenated with the previous content of Z , and then concatenated with c (it is represented by the c -edge from $(Z^{out}, 0)$ to $(Z^{out}, 1)$). Note that the invariant is satisfied. The content of variable X at position 5 is given by the label of the path from $(X^{in}, 5)$ to $(X^{out}, 5)$, which is c . Also note that some edges are labelled by strings with several letters, but there are finitely many possible such strings. In particular, we denote by O_T the set of all strings that appear in right-hand side of variable updates.

Let $T = (Q, q_0, \Sigma, \Gamma, \mathcal{X}, \delta, \rho, Q_f)$ be an SST. Let $u \in (\Gamma \cup X)^*$ and $s \in \Gamma^*$. The string s is said to *occur* in u if s is a factor of u . In particular, ϵ occurs in u for all u . Let O_T be the set of constant strings occurring in variable updates, i.e. $O_T = \{s \in \Gamma^* \mid \exists t \in \delta, s \text{ occurs in } \rho(t)\}$. Note that O_T is finite since δ is finite.

Let $w \in \text{dom}(T)$. The SST-output graph of w by T , denoted by $G_T(w)$, is defined as an infinite directed graph whose edges are labelled by elements of O_T . Formally, it is the graph $G_T(w) = (V, (E_\gamma)_{\gamma \in O_T})$ where $V = \{0, 1, \dots\} \times \mathcal{X} \times \{in, out\}$ is the set of vertices, $E := \bigcup_{\gamma \in O_T} E_\gamma \subseteq V \times V$

is the set of labelled edges defined as follows. Vertices $(i, X, d) \in V$ are denoted by (X^d, i) . Let $r = q_0 \dots q_n \dots$ be an accepting run of T on w . The set E is defined as the smallest set such that for all $X \in \mathcal{X}$,

1. $((X^{in}, 0), (X^{out}, 0)) \in E_\epsilon$ if $(X, 0)$ is useful,
2. for all i and $X \in \mathcal{X}$, if (X, i) is useful and if $\rho(q_i, w[i+1], q_{i+1})(X) = \gamma$, then $((X^{in}, i+1), (X^{out}, i+1)) \in E_\gamma$,
3. for all i and $X \in \mathcal{X}$, if (X, i) is useful and if $\rho(q_i, w[i+1], q_{i+1})(X) = \gamma_1 X_1 \dots \gamma_k X_k \gamma_{k+1}$ (with $k > 1$), then
 - $((X^{in}, i+1), (X_1^{in}, i)) \in E_{\gamma_1}$
 - $((X_k^{out}, i), (X^{out}, i+1)) \in E_{\gamma_{k+1}}$
 - for all $1 \leq j < k$, $((X_j^{out}, i), (X_{j+1}^{in}, i)) \in E_{\gamma_{j+1}}$

Note that since the transition monoid of T is 1-bounded, it is never the case that two copies of any variable (say X) flows into a variable (say Y), therefore this graph is well-defined and there are no multiple edges between two nodes.

We next show that the transformation that maps an ω -string s into its output structure is FO-definable, whenever the SST is 1-bounded and aperiodic. Using the fact that variable flow is FO-definable, we show that for any two variables X, Y , we can capture in FO, a path from (X^d, i) to (Y^e, j) for $d, e \in \{in, out\}$ in $G_T(s)$ and all positions i, j . We are in state q_i at position i .

For example,

1. There is a path from $(Z^d, 1)$ to $(Y^{out}, 5)$ for $d \in \{in, out\}$. This is because Z at position 1 flows into Z at position 4 (path $(Z^{in}, 4)$ to $(Z^{in}, 1)$, edge from $(Z^{in}, 1)$ to $(Z^{out}, 1)$, path from $(Z^{out}, 1)$ to $(Z^{out}, 4)$) this value of Z is used in updating Y at position 5 as $Y := bZc$. (edge from $(Z^{out}, 4)$ to $(Y^{out}, 5)$).
2. There is a path from $(Y^{in}, 5)$ to $(Z^d, 2)$. This is because Z at position 2 flows into Y at position 5 by the update $Y := YbZc$. (path from $(Z^{in}, 4)$ to $(Z^{in}, 2)$; path from $(Z^{in}, 2)$ to $(Z^{out}, 2)$ and path from $(Z^{out}, 2)$ to $(Z^{out}, 4)$; lastly, edge from $(Z^{out}, 4)$ to $(Y^{out}, 5)$). Also, note the edge from $(Y^{in}, 5)$ to $(Y^{in}, 4)$, and the path from $(Y^{in}, 4)$ to $(Y^{out}, 4)$, edge from $(Y^{out}, 4)$ to $(Z^{in}, 4)$).
3. There is a path from $(X^{in}, 3)$ to $(Z^{in}, 1)$ in Figure 7. However, Z at position 1 does not flow into X at position 3. Note that this is because of the update $X := XY$ at position 6, and Z at position 1 flows into Y at position 5, (note the path from $(Z^{out}, 1)$ to $(Z^{out}, 4)$, and the edge from $(Z^{out}, 4)$ to $(Y^{out}, 5)$) and X at position 3 flows into X at position 5 (note the path from $(X^{out}, 3)$ to $(X^{out}, 5)$) and X and Y are catenated in order at position 5 (the edge from $(X^{out}, 5)$ to $(Y^{in}, 5)$) to define X at position 6 (edge from $(Y^{out}, 5)$ to $(X^{out}, 6)$).

Thus, the SST output graphs have a nice property, which connects a path from (X^d, i) to $(Y^{d'}, j)$ based on the variable flow, and the catenation of variables in updates. Formally, let T be an **aperiodic, 1-bounded SST** T . Let $s \in \text{dom}(T)$, $G_T(s)$ its SST-output structure and $r = q_0 \dots q_n \dots$ the accepting run of T on s . For all variables $X, Y \in \mathcal{X}$, all positions $i, j \in \text{dom}(s) \cup \{0\}$, all $d, d' \in \{in, out\}$, there exists a path from node (X^d, i) to node $(Y^{d'}, j)$ in $G_T(s)$ iff (X, i) and (Y, j) are both useful and one of the following conditions hold: either

1. $(q_i, X) \rightsquigarrow_1^{s[i+1:j]} (q_j, Y)$ and $d' = out$, or
2. $(q_j, Y) \rightsquigarrow_1^{s[j+1:i]} (q_i, X)$ and $d = in$, or
3. there exists $k \geq \max(i, j)$ and two variables X', Y' such $(q_i, X) \rightsquigarrow_1^{s[i+1:k]} (q_k, X')$, $(q_j, Y) \rightsquigarrow_1^{s[j+1:k]} (q_k, Y')$ and X' and Y' are concatenated in this order¹ by r when reading $s[k+1]$.

H.4 Proof of Lemma 10

Let $s \in \text{dom}(T)$, $G_T(s)$ its SST-output structure and $r = q_0 \dots q_n \dots$ the accepting run of T on s . For all variables $X, Y \in \mathcal{X}$, all positions $i, j \in \text{dom}(s) \cup \{0\}$, all $d, d' \in \{in, out\}$, there exists a path from node (X^d, i) to node $(Y^{d'}, j)$ in $G_T(s)$ iff (X, i) and (Y, j) are both useful and one of the following conditions hold: either

¹ by concatenated we mean that there exists a variable update whose rhs is of the form $\dots X' \dots Y' \dots$

1. $(q_i, X) \rightsquigarrow_1^{s[i+1:j]} (q_j, Y)$ and $d' = out$, or
2. $(q_j, Y) \rightsquigarrow_1^{s[j+1:i]} (q_i, X)$ and $d = in$, or
3. there exists $k \geq \max(i, j)$ and two variables X', Y' such $(q_i, X) \rightsquigarrow_1^{s[i+1:k]} (q_k, X')$, $(q_j, Y) \rightsquigarrow_1^{s[j+1:k]} (q_k, Y')$ and X' and Y' are concatenated in this order² by r when reading $s[k+1]$.

