Variants of Weighted Automata

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Motivation

- classical automata with weight on transitions.
- weights may model
  - cost involved
  - amount of resources
  - probability or reliability of execution
- unifying framework using semiring
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- classical automata with weight on transitions.
- weights may model:
  - cost involved
  - amount of resources
  - probability or reliability of execution
- unifying framework using semiring
Preliminaries

- **Semiring**: structure \((K, +, \cdot, 0, 1)\)
  - \((K, +, 0)\) is a commutative monoid
  - \((K, \cdot)\) is a monoid
  - \(\cdot\) distributes over \(+\)
  - 0 is annihilator wrt \(\cdot\)
  - e.g., \(\mathbb{B} = (\{0, 1\}, \lor, \land, 0, 1)\)

- **Formal Power Series**: 
  - mapping \(S : A^* \rightarrow K\)
  - characteristic series \(1_L : A^* \rightarrow K\)
    - \((1_L, w) = 1\) if \(w \in L\)
    - \((1_L, w) = 0\) otherwise
Weighted Automata

- fix a semiring $K$ and an alphabet $A$.
- weighted finite automaton over $A$ and $K$ is a quadruple $A = (Q, \lambda, \mu, \gamma)$
- $Q$ - finite set of states
- $\mu : A \rightarrow K^{Q \times Q}$ - transition weight function
- $\lambda, \gamma : Q \rightarrow K$ - weight functions for entering and leaving a state
Example

e.g., Compute $f(a, b) = \text{maximum } n_b - n_a$ in any prefix

![Weighted automaton](image)

Figure 1: Weighted automaton $A$ to compute $f(a, b)$

$$\lambda(1) = 0 \quad \gamma(1) = -\infty$$
$$\lambda(2) = -\infty \quad \gamma(2) = 0$$

Over the semiring $(\mathbb{Z} \cup \{-\infty\}, \text{max}, +, -\infty, 0)$
Closure Properties

- The formal power series $\|A\|: A^* \rightarrow K$ is called the behavior of $A$.
- Recognizable if there exists a weighted automaton $A$ such that $S = \|A\|$.

Lemma 1
[Droste and Gastin, 2007]

1. Let $S, T \in K\langle A^* \rangle$ be recognizable, then $S + T$ is recognizable. If $K$ is commutative, then $S \odot T$ is also recognizable.

2. For any recognizable language $L \subseteq A^*$, the series $1_L$ is recognizable.
The syntax of weighted logic formula is given by

$$\varphi := k \mid P_a(x) \mid \neg P_a(x) \mid x \leq y \mid \neg (x \leq y) \mid x \in X \mid \neg (x \in X) \mid \varphi \lor \psi \mid \varphi \land \psi \mid \exists x. \varphi \mid \exists X. \varphi \mid \forall x. \varphi \mid \forall X. \varphi$$

where $k \in K$ and $a \in \Sigma$. $MSO(K, A)$ contains all such weighted logic formulas.
Weighted Logic Semantics

Let $\phi \in MSO(K, A)$ and $\mathcal{V}$ be a finite set of variables containing $\text{Free}(\phi)$. The $\mathcal{V}$ semantics of $\phi$ is a formal power series $[[\phi]]_{\mathcal{V}} \in A_{\mathcal{V}}^*$. If $\sigma$ is not a valid $(\mathcal{V}, w)$-assignment, then $[[\phi]]_{\mathcal{V}}(w, \sigma) = 0$. Otherwise define $[[\phi]]_{\mathcal{V}}(w, \sigma) \in K$ inductively as follows:

- $[[k]]_{\mathcal{V}}(w, \sigma) = k$
- $[[P_a(x)]]_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \text{if } w(\sigma(x)) = a \\ 0 & \text{otherwise} \end{cases}$
- $[[x \leq y]]_{\mathcal{V}}(w, \sigma) = \begin{cases} 1 & \text{if } \sigma(x) \leq \sigma(y) \\ 0 & \text{otherwise} \end{cases}$
Weighted Logic Semantics

\[ \nu(\sigma) = \begin{cases} 
1 & \text{if } \sigma(x) \in \sigma(X) \\
0 & \text{otherwise} 
\end{cases} \]

\[ \lnot \varphi(w, \sigma) = \begin{cases} 
1 & \text{if } \nu(\varphi(w, \sigma)) = 0 \text{ if } \varphi \text{ is of the form } P_a(x), \\
0 & \text{if } \nu(\varphi(w, \sigma)) = 1(x \leq y) \text{ or } (x \in X). 
\end{cases} \]

\[ \nu(\varphi \lor \psi)(w, \sigma) = \nu(\varphi(w, \sigma)) + \nu(\psi(w, \sigma)) \]

\[ \nu(\varphi \land \psi)(w, \sigma) = \nu(\varphi(w, \sigma)) \cdot \nu(\psi(w, \sigma)) \]
Weighted Logic Semantics

▶ $\exists x. \varphi \triangleright v(w, \sigma) = \sum_{1 \leq i \leq |w|} [\varphi]_{\nu \cup \{x\}}(w, \sigma[x \rightarrow i])$

▶ $\exists X. \varphi \triangleright v(w, \sigma) = \sum_{I \subseteq \{1, \ldots, |w|\}} [\varphi]_{\nu \cup \{X\}}(w, \sigma[X \rightarrow I])$

▶ $\forall x. \varphi \triangleright v(w, \sigma) = \prod_{1 \leq i \leq |w|} [\varphi]_{\nu \cup \{x\}}(w, \sigma[x \rightarrow i])$

▶ $\forall X. \varphi \triangleright v(w, \sigma) = \prod_{I \subseteq \{1, \ldots, |w|\}} [\varphi]_{\nu \cup \{X\}}(w, \sigma[X \rightarrow I])$
unrestricted universal quantification is too strong to preserve recognizability.