For all variables $X, Y \in \mathcal{X}$, we denote by $Cat_{X,Y}$ the set of pairs $(p, q, a) \in Q^2 \times \Sigma$ such that there exists a transition from p to q on a whose variable update concatenates X and Y (in this order). Define a formula for condition (3):

$$\Psi_3^{X,Y}(x, y) \equiv \exists z. x \preceq z \wedge y \preceq z \wedge \bigvee_{X', Y' \in \mathcal{X}, (p, q, a) \in Cat_{X', Y'}} [L_a(z) \wedge \phi_{X \rightsquigarrow X'}(x, z) \wedge \phi_{Y \rightsquigarrow Y'}(y, z) \wedge \phi_p(z) \wedge \phi_q(z+1)]$$

Then, formula $\text{path}_{X,Y,d,d'}(x, y)$ is defined by

$$\begin{aligned} \text{path}_{X,Y,in,in}(x, y) &\equiv \phi_{Y \rightsquigarrow X}(y, x) \vee \Psi_3^{X,Y} \\ \text{path}_{X,Y,in,out}(x, y) &\equiv \phi_{Y \rightsquigarrow X}(y, x) \vee \phi_{X \rightsquigarrow Y}(x, y) \vee \Psi_3^{X,Y} \\ \text{path}_{X,Y,out,in}(x, y) &\equiv \text{false} \\ \text{path}_{X,Y,out,out}(x, y) &\equiv \phi_{X \rightsquigarrow Y}(x, y) \vee \Psi_3^{X,Y} \end{aligned}$$

1. $\text{path}_{X,Y,in,in}(x, y)$: Recall that a path from (X^{in}, x) to (Y^{in}, y) always passes through (X^{out}, y) . Hence, $y \leq x$, (the X^{in} arrows move on the left) and it must be that there is an edge from (X^{out}, y) to (Y^{in}, y) , which happens when X and Y are concatenated in order. This is handled by $\Psi_3^{X,Y}$. The other possibility is when Y occurs in the right side of X at x ; in this case, we have an edge from (X^{in}, x) to some (Z^{in}, x) (Z could be Y), leading into a path to (Y^{in}, y) , $y \leq x$. Clearly, here Y flows into X between y and x .
2. $\text{path}_{X,Y,in,out}(x, y)$: One possibility is that there is a path from (X^{in}, x) to (X^{in}, z) , with $x > z$, and (X^{in}, z) has an edge to (X^{out}, z) . An edge from (X^{out}, z) to (Y^{in}, z) happens if X, Y are concatenated in order at z . A path from (Y^{in}, z) to (Y^{in}, y) and then (Y^{out}, y) , $z \geq y$ can happen. This is handled by $\Psi_3^{X,Y}$. A second case is similar to 1, where we have a path from (X^{in}, x) to (Y^{in}, y) , where Y flows into X between y and x , and then there is a path from (Y^{in}, y) to (Y^{out}, y) . A third case is when X occurs in the right of Y at y ; in this case, we have an edge from (X^{out}, y) to (Y^{out}, y) . Also, there is a path from (X^{in}, y) to (X^{out}, y) which goes through (X^{in}, x) , $x < y$. Clearly, X flows into Y between x and y .
3. Note that $\text{path}_{X,Y,out,in}(x, y)$ is false, since there is no path or edge from X^{out} to Y^{in} capturing flow; the edge from X^{out} to Y^{in} occurs only when a catenation of X, Y happens in order.
4. $\text{path}_{X,Y,out,out}(x, y)$: One case is when X, Y are catenated in order on the rightside of some variable Z . Then there is a path from (X^{out}, x) to (Y^{out}, y) (through (Y^{in}, y)), as handled by $\Psi_3^{X,Y}$. The other case is when X occurs on the right of Y at y . Then there is an edge from (X^{out}, x) to (Y^{out}, y) ($x = y - 1$) (may be through some (Z^{in}, x)).

H.5 Formal Construction of FOT from Aperiodic SST

We describe the construction of the FOT from the 1-bounded, aperiodic SST.

In the case of finite strings, if the output function of the accepting state is $X_1 \dots X_n$, then $(X_1^{in}, |s|)$ is the start node, on reading string s . The path from $(X_1^{in}, |s|)$ to $(X_1^{out}, |s|)$, followed by the path from $(X_2^{in}, |s|)$ to $(X_2^{out}, |s|)$ and so on gives the output.

Unlike the finite string case, one of the difficulties here, is to specify the start node of the FOT since the strings are infinite. The first thing we do, in order to specify the start node, is to identify the position where some Muller set starts, and the rest of the run stays in that set. This is done for instance, in Proposition 4 by the formula ψ_P^{Rec} . We can easily catch the first such position where ψ_P^{Rec} holds. This position will be labeled in the FOT with a unique symbol \perp . Let $O \subseteq \Gamma^*$

² by concatenated we mean that there exists a variable update whose rhs is of the form $\dots X' \dots Y' \dots$

be the finite set of output strings γ_i that appear in the variable updates of the SST. That is, $O = \{\gamma_i \mid \rho(q, a)(X) = \gamma_0 X_1 \gamma_1 \dots \gamma_{n-1} X_n \gamma_n\}$. We build an FOT that marks the first position where ψ_P^{Rec} evaluates to true as \perp , outputs the contents of X_1, \dots, X_{n-1} first till \perp , and then of X_n , where $F(P) = X_1 \dots X_{n-1} X_n$ is the output function of the SST.

The FOT is defined as $(\Sigma, O \cup \{\perp\}, \phi_{dom}, C, \phi_{pos}, \phi_{\prec})$ where:

- $\phi_{dom} = is_string \wedge \exists i \psi_P^{Rec}(i)$ where the FO formula is_string is a simple FO formula that says every position has a unique successor and predecessor. The conjunction with $\exists i \psi_P^{Rec}(i)$ says that there is a position i of the string from where ψ_P^{Rec} is true.
- $C = \mathcal{X} \times \{in, out\}$,
- $\phi_{pos} = \{\phi_\gamma^c(i) \mid c \in C, \gamma \in O \cup \{\perp\}\}$ is such that
 - $\phi_\perp^{(X_1, in)}(i) = \bigvee_{\{P \in F \mid F(P) = X_1 \dots X_n\}} [\psi_P^{Rec}(i) \wedge \forall j (j \prec i \rightarrow \neg \psi_P^{Rec}(j))]$. i is the first position from where a Muller set starts to hold continuously.
 - for $\gamma \in O$, we define $\phi_\gamma^c(i) = \neg \phi_\perp^c(i) \wedge \psi_\gamma^c(i)$, where $\psi_\gamma^c(i)$ is
 - true, if $(i = 0)$, $c = (X, in)$ and $\gamma = \epsilon$,
 - false, if $(i = 0)$, $c = (X, in)$ and $\gamma \neq \epsilon$,
 - $\bigvee_{q \in Q, a \in \Sigma} \phi_q(i-1) \wedge L_a(i-1)$ if $c = (X, in)$, $i > 0$ and $\rho(q, a)(X) = \gamma Y_1 \gamma_1 Y_2 \dots \gamma_n$. The formula $\phi_q(x)$ is defined in Proposition 4.
 - $\bigvee_{q \in Q, a \in \Sigma} \phi_q(i) \wedge L_a(i)$ if $c = (X, out)$, $i \geq 0$ and $\rho(q, a)(Y) = \gamma_0 Y_1 \gamma_1 Y_2 \dots Y_k \gamma_k \dots \gamma_n$, for a unique $Y \in \mathcal{X}$, and $Y_k = X$ and $\gamma = \gamma_k$ for some $1 \leq k \leq n$. Note that $Y \in \mathcal{X}$ is unique since the SST is 1-bounded (the SST output-structure does not contain useless variables, and if X appears in the rightside of two variables Y, Z then one of them will be useless for the output)
 - $\bigvee_{q \in Q, a \in \Sigma} \phi_q(i) \wedge L_a(i)$ if $c = (X, out)$, $i \geq 0$ and $\gamma = \epsilon$ if X does not appear in the update of any variable in $\rho(q, a)$.
- We next define $\phi^{c,d}(i, j) = \bigvee_{P \in F} [\psi_P^{Rec} \wedge \psi_P^{c,d}(i, j)]$, where $\psi_P^{c,d}(i, j)$ is defined as follows. Let $F(P) = X_1 X_2 \dots X_{n-1} X_n$. Let k be the earliest position where ψ_P^{Rec} holds. Note that all these edges are defined only when a copy is useful, that is, it contributes to the output. The notion of usefulness has been defined in section 5.
 - $true \wedge useful_X(i)$, if $c = (X, in)$, $d = (X, out)$ for some variable X , and $i = j = 0$. The formula $useful_X(i)$ is defined in Appendix H.2.
 - $\phi_q(i-1) \wedge L_a(i-1) \wedge useful_X(i)$ if $c = (X, in)$, $d = (X, out)$ $i = j > 0$ and $\rho(q, a)(X) = \gamma$.
 - $\phi_q(i) \wedge L_a(i) \wedge useful_X(i)$ if $c = (X, out)$, $d = (Y, in)$ for some variables X, Y , $i = j \geq 0$, and for some Z we have $\rho(q, a)(Z) = \gamma_0 Z_1 \gamma_1 \dots Z_n \gamma_n$ with $Z_\ell = X$ and $Z_{\ell+1} = Y$ for some $1 \leq \ell \leq n$
 - $\phi_q(i-1) \wedge L_a(i-1) \wedge useful_X(i)$ if $C = (X, in)$, $d = (Y, in)$ for some variables X, Y , $i > 0$, $j = i - 1$, $\rho(q, a)(X) = \gamma Y \gamma_1 Y_2 \dots Y_n \gamma_n$
 - $\phi_q(i) \wedge L_a(i) \wedge useful_X(i)$ if $c = (X, out)$, $d = (Y, out)$ for some variables X, Y , $i + 1 = j$ and $\rho(q, a)(Y) = \gamma X_1 \dots X \gamma_n$.
 - We do the above between copies upto position k , which influences the values of variables X_1, \dots, X_{n-1} , which produces the variable flow upto position k . Beyond position k , the contents of variables X_1, \dots, X_{n-1} remain unchanged. Hence, for all $i \geq k$, and $1 \leq j \leq n-1$, we dont have any edges from (X_j, out) at position i to (X_j, out) at position $i + 1$, and from (X_j, in) at position $i+1$ to (X_j, in) at position i . We simply add a transition from (X_j, out) to (X_{j+1}, in) at position k to seamlessly catenate the contents of X_1, \dots, X_{n-1}, X_n at position k .
 - From (X_n, in) at position k onwards, we simply follow the edges as defined above, to obtain at each position, the correct output $X_1 X_2 \dots X_n$. If at position k , $X_n := X_n \gamma_0 Y_1 \dots \gamma_{n-1} Y_n \gamma_n$ then we have the connections from (X_n, in) at position k to (X_n, in) at position $k - 1$, which will eventually reach (X_n, out) at position $k - 1$. This is then connected to (Y_1, in) at position $k - 1$, and so on till we reach (Y_n, out) at position $k - 1$. This is then connected to (X_n, out) at position k , rendering the correct output at position k . The same thing repeats for position $k + 1$ and so on.

Thanks to Lemma 10, the transitive closure between some copy (X, d) and (Y, d') is FO-definable as $\phi_{\prec}^{(X,d),(Y,d')} = \text{path}_{X,Y,d,d'}(x, y)$. This completes the construction of the FOT.

I Proofs from Section 6 : $2\text{WST}_{sf} \subset \text{SST}_{sf}$

In this section, we show that $2\text{WST}_{sf} \subset \text{SST}_{sf}$. Let the 2WST_{sf} be (T, A, B) with $T = (\Sigma, \Gamma, Q, q_0, \delta, F)$. We assume wlg that $L(A_p) \cap L(A_{p'}) = \emptyset$ for all $p, p' \in Q_A$ and $L(A_r) \cap L(A_{r'}) = \emptyset$ for all $r, r' \in Q_B$. Also, we assume that $F = 2^Q$, and check particular accepting conditions in the look-ahead automaton A .

The SST_{sf} we construct is $\mathcal{T} = ((\Sigma, \Gamma, Q', q'_0, \delta', X, \rho, F'), A, B)$ where $Q' = Q \times [Q \rightarrow Q]$ where \rightarrow represents a partial function. $q'_0 = (q_0, I)$ where I is the identity. The set of variables $\mathcal{X} = \{X_q \mid q \in Q\} \cup \{O\}$, and $F' : 2^{Q'} \rightarrow X^*$ is defined for all $P' \in F'$ as $F'(P') = O$. $\delta' : Q' \times Q_B \times \Sigma \times Q_A \rightarrow Q'$ is defined as follows: $\delta'((q, f), r, a, p) = (f'(q), f')$ where f' is defined as follows:

$$f'(q) = \begin{cases} t & \text{if } \delta(q, r, a, p) = (t, \gamma, 1), \\ f'(t) & \text{if } \delta(q, r, a, p) = (t, \gamma, 0), \\ f'(f(t)) & \text{if } \delta(q, r, a, p) = (t, \gamma, -1) \end{cases}$$

Thanks to the determinism of the 2WST_{sf} and the restriction that the entire input be read, f' is a well-defined, partial function. In each cell i , the state of the SST_{sf} will coincide with the state the 2WST_{sf} is in, when reading cell i for the first time.

The variable update ρ on each transition is defined as follows: for $\delta'((q, f), a) = (f'(q), f')$, the update $\rho((q, f), a)$ is defined as follows:

$$\rho((q, f), a)(Y) = \begin{cases} O\rho(X_q) & \text{if } Y = O, \\ \gamma & \text{if } Y = X_q \text{ and } \delta(q, r, a, p) = (t, \gamma, 1), \\ \gamma\rho(X_t) & \text{if } Y = X_q \text{ and } \delta(q, r, a, p) = (t, \gamma, 0), \\ \gamma X_t \rho(X_{f(t)}) & \text{if } Y = X_q \text{ and } \delta(q, r, a, p) = (t, \gamma, -1) \text{ and } f(t) \text{ is defined,} \\ \epsilon & \text{otherwise.} \end{cases}$$

At the moment, the SST_{sf} is not 1-bounded. One place where 1-boundedness is violated happens because it is possible to have $f(s_1) = f(s_2) = t$ for $s_1 \neq s_2$. In particular, assume $\delta(s_1, r, a, p) = \delta(s_2, r, a, p) = (t, \gamma, -1)$, and let $f(t) = q$. Then we have $X_{s_2}, X_{s_1} := \gamma X_t \rho(X_q)$. This would mean that we visit the same cell in the same state t twice, which would lead to non-determinism or not fully reading the input by the 2WST_{sf} . Thus, this cannot happen in an accepting run of the 2WST_{sf} . Hence, if we have $\delta'((q, f), r, a, p) = (f'(q), f')$ where f' is a partial function which is not 1-1, we can safely replace it with a subset f'' of f' which is 1-1. Since $f'(s_1) = f'(s_2)$ for $s_1 \neq s_2$ does not contribute to an accepting run, we can define f'' on those states which preserve the 1-1 property. The second place where 1-boundedness is violated is when the contents of $\rho(X_q)$ flows into O as well as X_q . This can be fixed by composition with another SST where on each transition, all variables other than X_q are unchanged ($X := X$ for $X \neq X_q$) and resetting X_q ($X_q := \epsilon$). Note that this does not affect the output since O is the only variable that is responsible for the output, and $\rho(X_q)$ has faithfully been reflected into it. With these two fixes, we regain the 1-boundedness property.