Definition 2
[Droste and Gastin, 2007] A formula \( \phi \in MSO(K, A) \) is called restricted, if it contains no universal set quantification of the form
\( \forall X. \psi \) and whenever \( \phi \) contains a universal first-order quantification \( \forall X. \psi \), then \( \llbracket \psi \rrbracket \) is a recognizable step function.

Theorem 3
[Droste and Gastin, 2007] Let \( K \) be a commutative semiring and \( A \) an alphabet. Then,
\[
K^{rec} \langle \langle A^* \rangle \rangle = K^{rmso} \langle \langle A^* \rangle \rangle = K^{remso} \langle \langle A^* \rangle \rangle.
\]
Weighted Automata - A general Model

- Finite Automata over $\text{Bool} = (\{0, 1\}, \lor, \land, 0, 1)$ semiring.
- Word Transducer over $(\Gamma, \cup, \cdot, \phi, \{\epsilon\})$ semiring
- Probabilistic Automata over $\text{Prob} = (\mathbb{R}_{\geq 0} \cap [0, 1], +, \cdot, 0, 1)$ semiring.
Probabilistic Automata and Stochastic Matrix

Definition 4

[Rabin, 1963] A \textit{probabilistic automaton} is a weighted automaton $A = (Q, A, \lambda, \mu, \gamma)$ over $\text{Prob} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$ such that

1. there is a single state $p \in Q$ such that $\lambda(p) = 1$ and, for all $q \in Q \setminus \{p\}$, $\lambda(q) = 0$,
2. for all $p \in Q$, $\gamma(p) \in \{0, 1\}$, and
3. for all $p \in Q$ and $a \in A$, we have $\sum_{q \in Q} \mu(p, a, q) = 1$.

$||A|| = \lambda \cdot \mu \cdot \gamma$.

- A matrix of $\mathbb{R}_{\geq 0}^{Q \times Q}$ is \textit{stochastic} if each of its rows sums to 1.

Proposition 5

The product of two stochastic matrices is again a stochastic matrix.
Stochastic Languages

Definition 6
A language $L \subseteq A^*$ is stochastic if there is a probabilistic automaton $A = (Q, A, \lambda, \mu, \gamma)$ and $\theta \in [0, 1]$ such that $L = L_{>\theta}(A)$.

Fundamental Results:

1. Every regular language is stochastic.
2. There is a stochastic language that is not recursively enumerable.
3. For isolated cut point, the associated threshold language is regular.

Definition 7 (isolated cut point)
Let $A = (Q, A, \lambda, \mu, \gamma)$ be a probabilistic automaton and let $\theta \in [0, 1]$. We can say $\theta$ is an *isolated cut point* of $A$ if there is $\delta > 0$ such that, for all $w \in A^*$, we have $|||A||(w) - \theta| \geq \delta$. 
Threshold Emptiness

<table>
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<th>0</th>
<th>1</th>
<th>$\theta \in (0, 1)$</th>
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**Figure 2**: Table about decidability of emptiness problem for probabilistic automata with different thresholds
It is natural, for a unary probabilistic automaton, and a relation $\sim \in \{>, \geq, =\}$, to consider the following problem:

**Input**

Unary Probabilistic automaton $A$ and $\theta 

**Threshold Emptiness**

Do we have $L_{\sim \theta}(A) \neq \phi$?

**Threshold Universality**

Do we have $L_{\sim \theta}(A) = A^*$?
Decision Problems:

Let \( C \) be a class of weighted automata over \( S \). The emptiness problem for \( C \) is given as follows:

**Input**

Unary Probabilistic automaton \( A \) and \( \theta \in (0, 1) \).

**Threshold Emptiness**

Do we have \( L_{\sim \theta}(A) \neq \phi \)?

**Threshold Universality**

Do we have \( L_{\sim \theta}(A) = A^* \)?

For a semiring \( S = (S, +, \cdot, 0, 1) \), a relation \( \nleq \subseteq S \times S \), and a class \( C \) of weighted automata over \( S \), consider the following problem:

**Input:**

Weighted automaton \( A \in C \) and \( \theta \in S \).

**Threshold Regularity wrt \( \nleq \)**

Is \( L_{\nleq \theta}(A) \) regular?
Decision Problems

We may also want to decide whether a threshold language empty or universal.

**Input:**
Weighted automaton $A \in \mathcal{C}$ and $\theta \in S$

**Threshold Emptiness wrt $1$**
Do we have $L_{\times \theta}(A) \neq \phi$?

**Threshold Universality wrt $1$**
Do we have $L_{\times \theta}(A) = A^*$?

Another question is to compare universally or existentially the semantics of two given automata.

**Input:**
Weighted Automata $A, B \in \mathcal{C}$.

**(IN)Equality wrt $\times$**
Do we have $\|A\|(w) \times \|B\|(w)$ for all $w \in A^*$?

**Existential (IN)Equality wrt $\times$**
Is there $w \in A^*$ such that $\|A\|(w) \times \|B\|(w)$?

Thank You. :}

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*Additional note to the recipient*