1.1 Connecting runs of the 2WST_{sf} and SST_{sf}

► **Lemma 27.** *Let $a_1 a_2 \dots \in \Sigma^\omega$, and let $r = (q_0, i_0 = 0) \xrightarrow{r_1, s(i_0), p_1 | z_1} (q_1, i_1) \xrightarrow{r_2, s(i_1), p_2 | z_2} (q_2, i_2) \dots$ be an accepting run of the 2WST_{sf} . Let $j_n = \min\{m \mid i_m = n\}$ be the index when cell n is read for the first time. Let $r' = (q'_0, f_0) \xrightarrow{r'_1, a_1, p'_1} (q'_1, f_1) \xrightarrow{r'_2, a_2, p'_2} (q'_2, f_2) \dots$ be the corresponding run in the constructed SST_{sf} . Then we have*

- $q'_i = q_{j_i}, a_i = s(j_i), r'_i = r_{j_i}, p'_i = p_{j_i}$,
- the output $\sigma'(O)$ till position i of the input is $z_1 z_2 \dots z_{j_i}$

- In short, if the $2WST_{sf}$ reads a_i with $r_i \in Q_B, p_i \in Q_A$, and state q , then the $2WST_{sf}$ will be in state $f_i(q)$ when it next reads a_{i+1} , and produces $\sigma_{r',i}(X_q)$ as output.

Proof. The proof is by induction on the length of the prefixes of the run r' . Consider a prefix of length 1. In this case, we are at the first position of the input, in initial state $q'_0 = q_0$, since the $2WST_{sf}$ moves right. We also have the states $r'_1 = r_1, p'_1 = p_1$, and the output obtained as $O := O\rho(X_{q_0})$, and $\rho(X_{q_0}) = z_1$. Thus, for a prefix of length 1, the states coincide, and the output in O coincides with the output produced so far.

Assume that upto a prefix of length k , we obtain the conditions of the lemma, and consider a prefix of length $k + 1$.

Let $r = (q_0, i_0 = 0) \xrightarrow{r_1, s(i_0), p_1 | z_1} (q_1, i_1) \dots (q_{k-1}, i_{k-1}) \xrightarrow{r_k, s(i_{k-1}), p_k | z_k} (q_k, i_k) \xrightarrow{r_{k+1}, s(i_k), p_{k+1} | z_{k+1}} (q_{k+1}, i_{k+1}) \dots$ be a prefix of length $k + 1$. The corresponding run r' in the SST_{sf} is $r' = (q'_0, f_0) \xrightarrow{r'_1, a_1, p'_1} (q'_1, f_1) \dots (q'_{k-1}, f_{k-1}) \xrightarrow{r'_k, a_k, p'_k} (q'_k, f_k) \xrightarrow{r'_{k+1}, a_{k+1}, p'_{k+1}} (q'_{k+1}, f_{k+1}) \dots$, where by inductive hypothesis, we know that $q'_k = q_{j_k}, a_k = s_{j_k}, r'_k = r_{j_k}, p'_k = p_{j_k}$, and the output $\sigma'(O)$ till position k of the input is $z_1 z_2 \dots z_{j_k}$. $q'_k = q_{j_k}$ is the state the $2WST_{sf}$ is in, when it moves to the right of cell k containing a_k , for the first time, and reads a_{k+1} .

By construction of the SST_{sf} , $\delta'((q_{j_k}, f), r_{j_k}, p_{j_k}) = (f'(q_{j_k}), f')$, where f' is defined based on the transition in the $2WST_{sf}$ from state q_{j_k} on reading a_{k+1} .

1. If $\delta(q_{j_k}, a, r_{j_k}, p_{j_k}) = (t, \gamma, 1)$, then by construction of the SST_{sf} , we obtain $f'(q_{j_k}) = t$ and $\rho((q_{j_k}, f), a)(O) = O\rho(X_{q_{j_k}}) = O\gamma$. By inductive hypothesis, the contents of O agrees with the output till k steps. By catenating γ , in this case also, we obtain the same situation.
2. If $\delta(q_{j_k}, a, r_{j_k}, p_{j_k}) = (t, \gamma, 0)$, then by construction of the SST_{sf} , we obtain $f'(q_{j_k}) = f'(t)$ and $\rho((q_{j_k}, f), a)(O) = O\gamma\rho(X_t)$. Inducting on the number of steps it takes for the $2WST_{sf}$ to move to the right of cell $k + 1$, we can show that the condition in the lemma holds.
 - If the $2WST_{sf}$ moves to the right of cell $k + 1$ in the next step (one step), in state $u = q'_{k+1}$, then we obtain $f'(t) = u$, where $\delta(t, a, r, p) = (u, \gamma', 1)$, in which case we obtain $\rho((q_{j_k}, f), a)(O) = O\gamma\gamma' (O\rho(X_t))$, where the current contents of O is the old contents of O followed by γ and $\rho(X_t) = \gamma'$ which indeed is the output obtained in the $2WST_{sf}$ when it moves to the right of cell $k + 1$.
 - Assume as inductive hypothesis, that the result holds (that is, the output O agrees with the output in the $2WST_{sf}$ so far, and that the state of the SST_{sf} agrees with the state of the $2WST_{sf}$) when it moves to the right of cell k in $\leq m$ steps. Consider now the case when it moves to the right of cell $k + 1$ in $m + 1$ steps. At the end of m steps, the $2WST_{sf}$ is on cell $k + 1$, in some state q , and in the constructed SST_{sf} , by hypothesis, the contents of O correctly reflect the output so far. By assumption, we have $\delta(q, a_{k+1}, r, p) = (q'', \gamma'', 1)$ for some state q'' . Then we obtain, by the inductive hypothesis applied to cell k at the end of m steps, and the construction of the SST_{sf} , the SST_{sf} to be in state $(f'(q), f')$, where $f'(q) = q''$, and $\rho((q'', f), a_{k+1})(O) = O\gamma''$. Since O is up-to-date with the output with respect to the $2WST_{sf}$, catenating γ'' indeed keeps it up-to-date, and the state of the SST_{sf} q'_{k+1} is indeed the state the $2WST_{sf}$ is in, when it moves to the right of cell $k + 1$ for the first time.
3. If $\delta(q_{j_k}, a, r_{j_k}, p_{j_k}) = (t, \gamma, -1)$, then by construction of the SST_{sf} , we obtain $f'(q_{j_k}) = f'(f(t))$ and $\rho((q_{j_k}, f), a)(O) = O\gamma X_t \rho(X_{f(t)})$. The $2WST_{sf}$ is in cell k at this point of time. Inducting on the number of steps it takes for the $2WST_{sf}$ to move to the right of cell $k + 1$ as above, we can show that the condition in the lemma holds.

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1.2 Aperiodicity of the SST_{sf}

Having constructed the SST_{sf} , the next task is to argue that the SST_{sf} is aperiodic.

- **Lemma 28.** *The constructed SST_{sf} is aperiodic.*

Proof. We start with an aperiodic $2WST_{sf} \mathcal{A}$ and obtain the $SST_{sf} \mathcal{T}$. Since \mathcal{A} is aperiodic, there exists an $m \in \mathbb{N}$ such that $u^m \in L_{pq}^{xy}$ iff $u^{m+1} \in L_{pq}^{xy}$ for all $p, q \in Q$ and $x, y \in \{\ell, r\}$. To show the aperiodicity of the SST_{sf} , we have to show that the transition monoid which captures state and variable flow is aperiodic. Lets consider a string $u \in \Sigma^*$ and a run in the $SST_{sf} (q, f) \xrightarrow{u} (q', f')$. Based on the transitions of the SST_{sf} , we make a basic observation on variable assignments as follows. We show a correspondence between the flow of states for each kind of traversal in the $2WST_{sf}$, and the respective state and variable flow in the constructed SST_{sf} . Using this information and the aperiodicity of the $2WST_{sf}$, we prove the aperiodicity of the SST_{sf} by showing the state, variable flow to be identical for strings w^m, w^{m+1} , for some chosen $m \in \mathbb{N}$, and any w .

We know already that if $u = a$, and if $\delta(q, r, a, p) = (\gamma, q', 1)$ in the $2WST_{sf} \mathcal{A}$, then in the SST_{sf} we have the variable update $\rho(X_q) = \gamma$. If $u = a$, and if $\delta(q, r, a, p) = (\gamma, q', -1)$ in the $2WST_{sf} \mathcal{A}$, then in the SST we have the variable update $\rho(X_q) = \gamma X_{q'} \rho(X_{f(q')})$. We now generalize the above for any $u \in \Sigma^*$ as follows:

- (a) If the run of u in the $2WST_{sf} \mathcal{A}$ starts at the rightmost letter of u in state q , and leaves it at the right with output γ , then we have $\rho(X_q) = \gamma$.
- (b) Assume the run of u in the $2WST_{sf} \mathcal{A}$ starts at the rightmost letter of u in state q , and leaves it at the left in state t_1 . Since the whole input is read, eventually the $2WST_{sf}$ will leave u at the right in state $f'(q)$. Then we have $\rho(X_q) = \gamma_0 X_{t_1} \gamma_1 X_{t_2} \dots \gamma_{n-1} X_{t_n} \gamma_n$ such that (i) $u \in L_{q, t_1}^{r\ell}$ with output γ_0 , (ii) $u \in L_{f(t_1), t_2}^{\ell\ell}$ with output γ_1 . In general, $u \in L_{f(t_i), t_{i+1}}^{\ell\ell}$ with output γ_i , and (iii) since \mathcal{A} completely reads the input, at some point, $u \in L_{f(t_n), f'(q)}^{\ell, r}$ with output γ_n . Also, in state t_i , we move one position left from some cell h , the leftmost position of u , and in state $f(t_i)$ we move to the right of cell h for the first time, and the output generated in the interim is obtained from $\rho(X_{t_i})$.

We will show the above (a), (b) by inducting on u . For the base case $u = a$, we already have the proof, by the construction of the updates in the SST_{sf} . Let p be any state. Consider the case when in the $2WST_{sf}$ we start at the right of ua in state p and leave it on the right.

- If we start at state p on a , and $\delta(p, a) = (\gamma, q, -1)$, then we are at the right of u in state q . We will be back on a in state $f(q)$. By inductive hypothesis on u , we have $\rho(X_q)$ as some string over Γ^* , since we will leave u at the right starting in state q in the right. Again, by inductive hypothesis for a , we have $\rho(X_p) = \gamma X_q \rho(X_{f(q)})$; by the inductive hypothesis on u , when we leave u on the right starting in state q on the rightmost symbol of u , we obtain $\rho(X_q)$ as a constant. We are now on a in state $f(q)$. If we leave a to the right starting at $f(q)$, then we have by inductive hypothesis, $\rho(X_{f(q)})$ is a constant, and then we obtain $\rho(X_p)$ as a constant, while considering moving to the right of ua starting at a in state p . This agrees with (a) above.
- If we start at state p on a , and $\delta(p, a) = (\gamma, q, 0)$. By construction of the SST_{sf} , we have $\rho(X_p) = \gamma \rho(X_q)$. As long as we stay in the same position, we continue obtaining the same situation. If we leave a to the right eventually, then by inductive hypothesis, we obtain $\rho(X_p)$ as a string over Γ^* . If from some state t , while on a , we move one position to the left, (on the rightmost symbol of u), then by inductive hypothesis on u , we obtain $\rho(X_t)$ as a constant; finally, since we move to the right of a starting in state $f(t)$, we obtain $\rho(X_{f(t)})$ as a string over Γ^* , and hence, $\rho(X_p)$ as well.

Thus, the inductive hypothesis works since $f(p) = f(q)$.

- If we start at state p on a , and $\delta(p, a) = (\gamma, q, 1)$, then we straightaway have $\rho(X_p)$ as a constant, and are done.

Now consider the case when we start at the right of ua and leave it at the left. Let us start in state p on a .

- If we start at state p on a , and $\delta(p, a) = (\gamma, q, -1)$, then we are at the right of u in state q . By inductive hypothesis for a we also have $\rho(X_p) = \gamma X_q \rho(X_{f'(q)})$. If we leave ua at the left in state r , then starting in the right of u in state q , we leave it at the left in r . By inductive hypothesis on u , we have $\rho(X_q) = \gamma_0 X_r \dots X_{r'} \gamma'$, such that $u \in L_{f(r'), f'(q)}^{\ell r}$. Leaving a on the right from state $f'(q)$ gives by inductive hypothesis $\rho(X_{f'(q)})$ as a constant string β .

On ua starting in state q on the right, the content of X_p is then $\gamma \gamma_0 X_r \dots X_{r'} \gamma' \rho(X_{f'(q)}) = \gamma \gamma_0 X_r \dots X_{r'} \gamma' \beta$, which agrees with (b), since r, \dots, r' are the states in which we leave u at the left.

- If we start at state p on a , and $\delta(p, a) = (\gamma, q, 0)$. Then the inductive hypothesis works since $f(p) = f(q)$.
- Lastly, the case $\delta(p, a) = (\gamma, q, 1)$ does not apply since we are leaving ua at the left starting on a in state p .

The aperiodicity of the SST_{sf} is now proved as follows. Since the $2WST_{sf}$ is aperiodic, we know that there is an $m \in \mathbb{N}$ such that $u^m \in L_{pq}^{xy}$ iff $u^{m+1} \in L_{pq}^{xy}$. Moreover, we also know that the matrices of u^m, u^{m+1} are identical in the $2WST_{sf}$, which tells us that the sequence of states seen in the left, right traversals of u^m, u^{m+1} are identical. If this is the case, then in the above argument, we obtain the variable substitutions and state, variable flow for u^m and u^{m+1} to be identical, since the state and variable flow of the SST_{sf} only depends on the state flow of the $2WST_{sf}$. Then we obtain the transition monoids of u^m, u^{m+1} to be the same in the SST_{sf} . The states of the look-behind and look-ahead automata in the SST_{sf} also follow the same sequence of states of the look-behind and look-ahead automata in the $2WST_{sf}$. Since the look-behind and look-ahead automata are aperiodic, we obtain $M_{u^m} = M_{u^{m+1}}$ in the SST_{sf} . Hence the SST_{sf} is aperiodic. ◀

J Proofs from Section 6 : $SST_{sf} \subseteq SST$

We now show that we can eliminate the star-free look-around from the $SST_{sf}(T, A, B)$ without losing expressiveness. Eliminating the look behind is easy: B can be simulated by computing for each state $p_B \in P_B$, the state p'_B of P_B reached by B starting in p_B on the current prefix, and whenever p'_B is a final state of B , the transition is triggered.

In order to remove the look-ahead, we need to keep track of $P_i \in 2^{P_A}$ at every step. On processing a string $s = a_1 a_2 a_3 \dots \in \Sigma^\omega$ in T starting with $P_0 = \emptyset$, we obtain successively P_1, P_2, \dots where $P_{i+1} = P'_{i+1} \cup \{p_{i+1}\}$ such that $\delta(q_i, r_{i+1}, a_{i+1}, p_{i+1}) = q_{i+1}$, and for all $p \in P_i$, $\delta_A(p, a_{i+1}) \in P'_{i+1}$. Thus, starting with $P_0 = \emptyset$ and $\delta(q_0, r_1, a_1, p_1) = q_1$, we have $P_1 = \{p_1\}$, $P_2 = \delta_A(p_1, a_2) \cup \{p_2\}$ and so on. A configuration of the SST_{sf} is thus a tuple $(q, (r'_1, \dots, r'_n), 2^{P_A})$ where r'_i is the state reached in B on reading the current prefix from state r_i , assuming $P_B = \{r_1, \dots, r_n\}$, and 2^{P_A} is a set of states in P_A obtained as explained above. We say that ρ is an accepting run of s if $(q_0, (r_1, \dots, r_n), P_0)$ is an initial configuration, i.e. $q_0 \in Q_0, P_0 = \emptyset$ and after some point, we only see all elements of exactly one Muller set M_i repeating infinitely often in Q as well as in P_A i.e. $\Omega(\rho)_1 = M_i$ from domain of F and $\Omega(\rho)_2 = M_j$ from P_f . Also, $(q_i, (r'_1, \dots, r'_n), P_i) \xrightarrow{r'_k, a, p_{i+1}} (q'_i, (r''_1, \dots, r''_n), P_{i+1})$ iff $\delta(q_i, a, r'_k, p_{i+1}) = q'_i$, $P_{i+1} = \delta_A^*(P_i, a) \cup \{p_{i+1}\}$ and $\delta_B(r'_j, a) = r''_j$ for $1 \leq j \leq n$ and the state r''_k is an accepting state of B .

A configuration in the SST_{sf} is said to be *accessible* if it can be reached from an initial configuration, and *co-accessible* if from it accepting configurations can be reached. It is *useful* if it is both accessible and co-accessible. Note that from the mutual-exclusiveness of look-arounds and the determinism of A, B , it follows that for any input string, there is at most one run of the SST_{sf} from and to useful configurations, as shown in Appendix J.1.

The concept of substitutions induced by a run can be naturally extended from SST to SST_{sf} . Also, we can define the transformation implemented by an SST_{sf} in a straightforward manner. The transition monoid of an SST_{sf} is defined by matrices indexed by configurations $(q_i, (r_1, \dots, r_n), P_i) \in Q \times P_B^n \times 2^{P_A}$, using the notion of run defined before, and the definition of aperiodicity of SST_{sf} follows that of SST .

J.1 Uniqueness of Accepting Runs in SST_{sf}

Let $a_1 a_2 \dots \in \Sigma^\omega$ and $\rho : (q_0, (r_1, \dots, r_n), P_0) \xrightarrow{\alpha_1} (q_1, (r_1^1, \dots, r_n^1), P_1) \xrightarrow{\alpha_2} (q_2, (r_1^2, \dots, r_n^2), P_2) \dots$ be an accepting run in the SST_{sf} . We show that ρ as well as the sequences of transitions associated with ρ are unique. Given a sequence of transitions of the SST_{sf} , it is clear that there is exactly one run since both A, B are deterministic. Lets assume that the sequence of transitions are not unique, that is there is another accepting run ρ' for $a_1 a_2 \dots$. Let i be the smallest index where ρ and ρ' differ. The $(i-1)$ th configuration is then some $(q_i, (r'_1, \dots, r'_n), P_i)$ in both ρ, ρ' . Let us assume that we have two transitions $\delta(q_i, r_j, a_i, p_i) = q_{i+1}$ and $\delta(q_i, r_k, a_i, p'_i) = q'_{i+1}$ enabled such that $r_j \neq r_k$ or $p_i \neq p'_i$ or $q_{i+1} \neq q'_{i+1}$. Assume $r_j \neq r_k$. Since both are triggered, we have the prefix upto now is

in $L(B_{r_j}) \cap L(B_{r_k})$ with $r_j \neq r_k$, which contradicts mutual exclusiveness of look-behind. If $p_i \neq p'_i$, then since both runs are accepting, we have the infinite suffix in $L(A_{p_i}) \cap L(A_{p'_i})$, which contradicts the mutual exclusiveness of look-ahead. If $r_j = r_k$ and $p_i = p'_i$, but $q_{i+1} \neq q'_{i+1}$, then δ is not a function, which is again a contradiction.

Variable Flow and Transition Monoid of SST_{sf}

Variable Flow and Transition Monoid. Let P_A, P_B represent the states of the (deterministic) look-ahead and look-behind automaton A, B , and Q denote states of the SST_{sf} .

The transition monoid of an SST_{sf} depends on its configurations and variables. It extends the notion of transition monoid for SST with look-behind, look-ahead states components but is defined only on *useful* configurations $(q, (r'_1, \dots, r'_n), P)$. A configuration $(q, (r'_1, \dots, r'_n), P)$ is *useful* iff it is *accessible* and *co-accessible*: that is, $(q, (r'_1, \dots, r'_n), P)$ is reachable from the initial configuration $(q_0, (r_1, \dots, r_n), \emptyset)$ (here, $P_B = \{r_1, \dots, r_n\}$) and will reach a configuration $(q', (r''_1, \dots, r''_n), P')$ from where on, some muller subset of Q is witnessed continuously, and some muller subset of P_A is witnessed continuously.

Note that given two useful configurations $(q, (r'_1, \dots, r'_n), P)$, $(q', (r''_1, \dots, r''_n), P')$ and a string $s \in \Sigma^*$, there exists *at most* one run from $(q, (r'_1, \dots, r'_n), P)$ to $(q', (r''_1, \dots, r''_n), P')$ on s . Indeed, since $(q, (r'_1, \dots, r'_n), P)$ and $(q', (r''_1, \dots, r''_n), P')$ are both useful, there exists $s_1, s_2 \in \Sigma^*$ such that $(q_0, (r_1, \dots, r_n), \emptyset) \rightsquigarrow^{s_1} (q, (r'_1, \dots, r'_n), P)$ and $(q', (r''_1, \dots, r''_n), P') \rightsquigarrow^{s_2} (q'', (r'''_1, \dots, r'''_n), P'')$ such that from $(q'', (r'''_1, \dots, r'''_n), P'')$, we settle in some Muller set of both Q and P_A reading some $w \in \Sigma^\omega$. If there are two runs from $(q, (r'_1, \dots, r'_n), P)$ to $(q', (r''_1, \dots, r''_n), P')$ on s , then there are two accepting runs for $s_1 s s_2 w$, which contradicts the fact that accepting runs are unique. We denote by $\text{useful}(T, A, B)$ the useful configurations of (T, A, B) .

Thanks to the uniqueness of the sequence of transitions associated with the run of an SST_{sf} from and to useful configurations on a given string, one can extend the notion of variable flow naturally by considering, as for SST , the composition of the variable updates along the run.

Assume that T has j muller sets and B has k muller sets. A string $s \in \Sigma^*$ maps to a square matrix M_s of dimension $|Q| \times (P_B \times \dots \times P_B) \times 2^{P_A} \cdot |\mathcal{X}|$ and is defined as

- $M_s[(q, (r'_1, \dots, r'_n), P), X][(q', (r''_1, \dots, r''_n), P'), X'] = n, \alpha_1, \alpha_2$ if there exists a run ρ from $(q, (r'_1, \dots, r'_n), P)$ to $(q', (r''_1, \dots, r''_n), P')$ on s such that n copies of X flows to X' over the run ρ , and $(q, (r_1, \dots, r_n), P)$ and $(q', (r'_1, \dots, r'_n), P')$ are both useful (which implies that the sequence of transitions of ρ from $(q, (r_1, \dots, r_n), P)$ to $(q', (r'_1, \dots, r'_n), P')$ is unique, as seen before), and
 - $\alpha_1 \in \{0, 1, \kappa\}^j$ and $\alpha_2 \in 2^{\{0, 1, \kappa\}^k}$. α_1 is a j -tuple keeping track for each of the j muller sets of T , whether the set of states seen on reading a string s is a muller set, a strict subset of it, or has seen a state outside of the muller set. Likewise, α_2 is a set of k -tuples doing the same thing based on the transition from set P to set P' . Note that since we keep a subset of states of P_A in the configuration, we need to keep track of this information for each state in the set. Thus, α_1 keeps track of the run from q and whether it witnesses a full muller set (1), a partial muller set (κ), or goes outside of a muller set (0). Likewise, α_2 keeps track of the same for each $p \in P_i$, the run from p . If a run is accepting, then α_2 will eventually become the singleton $\{(0, \dots, 0, 1, 0, \dots, 0)\}$ corresponding to some muller set of P_A .
 - $M_s[(q, (r'_1, \dots, r'_n), P), X][(q', (r''_1, \dots, r''_n), P'), X'] = \perp$, otherwise.

► **Lemma 29.** *For all aperiodic 1-bounded SST_{sf} with star-free look-around, there exists an equivalent aperiodic 1-bounded SST .*

Proof. Let (T, A, B) be an SST_{sf} , with $A = (P_A, \Sigma, \delta_A, P_f)$ a deterministic look-ahead muller automaton, $B = (P_B, \Sigma, \delta_B)$ be a deterministic look-behind automaton. Let $T = (\Sigma, \Gamma, Q, q_0, \delta, \mathcal{X}, \rho, F)$. Without loss of generality, we make the following *unique successor* assumption

- For all states $q, q', q'' \in Q$, and for all states $p, p' \in P_A$, and for all states $r, r' \in P_B$, and for any symbol $a \in \Sigma$, whenever $p \neq p'$ or $r \neq r'$, and if $\delta(q, r, a, p) = q'$, $\delta(q, r', a, p') = q''$, then $q' \neq q''$.

If this is not the case, say we have $p \neq p', r = r'$, and $q' = q''$. Then considering the state of T as (q, r) , we obtain $\delta((q, r), a, p) = (q', r) = \delta((q, r), a, p')$. However, it is easy to define transitions $\delta((q, r), a, p) = (q', r)$, $\delta((q, r), a, p') = (q'', r)$ by duplicating the transitions of q', q'' , without affecting anything else, especially the aperiodicity. The same argument can be given when $r \neq r'$ or both $(p \neq p'$ and $r \neq r')$. Thus, for our convenience, we assume the *unique successor* assumption without loss of generality.

J.2 Elimination of look-around from (T, A, B) : Construction of T'

We construct an aperiodic and 1-bounded SST T' equivalent to $\text{SST}_{sf}(T, A, B)$. As explained in definition of SST_{sf} , the unique run of a string s on (T, A, B) is not only a sequence of Q -states, but also a collection of the look ahead states 2^{P_A} , and maintains the reachable state from each state of P_B . At any time, the current state of Q , the n -tuple of reachable states of P_B , and the collection of look-ahead states $P \subseteq P_A$ is a configuration. For brevity of notation, let η (we also use ζ, η' in the sequel) denote the n -tuple of P_B states.

A configuration (q_1, η_1, P_1) , on reading a , evolves into $(q_2, \eta_2, P_2 \cup \{p_2\})$, where $\delta(q_1, r_2, a, p_2) = q_2$ is a transition in the SST_{sf} and $\delta_A(P_1, a) = P_2$, where δ_A is the transition function of the look ahead automaton A , and the state reached from r_2 is an accepting state of P_B . Note that the transition monoid of the SST_{sf} is aperiodic and 1-bounded by assumption. We now show how to remove the look-around, resulting in an equivalent SST T' whose transition monoid is aperiodic and 1-bounded.

While defining T' , we put together all the states resulting from transitions of the form (q, r, a, p, q') and (q, r', a, p', q'') in the SST_{sf} . We define $T' = (\Sigma, \Gamma, Q', q'_0, \delta', \mathcal{X}', \rho', Q'_f)$ with:

- $Q' = 2^{\text{useful}(T, A, B)}$ where $\text{useful}(T, A, B)$ are the useful configurations of (T, A, B) ;
(Note that we can pre-compute $\text{useful}(T, A, B)$ once we know (T, A, B)),
- $q'_0 = \{(q_0, \eta, \emptyset)\}$ ($\eta = (r_1, \dots, r_n)$ where $P_B = \{r_1, \dots, r_n\}$). We assume wlog that, (T, A, B) accepts at least one input. Therefore (q_0, η, \emptyset) is useful),
- Q'_f , the set of muller sets of T' , is defined by the set of pairs (S, R) where S is any of the j muller sets in F , and R is any of the k muller sets in P_f .
- $\mathcal{X}' = \{X_{q'} \mid X \in \mathcal{X}, q' \in \text{useful}(T, A, B)\}$,
- The transitions are defined as follows: $\delta'(S, a) = \bigcup_{(q, \eta, P) \in S} \Delta((q, \eta, P), a)$ where $\Delta((q, \eta, P), a) = \{(q', \eta', P' \cup \{p'\}) \mid \delta(q, r, a, p') = q' \text{ and } \delta_A(P, a) = P'\} \cap \text{useful}(T, A, B)$. Each component of η' is obtained from η using δ_B .

Before defining the update function, we first assume a total ordering $\preceq_{\text{useful}(T, A, B)}$ on $\text{useful}(T, A, B)$. For all (p, η, P) , we define the substitution $\sigma_{(p, \eta, P)}$ as $X \in \mathcal{X} \mapsto X_{(p, \eta, P)}$. Let (S, a, S') be a transition of T' . Given a state $(q', \eta', P') \in S'$, there might be several predecessor states $(q_1, \eta_1, P_1), \dots, (q_k, \eta_k, P_k)$ in S on reading a . The set $\{(q_1, \eta_1, P_1), \dots, (q_k, \eta_k, P_k)\} \subseteq S$ is denoted by $\text{Pre}_S((q', \eta', P'), a)$. Formally, it is defined by $\{(q, \eta, P) \in S \mid (q', \eta', P') \in \Delta((q, \eta, P), a)\}$.

We consider only the variable update of the transition from the minimal predecessor state. Indeed, since any string has at most one accepting run in the $\text{SST}_{sf}(T, A, B)$ (and at most one associated sequence of transitions), if two runs reach the same state at some point, they will anyway define the same output and therefore we can drop one of the variable update, as shown in [4]. Formally, the variable update $\rho'(S, a, S')(X_{(q', \eta', P')})$, for all $X_{(q', \eta', P')} \in \mathcal{X}'$ is defined by ϵ if $(q', \eta', P') \notin S'$, and by $\sigma_{(q, \eta, P)} \circ \rho(q, r, a, p, q')(X)$, where $(q, \eta, P) = \min \{(t, \zeta, R) \in S \mid (q', \eta', P') \in \Delta((t, \zeta, R), a)\}$, and $\delta(q, r, a, p) = q'$ (by the *unique successor* assumption the look-around states p, r are unique). It is shown in [4] that indeed T' is equivalent to T . We show here that the transition monoid of T' is aperiodic and 1-bounded.

For all $S \in Q'$, and $s \in \Sigma^*$, define $\Delta^*(S, s) = \{(q', \eta', P') \mid \exists (q, \eta, P) \in S \text{ such that } (q, \eta, P) \rightsquigarrow_{T, A, B}^s (q', \eta', P')\} \cap \text{useful}(T, A, B)$.

J.3 Connecting Transition Monoids of T' and (T, A, B)

► **Lemma 30.** *Let $M_{T'}$ be the transition monoid of T' and $M_{(T, A, B)}$ the transition monoid of (T, A, B) . Let $S_1, S_2 \in Q'$, $X_{(q, \eta, P)}, Y_{(q', \eta', P')} \in \mathcal{X}'$ and $s \in \Sigma^*$.*

Then $M_{T',s}[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$ iff $S_2 = \Delta^*(S_1, s)$ and one of the following hold:

1. either $i = 0$ and, $(q, \eta, P) \notin S_1$ or $(q', \eta', P') \notin S_2$, or
2. $(q, \eta, P) \in S_1$, $(q', \eta', P') \in S_2$, (q, η, P) is the minimal ancestor in S_1 of (q', η', P') (i.e. $(q, \eta, P) = \min \{(t, \zeta, R) \in S_1 \mid (q', \eta', P') \in \Delta^*((t, \zeta, R), s)\}$), and $M_{(T,A,B),s}[(q, \eta, P), X][(q', \eta', P'), Y] = i, \alpha_1, \alpha_2$.

Proof. It is easily shown that $M_{T',s}[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$ with $i \geq 0$ iff $S_2 = \Delta^*(S_1, s)$. Let us show the two other conditions.

Assume that $M_{T',s}[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$. The variable update function is defined in T' as follows: after reading string s , from state S_1 , all the variables $Z_{(t,\zeta,R)}$ such that $(t, \zeta, R) \notin S_2$ are reset to ϵ (and therefore no variable can flow from S_1 to them). In particular, if $(q', \eta', P') \notin S_2$, then no variable can flow in $Y_{(q',\eta',P')}$ and $i = 0$.

Now, assume that $(q', \eta', P') \in S_2$, and consider the sequence of states $S_1, S'_1, S'_2, \dots, S'_k, S_2$ of T' on reading s . By definition of the variable update, the variables that are used to update $Y_{(q',\eta',P')}$ on reading the last symbol of s from S'_k are copies of the form $Z_{(t,\zeta,R)}$ such that (t, ζ, R) is the minimal predecessor in S'_k of (q', η', P') (by Δ). By induction, it is easily shown that if some variable $Z_{(t,\zeta,R)}$ flows to $Y_{(q',\eta',P')}$ from S_1 to S_2 on reading s , then (t, ζ, R) is necessarily the minimal ancestor (by Δ^*) of (q', η', P') on reading s . In particular if $(q, \eta, P) \notin S_1$, then $i = 0$.

Finally, if $i > 0$, then necessarily (q, η, P) is the minimal ancestor in S_1 of (q', η', P') on reading s , from S_1 to S_2 , and since T' mimics the variable update of (T, A, B) on the copies, we get that $M_{(T,A,B),s}[(q, \eta, P), X][(q', \eta', P'), Y] = i, \alpha_1, \alpha_2$.

The converse is shown similarly. \blacktriangleleft

J.4 1-boundedness and aperiodicity of T'

1-boundedness is an obvious consequence of the claim and the fact that (T, A, B) is 1-bounded. Let us show that $M_{T'}$ is aperiodic. We know that $M_{(T,A,B)}$ is aperiodic. Therefore there exists $n \in \mathbb{N}$ such that for all strings $s \in \Sigma^*$, $M_{(T,A,B),s}^n = M_{(T,A,B),s}^{n+1}$.

We first show that $\forall S_1, S_2 \in Q'$, and all strings $s \in \Sigma^*$, $\Delta^*(S_1, s^n) = S_2$ iff $\Delta^*(S_1, s^{n+1}) = S_2$. Indeed,

- $S_2 = \Delta^*(S_1, s^n)$, iff

$$S_2 = \{(q', \eta', P') \in \text{useful}(T, A, B) \mid \exists (q, \eta, P) \in S_1, (q, \eta, P) \rightsquigarrow_{T,A,B}^{s^n} (q', \eta', P')\},$$

iff

- $S_2 = \{(q', \eta', P') \in \text{useful}(T, A, B) \mid \exists (q, \eta, P) \in S_1, M_{(T,A,B),s}^n[(q, \eta, P), X][(q', \eta', P'), Y] = i, \alpha_1, \alpha_2, i \geq 0 \text{ for some } X, Y \in \mathcal{X}\}$, iff
- by aperiodicity of $M_{(T,A,B)}$,

$$S_2 = \{(q', \eta', P') \in \text{useful}(T, A, B) \mid \exists (q, \eta, P) \in S_1,$$

$$M_{(T,A,B),s}^{n+1}[(q, \eta, P), X][(q', \eta', P'), Y] = i, \alpha_1, \alpha_2, i \geq 0 \text{ for some } X, Y \in \mathcal{X}\}$$

iff

- $S_2 = \Delta^*(S_1, s^{n+1})$.

Let $S_1, S_2 \in Q'$ and $X_{(q,\eta,P)}, Y_{(q',\eta',P')} \in \mathcal{X}$. Let also $s \in \Sigma^*$. We now show that the matrices corresponding to s^n, s^{n+1} coincide, using Lemma 30. By the first condition of Lemma 30, we have

$M_{T',s}^n[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$ and condition (1) of Lemma 30 holds, iff

$M_{T',s}^{n+1}[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$ and condition (1) of Lemma 30 holds.

- Indeed, $M_{T',s}^n[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = 0, \alpha_1, \alpha_2$ and, $(q, \eta, P) \notin S_1$ or $(q', \eta', P') \notin S_2$ iff by Lemma 30, $\Delta^*(S_1, s^n) = S_2$, and $(q, \eta, P) \notin S_1$ or $(q', \eta', P') \notin S_2$, iff by what we just showed, $\Delta^*(S_1, s^{n+1}) = S_2$, and $(q, \eta, P) \notin S_1$ or $(q', \eta', P') \notin S_2$, iff by Lemma 30, $M_{T',s}^{n+1}[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = 0, \alpha_1, \alpha_2$ and condition (1) of Lemma 30 holds.

Let us now look at condition (2) of Lemma 30, and show that

$M_{T',s}^n[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$ and condition (2) of Lemma 30 holds, iff
 $M_{T',s}^{n+1}[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$ and condition (2) of Lemma 30 holds.

- We only show one direction, the other being proved exactly similarly. Assume that $M_{T',s}^n[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$ and $(q, \eta, P) \in S_1$, $(q', \eta', P') \in S_2$, and (q, η, P) is the minimal ancestor in S_1 of (q', η', P') , and $M_{(T,A,B),s}^n[(q, \eta, P), X][(q', \eta', P'), Y] = i, \alpha_1, \alpha_2$. By Lemma 30, we have $\Delta^*(S_1, s^n) = S_2$, and therefore $\Delta^*(S_1, s^{n+1}) = S_2$. Now, we have $(q, \eta, P) = \min \{(t, \zeta, R) \in S_1 \mid (q', \eta', P') \in \Delta^*((t, \zeta, R), s^n)\}$. Since $\Delta^*((t, \zeta, R), s^n) = \Delta^*((t, \zeta, R), s^{n+1})$ for all $(t, \zeta, R) \in S_1$, we have $(q, \eta, P) = \min \{(t, \zeta, R) \in S_1 \mid (q', \eta', P') \in \Delta^*((t, \zeta, R), s^{n+1})\}$.

Finally, $M_{(T,A,B),s}^{n+1}[q, \eta, P, X][q', \eta', P', Y] = M_{(T,A,B),s}^n[q, \eta, P, X][q', \eta', P', Y]$ (by aperiodicity of (T, A, B)). By Lemma 30, we obtain $M_{T',s}^{n+1}[S_1, X_{(q,\eta,P)}][S_2, Y_{(q',\eta',P')}] = i, \alpha_1, \alpha_2$ and hence condition (2) of Lemma 30 is satisfied.

Using Lemma 30 and the aperiodicity of $M_{(T,A,B)}$, we obtain the aperiodicity of $M_{T'}$. ◀